ON A CURIOUS PROPERTY OF F_{184}

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Abstract. We prove that the density of the set $\mathcal{N}$ of n such that F_n has no nonzero digit in its base 10 expansion is zero. We give some heuristics that the set $\mathcal{N}$ is finite with the largest member being $n = 184$.

1. Introduction

The number

$$F_{184} = 127127879743834334146972278486287885163$$

has all its base 10 digits different than 0. This might not seem strange until one learns that $n = 184$ is the largest $n \leq 10^4$ with this property. That is, for each $n \in (184, 10000]$, F_n has at least one digit equal to 0. We offer the following conjecture.

Conjecture 1.1. *If $n > 184$, then F_n has a digit equal to 0 in its base 10 expansion.*

Let

$$\mathcal{N} = \{n : F_n \text{ has only nonzero digits in base } 10\}.$$

Although we cannot prove that $\mathcal{N}$ is finite, we can at least prove that it is thin. For a positive real number x, let

$$\mathcal{N}(x) = \mathcal{N} \cap [1, x].$$

We use the Landau symbols O and o and the Vinogradov symbol $\ll$, $\gg$, $\asymp$ with the usual meaning. Recall that $f(x) = O(g(x)$, $f(x) \ll g(x)$ and $g(x) \gg f(x)$ are all equivalent to $|f(x)| < Kg(x)$, which holds with some constant K for all $x > x_0$, whereas $f(x) \asymp g(x)$ means that $f(x) \ll g(x)$ and $g(x) \ll f(x)$ hold. Further, $f(x) = o(g(x))$ if $f(x)/g(x) \to 0$ as $x \to \infty$. We have the following theorem.

Theorem 1.2. *The estimate*

$$\#\mathcal{N}(x) \ll x^{1-c}$$

holds with $c = 1 - \log 9/\log 10 = 0.045757\ldots$.

In particular, by the Abel summation formula, we have

$$\sum_{n \in \mathcal{N}} \frac{1}{n} = O(1).$$

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2. The Proof of Theorem 1.2

It is well-known that the sequence $\{F_n\}_{n\ge 0}$ is periodic modulo m with some period $\rho(m)$ (see [1]). That is, the congruence $F_{n+\rho(m)} \equiv F_n \pmod m$ holds for all $n \ge 0$. Further, $\rho(5^k) = 4\times 5^k$, whereas $\rho(2^k) = 3\times 2^{k-1}$. Since $\rho(\mathrm{lcm}[m_1, m_2]) = \mathrm{lcm}[\rho(m_1), \rho(m_2)]$, it follows that

$$\rho(10^k) = 15 \times 10^{k-1} \qquad \text{for all} \qquad k \ge 3.$$

Lemma 2.1. *Let $k \ge 5$. For each nonzero residue class $a \pmod{10^k}$, there are at most 16 values of n modulo $\rho(10^k)$ such that $F_n \equiv a \pmod{10^k}$.*

Proof. Assume that there are 17 values of $n \pmod{\rho(10^k)}$ such that $F_n \equiv a \pmod{10^k}$. Then, there are at least five of them, $0 \le n_1 < n_2 < \cdots < n_5 \le 15 \times 10^{k-1} - 1$, such that n_i are in the same residue class modulo 4 for $i = 1, 2, 3, 4, 5$. Clearly, $n_1 > 0$ because a is nonzero. It is well-known that if $u < v$ and $u \equiv v \pmod 4$, then

$$F_v - F_u = F_{(v-u)/2}L_{(v+u)/2},$$

where $\{L_m\}_{m\ge 0}$ is the Lucas companion of the Fibonacci sequence. Applying this with $u = n_1$ and $v = n_i$ for $i = 2, 3, 4, 5$, we get that

