A FAMILY OF NONLINEAR RECURRENCES AND THEIR LINEAR SOLUTIONS

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Abstract. We solve a second order recurrence; the solutions are second order linear recurrences. Some solutions are related to Fibonacci sequences.

1. Nonlinear Recurrence

We consider a nonlinear recurrence relation (with parameters p and q) and show that its sequence solutions are second order linear recurrences. For special values of the parameters this is related to Fibonacci sequences, see Section 2.

For constants p and q with initial values $u_0 = a$ and $u_1 = b$, the nonlinear recurrence relation is

$$u_{n+1}(u_n - u_{n-1}) = pu_n^2 - u_n u_{n-1} - q. \tag{1}$$

Lemma 1.1.

$$\frac{u_{n-1}^2 + u_n^2 - (p+1)u_n u_{n-1} + q}{u_{n-1} - u_n} = (-1)^n \frac{a^2 + b^2 - (p+1)ab + q}{b - a}.$$

Proof. For $n = 1$, the equation of the lemma follows immediately from the definition.

Assume this has been shown for n. We will check this for $n + 1$. Substitute the recurrence relation

$$u_{n+1} = \frac{pu_n^2 - u_n u_{n-1} - q}{(u_n - u_{n-1})}$$

into

$$\frac{u_n^2 + u_{n+1}^2 - (p+1)u_{n+1}u_n + q}{u_n - u_{n+1}}$$

and simplify to get

$$-\frac{u_{n-1}^2 + u_n^2 - (p+1)u_n u_{n-1} + q}{u_{n-1} - u_n}.$$

The result follows. □

Theorem 1.2. *The solutions to this nonlinear recurrence satisfy the second order linear recurrence*

$$u_{n+1} = (p+1)u_n - u_{n-1} + (-1)^n \frac{a^2 + b^2 - (p+1)ab + q}{b - a}. \tag{2}$$

Proof. Using

$$\begin{aligned} u_{n+1} - (p+1)u_n + u_{n-1} &= \frac{pu_n^2 - u_n u_{n-1} - q}{(u_n - u_{n-1})} - (p+1)u_n + u_{n-1} \\ &= -\frac{u_{n-1}^2 + u_n^2 - (p+1)u_n u_{n-1} + q}{u_{n-1} - u_n} \end{aligned}$$

and Lemma 1.1 gives the desired result. □

1.1. $p = 0$. The sequence is periodic of period 6.

1.2. $p = -1$. The sequence is periodic of period 4.

1.3. $p = -2$. The sequence is periodic of period 6.

2. Fibonacci-Related Sequences

Theorem 2.1. *For $p = 2$, $q = 1$, and $n \geq 0$,*

$$u_{n+2} = \frac{A_n a^2 + B_n b^2 + C_n ab + D_n}{b - a}.$$

Proof. It can be shown that $u_2 = \frac{2b^2 - ab - 1}{b-a}$ and $u_3 = \frac{a^2 + 6b^2 - 5ab - 2}{b-a}$, so the base case for an induction is valid and

$$A_0 = 0,\ A_1 = 1,\ B_0 = 2,\ B_1 = 6,\ C_0 = -1,\ C_1 = -5,\ D_0 = -1,\ D_1 = -2.$$

Assume that the cases of n and $n + 1$ have been shown. Then,

$$u_{n+2} = 3u_{n+1} - u_n + (-1)^{n+1} \frac{a^2 + b^2 - 3ab + 1}{b - a}.$$

Thus, A_n, B_n, and D_n satisfy

$$\alpha_n = 3\alpha_{n-1} - \alpha_{n-2} + (-1)^{n+1}.$$

Its generating series $\alpha(x)$ satisfies

$$\sum_{n\geq 0} \alpha_{n+2} x^{n+2} = 3x \sum_{n\geq 0} \alpha_{n+1} x^{n+1} - x^2 \sum_{n\geq 0} \alpha_n x^n - x^2 \sum_{n\geq 0} (-1)^n x^n,$$

which gives

$$\alpha(x) - \alpha_0 - \alpha_1 x = 3x(\alpha(x) - \alpha_0) - \alpha(x)x^2 - \frac{x^2}{1 + x}.$$

So,

$$\alpha(x) = \frac{\alpha_0 + (\alpha_1 - 2\alpha_0)x + (-1 - 3\alpha_0 + \alpha_1)x^2}{(1 + x)(1 - 3x + x^2)}.$$

Thus,

$$D(x) = \frac{-1}{(1 + x)(1 - 3x + x^2)},$$

$$B(x) = \frac{2 + 2x - x^2}{(1 + x)(1 - 3x + x^2)},$$

$$A(x) = \frac{x}{(1 + x)(1 - 3x + x^2)}.$$

Let $c_n = -C_n$. Summing the generating function

$$\sum_{n\geq 2} c_n x^n = 3x \sum_{n\geq 2} c_{n-1} x^{n-1} - x^2 \sum_{n\geq 2} c_{n-2} x^{n-2} - 3 \sum_{n\geq 2} (-1)^n x^n.$$

So,

$$c(x) - c_0 - c_1 x = 3x(c(x) - c_0) - c(x)x^2 - \frac{3x^2}{1 + x}.$$

Hence,

$$C(x) = \frac{-1-3x+x^2}{(1+x)(1-3x+x^2)}.$$

□

The sequence given by D_{n-1} is $-F_nF_{n+1}$, $n \geq 1$ [1, A001654], and C_{n-1} is $F_{n+1}^2 - F_n^2 + F_{n+1}F_n$, $n \geq 1$ [1, A236428], for the Fibonacci sequence F_n. This leads to other Fibonacci identities as illustrated by the examples below.

