SOME EXTENDED GIBONACCI POLYNOMIAL SUMS WITH DIVIDENDS

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Abstract. We investigate some gibonacci polynomial sums, and extract their implications to the Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, Vieta, Vieta-Lucas, and Chebyshev families. We also explore the graph-theoretic interpretations of the gibonacci polynomial sums and the corresponding Jacobsthal versions.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary complex variable; $a(x), b(x), z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 0$ [8, 9].

Fibonacci, *Lucas*, *Pell*, *Pell-Lucas*, *Jacobsthal*, *Jacobsthal-Lucas*, *Vieta*, *Vieta-Lucas*, and *Chebyshev polynomials* of both kinds belong to this extended family. They are denoted by $f_n(x)$, $l_n(x)$, $p_n(x)$, $q_n(x)$, $J_n(x)$, $j_n(x)$, $V_n(x)$, $v_n(x)$, $T_n(x)$, and $U_n(x)$, respectively. Correspondingly, we have the numeric counterparts $F_n = f_n(1)$, $L_n = l_n(1)$, $P_n = p_n(1)$, $Q_n = 2q_n(1)$, $J_n = J_n(2)$, and $j_n = j_n(2)$ [4, 5, 8, 9]. Clearly, $J_n(1) = F_n$ and $j_n(1) = L_n$.

1.1. Bridges Among the Subfamilies.

By virtue of the relationships in Table 1, every ginonacci result has a Jacobsthal, Jacobsthal-Lucas, Vieta, Vieta-Lucas, and Chebyshev companion, where $i = \sqrt{-1}$ [5, 8, 9].

$J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$	$j_n(x) = x^{n/2} l_n(1/\sqrt{x})$
$V_n(x) = i^{n-1} f_n(-ix)$	$v_n(x) = i^n l_n(-ix)$
$U_n(x) = V_{n+1}(2x)$	$2T_n(x) = v_n(2x)$

Table 1: Links Among the Subfamilies

In the interest of brevity, clarity, and convenience, we *omit* the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. We let $g_n = f_n$ or l_n and *omit* much of the basic algebra.

Finally, let t_n denote the nth *triangular number* $n(n+1)/2$, where $n \geq 1$.

2. Gibonacci Sums

With this background, we begin our investigation of four gibonacci sums. Our discourse hinges on recursive technique [1, 10] and the following gibonacci properties [10], where $\Delta^2 =$

$x^2 + 4$:

$$f_{2n} = f_n l_n \qquad xf_n + l_n = 2f_{n+1}$$
$$l_n^2 - \Delta^2 f_n^2 = 4(-1)^n \qquad f_{a+b}^2 - f_{a-b}^2 = f_{2a} f_{2b}$$
$$l_{a+b}^2 - l_{a-b}^2 = \Delta^2 f_{2a} f_{2b} \qquad f_{2n} + xf_n^2 = 2f_n f_{n+1}$$
$$f_{2n} - xf_n^2 = 2f_n f_{n-1}.$$

Theorem 2.1. *Let m be a positive integer. Then,*

$$\sum_{k=1}^{n} f_{2mk^2} f_{2mk} = f_{2mt_n}^2. \tag{2.1}$$

Proof. Let A_n and B_n denote the left side and right side of the summation formula, respectively. Then,

$$\begin{aligned} B_n - B_{n-1} &= f_{mn(n+1)}^2 - f_{mn(n-1)}^2 \\ &= f_{mn^2+mn}^2 - f_{mn^2-mn}^2 \\ &= f_{2mn^2} f_{2mn} \\ &= A_n - A_{n-1}. \end{aligned}$$

So $A_n - B_n = A_{n-1} - B_{n-1}$, and hence $A_n - B_n = A_1 - B_1 = f_{2m}^2 - f_{2m}^2 = 0$. Thus, $A_n = B_n$ as desired. □

In particular, we have

$$\begin{aligned} \sum_{k=1}^{n} f_{2k^2} f_{2k} &= f_{2t_n}^2; \\ \sum_{k=1}^{n} f_{4k^2} f_{4k} &= f_{4t_n}^2. \end{aligned}$$

It then follows that $\sum_{k=1}^{n} F_{2k^2} F_{2k} = F_{2t_n}^2$ [12] and $\sum_{k=1}^{n} F_{4k^2} F_{4k} = F_{4t_n}^2$, respectively.

