PERIODICITY OF ONES DIGITS IN JACOBSTHAL NUMBERS WITH TRIANGULAR AND JACOBSTHAL SUBSCRIPTS

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Abstract. We explore the periodicity of the ones digits in the sequences $\{C_n\}$, $\{C_{t_n}\}$, $\{C_{2t_n}\}$, $\{C_{4t_n}\}$, $\{C_{t_{t_n}}\}$, $\{C_{8t_{t_n}}\}$, $\{C_{t_{n^2}}\}$, and C_{C_n}, where C_n denotes the nth Jacobsthal or Jacobsthal-Lucas number, and t_n the nth triangular number.

1. Introduction

Extended gibonacci numbers z_n are defined by the recurrence $z_n = az_{n-1} + bz_{n-2}$, where a, b, z_1, and z_2 are arbitrary integers; and $n \geq 3$. Suppose $a = 1$ and $b = 2$. When $z_1 = 1 = z_2$, $z_n = J_n$, the nth *Jacobsthal number*; and when $z_1 = 1$ and $z_2 = 5$, $z_n = j_n$, the nth *Jacobsthal-Lucas number* [1, 4].

The numbers J_n and j_n can also be defined explicitly by the *Binet-like formulas* [3, 4]

$$3J_n = 2^n - (-1)^n \text{ and } j_n = 2^n + (-1)^n.$$

Clearly, J_n and j_n are odd; and j_n is a *Mersenne number* when n is odd. In the interest of brevity and convenience, we let $C_n = J_n$ or j_n, and $n \geq 1$.

1.1. **A Digraph Model for C_n.** To see the graph-theoretic interpretation of C_n, consider the weighted digraph D in Figure 1 with vertices v_1 and v_2. The number of closed walks of length n, originating at v_1, is J_{n+1} and that of those originating at v_2 is $2J_{n-1}$. So, the total number of walks of length n is $J_{n+1} + 2J_{n-1} = j_n$ [5, 4].

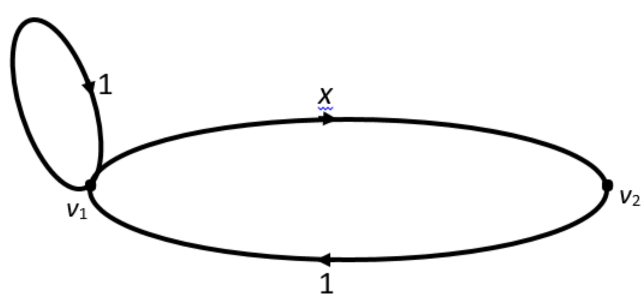

Figure 1. Jacobsthal Digraph D

Next, we study the periodicity of the ones digits in Jacobsthal and Jacobsthal-Lucas numbers.

2. Periodicity of the Sequence $\{C_n \pmod{10}\}$

Let $G_n = F_n$ or L_n, where F_n denotes the nth Fibonacci number and L_n denotes the nth Lucas number. It is well-known that the sequence $\{G_n \pmod{10}\}$ is periodic with

$$\text{period} = \begin{cases} 60 & \text{if } G_n = F_n \\ 12 & \text{otherwise;} \end{cases}$$

see Corollary 23.4 in [3], and [7]. This result has a Jacobsthal companion, as the following theorem shows.

Theorem 2.1. *The sequence $\{C_n \pmod{10}\}$ is periodic with period 4. The repeating block is*

$$\begin{cases} 1135 & \text{if } C_n = J_n \\ 1577 & \text{otherwise.} \end{cases}$$

Proof. The proof employs the following facts [2]:

- If $ab \equiv ac \pmod{m}$ and $(a, m) = 1$, then $b \equiv c \pmod{m}$.
- If $(a, m) = 1$, then a is invertible modulo m.

Suppose $C_n = J_n$ and $n = 4k + r$, where $0 \le r \le 3$. Then,

$$\begin{aligned} 3J_n &= 2^{4k+r} - (-1)^{4k+r} \\ &\equiv 6 \cdot 2^r - (-1)^r \pmod{10} \\ J_n &\equiv 2^{r+1} - 7(-1)^r \pmod{10}. \end{aligned}$$

When $r = 1$, $J_n \equiv 4 + 7 \equiv 1 \pmod{10}$; when $r = 2$, $J_n \equiv 8 - 7 \equiv 1 \pmod{10}$; when $r = 3$, $J_n \equiv 6 + 7 \equiv 3 \pmod{10}$; and when $r = 0$, $J_n \equiv 2 - 7 \equiv 5 \pmod{10}$.