2.1. Examples from OEIS[1].

- $p = 2$, $q = 1$, $a = 0$, $b = 1$ A007598, squares of Fibonacci
- $p = 2$, $q = 1$, $a = 1$, $b = 2$ A001519
- $p = 2$, $q = 1$, $a = 1$, $b = 3$ A061646
- $p = 2$, $q = 1$, $a = 1$, $b = 5$ A236428
- $p = 2$, $q = 1$, $a = 2$, $b = 3$ A248161

For the proof of the first item $p = 2$, $q = 1$, $a = 0$, and $b = 1$, using the formulas above,

$$\begin{aligned} B_{n-1} + D_{n-1} &= F_{n+1}F_n + 2F_{n-1}F_n - F_{n-2}F_{n-1} \\ &= F_{n+1}F_n + F_{n-1}F_n + F_{n-1}(F_n - F_{n-2}) \\ &= F_n(F_{n+1} + F_{n-1}) + F_{n-1}^2 = F_n(F_n + 2F_{n-1}) + F_{n-1}^2 \\ &= F_{n+1}^2. \end{aligned} \tag{3}$$

2.2. $q = 0$, $p = 2$.

Theorem 2.2. *If $q = 0$, $p = 2$, and $n \geq 0$,*

$$u_{n+2} = \frac{(F_{n+1}a - F_{n+3}b)(F_na - F_{n+2}b)}{b-a}.$$

Proof. We can see that $u_2 = -\frac{b(a-2b)}{b-a}$ and $u_3 = \frac{(a-2b)(a-3b)}{b-a}$. Using the same notation as in the proof of the previous theorem, we have that $A_0 = 0$, $A_1 = 1$, $B_0 = 2$, $B_1 = 6$, $C_0 = -1$, and $C_1 = -5$, so as before $A(x) = \frac{x}{(1+x)(1-3x+x^2)}$, $A_n = F_nF_{n+1}$, $B(x) = \frac{2+2x-x^2}{(1+x)(1-3x+x^2)}$, and $C(x) = \frac{-1-3x+x^2}{(1+x)(1-3x+x^2)}$. Thus,

$$\begin{aligned} B_{n-1} &= 2F_nF_{n+1} + 2F_{n-1}F_n - F_{n-2}F_{n-1} \\ &= F_nF_{n+1} + F_{n+1}^2, \quad (3) \\ &= F_{n+1}F_{n+2}, \end{aligned}$$

and

$$\begin{aligned} C_{n-1} &= -F_nF_{n+1} - 3F_{n-1}F_n + F_{n-2}F_{n-1} \\ &= -F_{n-1}F_n - F_{n+1}^2 \\ &= -F_{n-1}F_{n+1} - F_{n-1}F_n - F_nF_{n+1} \\ &= -F_{n-1}F_{n+2} - F_nF_{n+1}, \end{aligned}$$

and the desired result follows. □

2.3. $p = 2$, $q = 1$.

Corollary 2.3. *If $q = 1$, $p = 2$, and $n \geq 0$,*

$$u_{n+2} = \frac{(F_{n+1}a - F_{n+3}b)(F_na - F_{n+2}b) - F_{n+2}F_{n+1}}{b-a}.$$

3. Integer Sequence Examples

Suppose a and b are integers.

Proposition 3.1. *The constant term in the formula for u_{n+1} is an integer for given integers p, q, a, and b if and only if $(a-b) \mid (p-1)ab-q$. For $b = a \pm 1$, this is always true.*

Proof. The conclusion follows immediately from

$$\begin{aligned} u_{n+1} &= (p+1)u_n - u_{n-1} + (-1)^{n-1}\frac{a^2+b^2-2ab-(p-1)ab+q}{a-b} \\ &= (p+1)u_n - u_{n-1} + (-1)^{n-1}\left(a-b-\frac{ab(p-1)-q}{a-b}\right). \end{aligned}$$

□

Corollary 3.2. *Suppose $q = p-1$ and $a = 1$. Then the sequence is integral.*

Corollary 3.3. *For $a = 1$, $b = p$, and $q = p-1$, the sequence is $u_{n+1} = pu_n - u_{n-1}$.*

Proof. If $a = 1$, $b = p$, and $q = p-1$, then $a^2+b^2-(p+1)ab+q = 1+p^2-p(p+1)+p-1 = 0$. □

3.1. Examples from OEIS[1].

- $p = 3$, $q = 2$, $a = 1$, $b = 3$ A001835
- $p = 3$, $q = 2$, $a = 1$, $b = 4$ A214998
- $p = 3$, $q = 2$, $a = 1$, $b = 5$ A120893
- $p = 3$, $q = 2$, $a = 1$, $b = 7$ A217233
- $p = 5$, $q = 4$, $a = 1$, $b = 5$ A001653
- $p = 5$, $q = 4$, $a = 1$, $b = 6$ A218990
- $p = 10$, $q = 9$, $a = 1$, $b = 10$ A078922

References

[1] N. J. A. Sloane, *The On-Line Encyclopedia Of Integer Sequences*, http://oeis.org

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