Theorem 2.2. *Let m be a positive integer. Then,*

$$\sum_{k=1}^{n} f_{2mxf_k^2} f_{2mf_{2k}} = f_{2mf_n f_{n+1}}^2. \tag{2.2}$$

Proof. Let A_n and B_n denote the left side and right side of the formula, respectively. Then,

$$\begin{aligned} B_n - B_{n-1} &= f_{2mf_n f_{n+1}}^2 - f_{2mf_n f_{n-1}}^2 \\ &= f_{mf_{2n}+mxf_n^2}^2 - f_{mf_{2n}-mxf_n^2}^2 \\ &= f_{2mf_{2n}} f_{2mxf_n^2} \\ &= A_n - A_{n-1}. \end{aligned}$$

Consequently, $A_n - B_n = A_{n-1} - B_{n-1} = A_1 - B_1 = f_{2mx}^2 - f_{2mx}^2 = 0$. So, $A_n = B_n$ as desired. □

It follows from formula (2.2) that

$$\begin{aligned}\sum_{k=1}^{n} f_{2xf_k^2} f_{2f_{2k}} &= f^2_{2f_n f_{n+1}};\\ \sum_{k=1}^{n} f_{4xf_k^2} f_{4f_{2k}} &= f^2_{4f_n f_{n+1}}.\end{aligned}$$

Consequently, $\sum_{k=1}^{n} F_{2F_k^2} F_{2F_{2k}} = F^2_{2F_n F_{n+1}}$ [12] and $\sum_{k=1}^{n} F_{4F_k^2} F_{4F_{2k}} = F^2_{4F_n F_{n+1}}$, respectively. For example,

$$\sum_{k=1}^{3} F_{4F_k^2} F_{4F_{2k}} = 2,149,991,424 = F^2_{4F_3 F_4}.$$

The next two theorems establish the Lucas companions of Theorems 2.1 and 2.2.

Theorem 2.3. *Let m be a positive integer. Then,*

$$\Delta^2 \sum_{k=1}^{n} f_{2mk^2} f_{2mk} = l^2_{2mt_n} - 4. \tag{2.3}$$

Proof. Let A_n be the left side of the summation formula and B_n its right side. We then have

$$\begin{aligned}B_n - B_{n-1} &= l^2_{mn(n+1)} - l^2_{mn(n-1)}\\ &= l^2_{mn^2+mn} - l^2_{mn^2-mn}\\ &= \Delta^2 f_{2mn^2} f_{2mn}\\ &= A_n - A_{n-1}.\end{aligned}$$

Then $A_n - B_n = A_{n-1} - B_{n-1} = A_1 - B_1 = \Delta^2 f^2_{2m} - \left(l^2_{2m} - 4\right) = 0$. Thus, $A_n = B_n$ as expected. □

Because l_{2mt_n} ends in 2 [10], the right side does not contain a constant term, which is consistent with the left side.

In particular, formula (2.3) implies

$$5 \sum_{k=1}^{n} F_{2mk^2} F_{2mk} = L^2_{2mt_n} - 4.$$

Consequently, $L^2_{2mt_n} \equiv 4 \pmod 5$.

Theorem 2.4. *Let m be a positive integer. Then,*

$$\Delta^2 \sum_{k=1}^{n} f_{2mxf_k^2} f_{2mf_{2k}} = l^2_{2mf_n f_{n+1}} - 4. \tag{2.4}$$

Proof. Let A_n denote the left side of the formula and B_n its right side. Then,

$$\begin{aligned}B_n - B_{n-1} &= l^2_{2mf_n f_{n+1}} - l^2_{2mf_n f_{n-1}}\\ &= l^2_{mf_{2n}+mxf_n^2} - l^2_{mf_{2n}-mxf_n^2}\\ &= \Delta^2 f_{2mf_{2n}} f_{2mxf_n^2}\\ &= A_n - A_{n-1}.\end{aligned}$$