On the other hand, let $C_n = j_n$. Then, $j_n = 2^{4k+r} + (-1)^{4k+r} \equiv 6 \cdot 2^r + (-1)^r \pmod{10}$.

When $r = 1$, $j_n \equiv 6 \cdot 2 - 1 \equiv 1 \pmod{10}$; when $r = 2$, $j_n \equiv 6 \cdot 4 + 1 \equiv 5 \pmod{10}$; when $r = 3$, $j_n \equiv 6 \cdot 8 - 1 \equiv 7 \pmod{10}$; and when $r = 0$, $j_n \equiv 6 + 1 \equiv 7 \pmod{10}$.

The given result now follows by combining the two cases. □

3. Periodicity of the Sequence $\{C_{t_n} (\bmod\ 10)\}$

We now investigate the periodicity of the sequence $\{C_{t_n} \pmod{10}\}$, where t_n denotes the nth *triangular number* $n(n+1)/2$. It follows from the digraph model that the digraph contains J_{t_n} closed walks of length $t_n - 1$ originating at v_1, and j_{t_n} closed walks of length t_n originating at v_2.

The following theorem gives the period and the repeating block of the sequence.

Theorem 3.1. *The sequence $\{C_{t_n} (\bmod\ 10)\}$ is periodic with period 8. The repeating block is*

$$\begin{cases} 13113155 & \text{if } C_n = J_n \\ 17557177 & \text{otherwise.} \end{cases}$$

Proof. The proof will use the explicit formula for C_n and congruence modulo 10.

Part 1. Let $C_n = J_n$ and $n = 8k + r$, where $0 \le r \le 7$. Because $6^n \equiv 6 \pmod{10}$ for $n \ge 1$, we have

$$\begin{aligned}
3J_{t_n} &= 2^{(n^2+n)/2} - (-1)^{(n^2+n)/2} \\
&= 2^{[(8k+r)^2+(8k+r)]/2} - (-1)^{(8k+r)(8k+r+1)/2} \\
&= 2^{32k^2} \cdot 2^{4k(2r+1)} \cdot 2^{r(r+1)/2} - (-1)^{r(r+1)/2} \\
&\equiv 6^{k^2} \cdot 6^{k(2r+1)} \cdot 2^{r(r+1)/2} - (-1)^{r(r+1)/2} \pmod{10} \\
&\equiv 6 \cdot 2^{r(r+1)/2} - (-1)^{r(r+1)/2} \pmod{10} \\
J_{t_n} &\equiv 7 \cdot 6 \cdot 2^{r(r+1)/2} - 7(-1)^{r(r+1)/2} \pmod{10} \\
&\equiv 2 \cdot 2^{r(r+1)/2} - 7(-1)^{r(r+1)/2} \pmod{10}. \qquad (3.1)
\end{aligned}$$

Now, consider congruence (3.1) for each value of r:
When $r = 1$ and when $r = 6 \equiv -2 \pmod 8$, $J_{t_n} \equiv 2 \cdot 2 + 7 \equiv 1 \pmod{10}$.
When $r = 2$ and when $r = 5 \equiv -3 \pmod 8$, $J_{t_n} \equiv 2 \cdot 8 + 7 \equiv 3 \pmod{10}$.
When $r = 3$, $J_{t_n} \equiv 2 \cdot 2^6 - 7 \equiv 8 - 7 \equiv 1 \pmod{10}$.
When $r = 4$, $J_{t_n} \equiv 2 \cdot 2^{10} - 7 \equiv 8 - 7 \equiv 1 \pmod{10}$.
When $r = 7 \equiv -1 \pmod 8$ and when $r = 0$, $J_{t_n} \equiv 2 - 7 \equiv 5 \pmod{10}$.
Combining all cases, we get the desired result.