$$F_{n_i} - F_{n_1} = F_{(n_i-n_1)/2}L_{(n_i+n_1)/2}. \tag{2.1}$$

In (2.1), the left side is a multiple of 10^k. Thus, $10^k \mid F_{(n_i-n_1)/2}L_{(n_i+n_1)/2}$. It is easy to check that $5 \nmid L_m$ for any m. Furthermore, $8 \nmid L_m$ for any m. Thus, $5^k \mid F_{(n_i-n_1)/2}$ and $2^{k-2} \mid F_{(n_i-n_1)/2}$. Recall that the index of appearance of the positive integer m in $\{F_n\}_{n\ge 0}$ is the minimal k such that $m \mid F_k$. This number k always exists and it is denoted by $z(m)$. Since $z(5^k) = 5^k$ and $z(2^k) = 3 \times 2^{k-2}$ for $k \ge 3$, it follows that $5^k \mid (n_i - n_1)/2$ and $3 \times 2^{k-4} \mid (n_i - n_1)/2$. Thus, $5^k \mid (n_i - n_1)$ and also $3 \times 2^{k-3} \mid (n_i - n_1)$, which show that $\mathrm{lcm}[5^k, 3\times 2^{k-3}] = 15\times 10^{k-1}/4$ divides $n_i - n_1$. Thus, the only possibilities of $n_i \le 15\times 10^{k-1}$ are

$$n_1 + \lambda \times 15 \times 10^{k-1}/4$$

with $\lambda \in \{1, 2, 3\}$. Thus, there are at most three such n_i and not four of them, which is the desired contradiction. □

Now, let x be large and let k be a positive integer such that $15\times 10^{k-1} \le x$. Assume $k \ge 5$. If $n \in \mathcal{N}(x)$, then it follows that none of the digits in the base 10 expansion of F_n is zero. In particular, none of its last k digits is zero. Thus, $F_n \equiv a \pmod{10^k}$, where a belongs to the set of numbers with exactly k-digits, none of which is zero, which has 9^k elements. For each such a, by Lemma 2.1, the equation $F_n \equiv a \pmod{10^k}$ has at most 16 solutions modulo $\rho(m) = 15\times 10^{k-1}$. Thus, choosing k maximal with the above property (namely, $15\times 10^{k-1} \le x$ but $15 \times 10^k > x$), it follows that the interval $[1, x]$ contains less than 10 multiples of $\rho(m)$, so there are at most 160 values of $n \le x$ such that $F_n \equiv a \pmod{10^k}$. This shows that

$$\#\mathcal{N}(x) \le 160 \times 9^k = (160 \times 9)\left(10^{k-1}\right)^{1-c} = \left(\frac{160 \times 9}{15^{1-c}}\right) x^{1-c} < 110x^{1-c},$$

which is what we wanted to prove.

3. Heuristics

By the same argument, the counting function of the set of $n \le x$ with only nonzero digits in base 10 is $O(x^{1-c})$. This can be interpreted by saying that the "expectation" or "probability" that n has only nonzero digits is $O(1/n^c)$. Assuming that a Fibonacci number F_n behaves like a regular integer with respect to the above property, we then expect that F_n has only nonzero digits with a frequency of $O(1/F_n^c)$. Since $F_n \asymp \alpha^n/\sqrt{5}$, where $\alpha = (1+\sqrt{5})/2$, it follows that the number of Fibonacci numbers that have only nonzero digits in their base 10 expansion should be

$$\ll \sum_{n\ge 1} \frac{1}{F_n^c} \ll \sum_{n\ge 1} \frac{1}{(\alpha^c)^n} \ll \frac{1}{\alpha^c - 1}.$$

Strangely enough, $1/(\alpha^c - 1) = 44.9171\ldots$, and there are exactly 45 known values of F_n that have only nonzero digits, and these correspond to the following values of n:

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 22, 23, 24, 26, 27, 28,$$
$$29, 31, 33, 35, 37, 39, 42, 43, 53, 54, 55, 56, 57, 58, 78, 80, 85, 87, 97, 125, 184$$

(we do not count $n = 1$ because $F_1 = F_2 = 1$). Similar observations apply to other bases or other similar problems, like asking if F_n has only odd digits or only even digits in its base 10 expansion. Computations up to $n \le 10^4$ revealed that the largest n in this range for which F_n has only odd digits is $n = 22$ with $F_{22} = 17711$, the largest n in this range for which F_n has only even digits is $n = 6$ with $F_n = 8$, and the largest n for which F_n has only prime digits is $n = 14$ for which $F_{14} = 377$.

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References

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