So $A_n - B_n = A_{n-1} - B_{n-1} = A_1 - B_1 = \Delta^2 f_{2mx}^2 - \left(l_{2mx}^2 - 4\right) = 0$. Thus, $A_n = B_n$ as desired. □

It follows by Theorem 2.4 that

$$5\sum_{k=1}^{n} F_{2mF_k^2} F_{2mF_{2k}} = L_{2mF_nF_{n+1}}^2 - 4.$$

2.1. **Graph-Theoretic Interpretations.** Theorems 2.1 through 2.4 can be interpreted using graph-theoretic tools. To this end, consider the *weighted digraph* D_1 with vertices v_1 and v_2 in Figure 1 [7]. It follows by induction

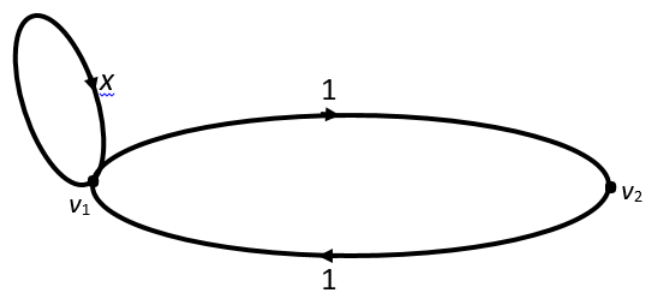

FIGURE 1. Fibonacci Digraph D_1

from its *weighted adjacency matrix* $Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ that

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \geq 1$ [7].

A *walk* from vertex v_i to vertex v_j is a sequence v_i-e_i-v_{i+1}-$\cdots$-v_{j-1}-e_{j-1}-v_j of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is *closed* if $v_i = v_j$; otherwise, it is *open*. The *length* of a walk is the number of edges in the walk. The *weight* of a walk is the product of the weights of the edges along the walk.

The following theorem provides a powerful tool for computing the sum of the weights of walks of length n from v_i to v_j [7].

Theorem 2.5. *Let A be the weighted adjacency matrix of a weighted and connected digraph with vertices $v_1, v_2, \ldots, v_k$. Then the ijth entry of the matrix A^n gives the sum of the weights of all walks of length n from v_i to v_j, where $n \geq 1$.*

The next result follows from this theorem.

Corollary 2.6. *The ijth entry of Q^n gives the sum of the weights of all walks of length n from v_i to v_j in the weighted digraph D_1, where $1 \leq i, j \leq n$.*

Consequently, the sum of the weights of all closed walks of length n originating at v_1 is f_{n+1}, and that of closed walks of length n originating at v_2 is f_{n-1}. So, the sum of the weights of all closed walks of length n is $f_{n+1} + f_{n-1} = l_n$.

With this background, we are ready for the interpretations.

Formula (2.1): Let a_k and b_k denote the sums of the weights of closed walks of lengths $2mk^2-1$ and $2mk-1$ originating at v_1, respectively. Then,

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= \sum_{k=1}^{n} f_{2mk^2} f_{2mk}\\ &= f_{2mt_n}^2\\ &= \begin{pmatrix}\text{sum of the weights of closed walks of}\\ \text{length } \ 2mt_n - 1 \text{ originating at } v_1\end{pmatrix}^2.\end{aligned}$$

For example, when $m = 1$ and $n = 3$, we have

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= f_2^2 + f_8 f_4 + f_{18} f_6\\ &= x^{22} + 20x^{20} + 172x^{18} + 832x^{16} + 2486x^{14} + 4744x^{12}\\ &\quad + 5776x^{10} + 4352x^8 + 1897x^6 + 420x^4 + 36x^2\\ &= f_{3\cdot 4}^2,\end{aligned}$$

as expected.

Formula (2.3): Let c_k denote the sum of the weights of closed walks of length f_{2mk^2-1} and d_k that of those of length f_{2mk-1} originating at v_1. Then,

$$\begin{aligned}\Delta^2 \sum_{k=1}^{n} c_k d_k &= \Delta^2 \sum_{k=1}^{n} f_{2mk^2} f_{2mk}\\ &= l_{2mt_n}^2 - 4\\ &= \begin{pmatrix}\text{sum of the weights of closed walks}\\ \text{of length } 2mt_n \text{ originating at } v_1\end{pmatrix}^2 - 4.\end{aligned}$$

Formulas (2.2) and (2.4) can be interpreted similarly.

2.2. **Pell Implications.** Because $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, it follows from Theorems 2.1 through 2.4 that

$$\begin{aligned}\sum_{k=1}^{n} p_{2mk^2} p_{2mk} &= p_{2mt_n}^2;\\ \sum_{k=1}^{n} p_{2mxf_k^2} p_{2mf_{2k}} &= p_{2mf_n f_{n+1}}^2;\\ 4(x^2+1) \sum_{k=1}^{n} p_{2mk^2} p_{2mk} &= q_{2mt_n}^2 - 4;\\ 4(x^2+1) \sum_{k=1}^{n} p_{2mxf_k^2} p_{2mf_{2k}} &= q_{2mf_n f_{n+1}}^2 - 4.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\sum_{k=1}^{n} P_{2mk^2}P_{2mk} &= P^2_{2mt_n};\\ \sum_{k=1}^{n} P_{2mF_k^2}P_{2mF_{2k}} &= P^2_{2mF_nF_{n+1}};\\ 2\sum_{k=1}^{n} P_{2mk^2}P_{2mk} &= Q^2_{2mt_n}-1;\\ 2\sum_{k=1}^{n} P_{2mF_k^2}P_{2mF_{2k}} &= Q^2_{2mF_nF_{n+1}}-1.\end{aligned}$$

As an example, suppose we let $m = 2$ and $n = 3$. Then,

$$\begin{aligned}2\sum_{k=1}^{3} P_{4F_k^2}P_{4F_{2k}} &= 590,436,102,659,356,800\\ &= Q^2_{4F_3F_4}-1,\end{aligned}$$

as expected.

3. Jacobsthal Implications

It follows by the relationships $J_n(x) = x^{(n-1)/2}f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2}l_n(1/\sqrt{x})$ that Theorems 2.1 through 2.4 have Jacobsthal consequences as well:

$$\sum_{k=1}^{n} x^{m(n-k)(n+k+1)}J_{2mk^2}(x)J_{2mk}(x) = J^2_{2mt_n}(x); \quad (3.1)$$

$$\sum_{k=1}^{n} x^{2m(f_nf_{n+1}-f_kf_{k+1})}J_{2mxf_k^2}(x)J_{2mf_{2k}}(x) = J^2_{2mf_nf_{n+1}}(x); \quad (3.2)$$

$$(4x+1)\sum_{k=1}^{n} x^{m(n-k)(n+k+1)}J_{2mk^2}(x)J_{2mk}(x) = j^2_{2mt_n}(x) - 4x^{2mt_n}; \quad (3.3)$$

$$(4x+1)\sum_{k=1}^{n} x^{2m(f_nf_{n+1}-f_kf_{k+1})}J_{2mxf_k^2}(x)J_{2mf_{2k}}(x) = j^2_{2mf_nf_{n+1}}(x) - 4x^{2mf_nf_{n+1}}. \quad (3.4)$$

The proofs are straightforward. In the interest of brevity, we will confirm formulas (3.3) and (3.4) only, and omit the other two.

Proof of Formula (3.3): Replace x with $1/\sqrt{x}$ in Formula (2.3). Multiplying the resulting equation by $x^{mn(n+1)}$ yields

$$\begin{aligned}&(4x+1)\sum_{k=1}^{n} x^{m(n-k)(n+k+1)}\left[x^{(2mk^2-1)/2}f_{2mk^2}\right]\left[x^{(2mk-1)/2}f_{2mk}\right]\\ &= \left[x^{mn(n+1)/2}l_{mn(n+1}\right]^2 - \ 4x^{mn(n+1)}\\ &(4x+1)\sum_{k=1}^{n} x^{m(n-k)(n+k+1)}J_{2mk^2}(x)J_{2mk}(x) = j^2_{2mt_n}(x) - 4x^{2mt_n},\end{aligned}$$

where $g_n = g_n(1/\sqrt{x})$.

Proof of Formula (3.4): Replace x with $u = 1/\sqrt{x}$ in formula (2.4). Multiply the resulting equation by $x^{2mf_nf_{n+1}}$. We then obtain

$$(4x+1)\sum_{k=1}^{n} x^{2m(f_nf_{n+1}-f_kf_{k+1})}\left[x^{(2mxf_k^2-1)/2}f_{2mxf_k^2}(u)\right]\left[x^{(2mf_{2k}-1)/2}f_{2mf_{2k}}(u)\right]$$
$$= \left[x^{(2mf_nf_{n+1})/2}l_{2mf_nf_{n+1}}(u)\right]^2 - 4x^{2mf_nf_{n+1}}$$
$$(4x+1)\sum_{k=1}^{n} x^{2m(f_nf_{n+1}-f_kf_{k+1})}J_{2mxf_k^2}(x)J_{2mf_{2k}}(x) = j^2_{2mf_nf_{n+1}}(x) - 4x^{2mf_nf_{n+1}}.$$

3.1. **Graph-Theoretic Interpretations.** Next, we interpret formulas (3.1) and (3.2) using the weighted digraph D_2 in Figure 2 with vertices v_1 and v_2. Its weighted adjacency matrix $M = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix}$ yields

$$M^n = \begin{bmatrix} J_{n+1}(x) & xJ_n(x) \\ J_n(x) & xJ_{n-1}(x) \end{bmatrix}.$$

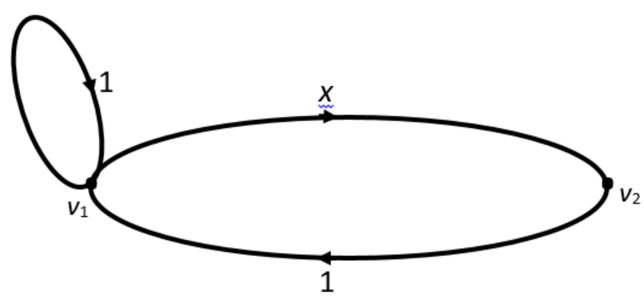

FIGURE 2. Jacobsthal Digraph D_2

So, the sum of the closed walks of length n from v_1 to itself is $J_{n+1}(x)$, and that from v_2 to itself is $xJ_{n-1}(x)$. Consequently, the sum of the weights of all closed walks of length n is $J_{n+1}(x) + xJ_{n-1}(x) = j_n(x)$ [10].

We are now ready for the interpretations.

Formula (3.1): Let a_k and b_k denote the sums of the weights of closed walks of lengths $2mk^2-1$ and $2mk-1$ originating at v_1, respectively. Then,

$$\begin{aligned} \sum_{k=1}^{n} x^{m(n-k)(n+k+1)}a_kb_k &= \sum_{k=1}^{n} x^{m(n-k)(n+k+1)}J_{2mk^2}(x)J_{2mk}(x) \\ &= J^2_{2mt_n}(x) \\ &= \begin{pmatrix}\text{sum of the weights of closed walks} \\ \text{of length } 2mt_n-1 \text{ originating at } v_1\end{pmatrix}^2. \end{aligned}$$

Formula (3.2): Let c_k be the sum of the weights of closed walks of length $f_{2mxf_k^2-1}$, and d_k that of those of length $f_{2mf_{2k}-1}$, all originating at v_1. Then,

$$\begin{aligned}\sum_{k=1}^{n} x^{2m(f_nf_{n+1}-f_kf_{k+1})}c_kd_k &= \sum_{k=1}^{n} x^{2m(f_nf_{n+1}-f_kf_{k+1})}J_{2mxf_k^2}(x)J_{2mf_{2k}}(x)\\ &= J^2_{2mf_nf_{n+1}}(x)\\ &= \begin{pmatrix}\text{sum of the weights of closed walks of}\\ \text{length } 2mf_nf_{n+1}-1 \text{ originating at } v_1\end{pmatrix}^2.\end{aligned}$$

The interpretations of formulas (3.3) and (3.4) follow similarly.

3.2. **Some Special Cases.** It follows from formula (3.1) that

$$\sum_{k=1}^{n} 2^{(n-k)(n+k+1)}J_{2k^2}J_{2k} = J^2_{2t_n}.$$

For example,

$$\sum_{k=1}^{4} 2^{(4-k)(5+k)}J_{2k^2}J_{2k} = 122,167,725,625 = J^2_{4\cdot 5}.$$

Formula (3.2) implies that

$$\sum_{k=1}^{n} 4^{f_nf_{n+1}-f_kf_{k+1}}J_{2xf_k^2}J_{2f_{2k}} = J^2_{2f_nf_{n+1}}.$$

Consequently,

$$\sum_{k=1}^{n} 4^{F_nF_{n+1}-F_kF_{k+1}}J_{2F_k^2}J_{2F_{2k}} = J^2_{2F_nF_{n+1}}.$$

For example,

$$\sum_{k=1}^{4} 4^{F_4F_5-F_kF_{k+1}}J_{2F_k^2}J_{2F_{2k}} = 128,102,389,162,151,481 = 357,913,941^2 = J^2_{2F_4F_5}.$$

It also follows from formula (3.2) that

$$\sum_{k=1}^{n} 4^{m(P_nP_{n+1}-P_kP_{k+1})}J_{4mP_k^2}J_{2mP_{2k}} = J^2_{2mP_nP_{n+1}}.$$

As an example,

$$\sum_{k=1}^{2} 4^{P_2P_3-P_kP_{k+1}}J_{4P_k^2}J_{2P_{2k}} = 122,167,725,625 = J^2_{2P_2P_3}.$$

From formula (3.3), we get

$$9\sum_{k=1}^{n} 2^{(n-k)(n+k+1)}J_{2k^2}J_{2k} = j^2_{2t_n} - 4^{t_n+1}. \tag{3.5}$$

For example,

$$9\sum_{k=1}^{n} 2^{(4-k)(5+k)}J_{2k^2}J_{2k} = 1,099,509,530,625 = j^2_{2t_4} - 4^{t_4+1}.$$

Formula (3.5) has an interesting byproduct:

$$j_{2t_n}^2 \equiv \begin{cases} 7 \pmod 9 & \text{if } n \equiv 1 \pmod 3 \\ 4 \pmod 9 & \text{otherwise.} \end{cases}$$

It follows from formula (3.4) that

$$9\sum_{k=1}^{n} 4^{F_nF_{n+1}-F_kF_{k+1}} J_{2F_k^2} J_{2F_{2k}} = j_{2F_nF_{n+1}}^2 - 4^{F_nF_{n+1}+1}.$$

For example,

$$9\sum_{k=1}^{4} 4^{F_4F_5-F_kF_{k+1}} J_{2F_k^2} J_{2F_{2k}} = 1,152,921,502,459,363,329 = j_{2F_4F_5}^2 - 4^{F_4F_5+1}.$$

It follows from equations (3.1) and (3.3), and (3.2) and (3.4) that

$$(4x+1)J_{mn(n+1)}^2(x) = j_{2mt_n}^2(x) - 4x^{2mt_n}; \tag{3.6}$$

$$(4x+1)J_{2mf_nf_{n+1}}^2(x) = j_{2mf_nf_{n+1}}^2(x) - 4x^{2mf_nf_{n+1}}, \tag{3.7}$$

respectively.

Let $\lambda = t_n$ or F_nF_{n+1}. It then follows from equations (3.6) and (3.7) that

$$9J_{2\lambda}^2 = j_{2\lambda}^2 - 4^{\lambda+1}; \tag{3.8}$$

$$9J_{4\lambda}^2 = j_{4\lambda}^2 - 4^{2\lambda+1}, \tag{3.9}$$

respectively.

For example,

$$\begin{aligned} j_{2t_4}^2 - 4\cdot 4^{t_4} &= 1{,}099{,}509{,}530{,}625 = 9J_{2t_4}^2; \\ j_{4t_3}^2 - 4\cdot 4^{2t_3} &= 281{,}474{,}943{,}156{,}225 = 9J_{4t_3}^2; \\ j_{2F_4F_5}^2 - 4\cdot 4^{F_4F_5} &= 1{,}152{,}921{,}502{,}459{,}363{,}329 = 9J_{2F_4F_5}^2; \\ j_{4F_3F_4}^2 - 4\cdot 4^{2F_3F_4} &= 281{,}474{,}943{,}156{,}225 = 9J_{4F_3F_4}^2. \end{aligned}$$

Because $J_{2n} = J_nj_n$ [10], equations (3.8) and (3.9) imply that

$$j_{2\lambda}^4 = j_{4\lambda}^2 + 4^{\lambda+1}j_{2\lambda}^2 - 4^{2\lambda+1}.$$

In addition, they yield two interesting congruences:

$$J_{2\lambda}^2 + j_{2\lambda}^2 \equiv \begin{cases} 6 \pmod{10} & \text{if } \lambda \text{ is odd} \\ 4 \pmod{10} & \text{otherwise;} \end{cases}$$

$$J_{4\lambda}^2 + j_{4\lambda}^2 \equiv 4 \pmod{10}.$$

For instance, when $n = 3$, $\lambda = 6$. Then $J_{4\lambda} \equiv 5 \pmod{10}$ and $j_{4\lambda} \equiv 7 \pmod{10}$; so $J_{24}^2 + j_{24}^2 \equiv 4 \pmod{10}$.

4. Vieta and Chebyshev Implications

Finally, it follows by the relationships in Table 1 that formulas (2.1) through (2.4) also have Vieta and Chebyshev companions:

$$\sum_{k=1}^{n}(-1)^{m(n-k)(n+k+1)}V_{2mk^2}(x)V_{2mk}(x) = V^2_{2mt_n}(x);$$

$$\sum_{k=1}^{n}V_{2mxf_k^2}(x)V_{2mf_{2k}}(x) = V^2_{2mf_nf_{n+1}}(x);$$

$$(x^2-4)\sum_{k=1}^{n}(-1)^{m(n-k)(n+k+1)}V_{2mk^2}(x)V_{2mk}(x) = v^2_{2mt_n}(x)-4;$$

$$(x^2-4)\sum_{k=1}^{n}V_{2mxf_k^2}(x)V_{2mf_{2k}}(x) = v^2_{2mf_nf_{n+1}}(x)-4.$$

In the interest of brevity, we omit their confirmations.

The Chebyshev counterparts now follow by the relationships $U_n(x) = V_{n+1}(2x)$ and $2T_n(x) = v_n(2x)$.

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References

[1] S. Clary and P. D. Hemenway, *On sums of cubes of Fibonacci numbers*, Applications of Fibonacci Numbers (ed. G. E. Bergum et al.), **5** (1993), Kluwer, Dordrecht, 123–136.
[2] K. B. Davenport, *Problem B-1235*, The Fibonacci Quarterly, **56.3** (2018), 276.
[3] A. F. Horadam, *Jacobsthal representation numbers*, The Fibonacci Quarterly, **34.1** (1996), 40–54.
[4] A. F. Horadam, *Jacobsthal representation polynomials*, The Fibonacci Quarterly, **35.2** (1997), 137–148.
[5] A. F. Horadam, *Vieta polynomials*, The Fibonacci Quarterly, **40.3** (2002), 223–232.
[6] T. Koshy, *Polynomial extensions of the Ginsburg and Lucas identities*, The Fibonacci Quarterly, **52.2** (2014), 141–147.
[7] T. Koshy, *Graph-theoretic models for the univariate Fibonacci family*, The Fibonacci Quarterly, **53.2** (2015), 135–146.
[8] T. Koshy, *Polynomial extensions of the Lucas and Ginsburg identities revisited: Additional dividends I*, The Fibonacci Quarterly **56.2** (2018), 106–112.
[9] T. Koshy, *Differences of gibonacci polynomial products of orders 2, 3, and 4*, The Fibonacci Quarterly **56.3** (2018), 212–220.
[10] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Volume II, Wiley, Hoboken, New Jersey, 2019.
[11] H. Ohtsuka, *Problem B-1211*, The Fibonacci Quarterly, **55.3** (2017), 276.
[12] H. Ohtsuka, *Problem B-1233*, The Fibonacci Quarterly, **56.3** (2018), 276.

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