Part 2. Let $C_n = j_n$ and $n = 8k + r$, where $0 \le r \le 7$. Then,

$$j_{t_n} \equiv 6 \cdot 2^{r(r+1)/2} + (-1)^{r(r+1)/2} \pmod{10}. \qquad (3.2)$$

Consequently, by congruence (3.2), we have:
When $r = 1$ and when $r = 6 \equiv -2 \pmod 8$, $j_{t_n} \equiv 6 \cdot 2 - 1 \equiv 1 \pmod{10}$.
When $r = 2$ and when $r = 5 \equiv -3 \pmod 8$, $j_{t_n} \equiv 6 \cdot 2^3 - 1 \equiv 7 \pmod{10}$.
When $r = 3$, $j_{t_n} \equiv 6 \cdot 2^6 + 1 \equiv 5 \pmod{10}$.
When $r = 4$, $j_{t_n} \equiv 6 \cdot 2^{10} + 1 \equiv 5 \pmod{10}$.
When $r = 7 \equiv -1 \pmod 8$ and when $r = 0$, $j_{t_n} \equiv 6 + 1 \equiv 7 \pmod{10}$.
The given result now follows by combining the eight cases. □

For example, $J_{t_{12}} = J_{78} \equiv 1 \pmod{10}$; $J_{t_{13}} = J_{91} \equiv 3 \pmod{10}$; $j_{t_{12}} = j_{78} \equiv 5 \pmod{10}$; and $j_{t_{13}} = j_{91} \equiv 7 \pmod{10}$, as expected.
An Observation. Let $x_1x_2 \ldots x_8$ and $w_1w_2 \ldots w_8$ denote the repeating blocks in $\{J_{t_n} \pmod{10}\}$ and $\{j_{t_n} \pmod{10}\}$, respectively. Considering each block as a *word*, the subwords $x_1x_2 \ldots x_6, x_7x_8, w_1w_2 \ldots w_6$, and w_7w_8 are all palindromic.
Next, we establish an elegant theorem. Its proof hinges on the following lemma.

Lemma 3.2. *Let n be a positive integer. Then,* $5 \cdot 2^{t_n} \equiv 0 \pmod{10}$.

The proof is a straightforward application of induction; so we omit it here. We are now ready for the theorem.

Theorem 3.3. *If $u \equiv v \pmod 8$, then $C_{t_u} \equiv C_{t_v} \pmod{10}$.*
Proof. Because $u \equiv v \pmod 8$, $u = v + 8m$ for some integer m. Then, $t_u = t_{v+8m}$.

Suppose $C_n = J_n$. By Lemma 3.2, we have

$$\begin{aligned}
3J_{t_v} &= 2^{t_v} - (-1)^{t_v} \\
J_{t_v} &\equiv 7 \cdot 2^{t_v} - 7(-1)^{t_v} \pmod{10}; \\
3J_{t_u} &= 2^{t_u} - (-1)^{t_u} \\
&= 2^{t_v+8m} - (-1)^{t_v+8m} \\
&\equiv 6 \cdot 2^{t_v} - (-1)^{t_v} \pmod{10} \\
J_{t_u} &\equiv 2 \cdot 2^{t_v} - 7(-1)^{t_v} \pmod{10}; \\
J_{t_u} - J_{t_v} &\equiv -5 \cdot 2^{t_v} \pmod{10} \\
&\equiv 0 \pmod{10}.
\end{aligned}$$

Thus, $J_{t_u} \equiv J_{t_v} \pmod{10}$.

On the other hand, let $C_n = j_n$. Again by Lemma 3.2, we have

$$\begin{aligned}
j_{t_v} &= 2^{t_v} + (-1)^{t_v}; \\
j_{t_u} &= 2^{t_u} + (-1)^{t_u} \\
&= 2^{t_v+8m} + (-1)^{t_v+8m} \\
&\equiv 6 \cdot 2^{t_v} + (-1)^{t_v} \pmod{10} \\
j_{t_u} - j_{t_v} &\equiv 5 \cdot 2^{t_v} \pmod{10} \\
&\equiv 0 \pmod{10} \\
j_{t_u} &\equiv j_{t_v} \pmod{10}.
\end{aligned}$$

Combining the two cases, we get the desired result. □

Theorem 3.1 is a byproduct of this theorem, as the following corollary reveals.

Corollary 3.4. *The sequence $\{C_{t_n} (\text{mod } 10)\}$ is periodic with period 8.*

Proof. Let $u = n + 8$ and $v = n$. Then, by Theorem 3.3, $C_{t_{n+8}} \equiv C_{t_n} (\text{mod } 10)$. When $0 \le k < 8$, $(n + k) - n \not\equiv 0 \pmod 8$; consequently, $\{C_{t_n} (\text{mod } 10)\}$ is periodic with period 8, as desired. □

Theorem 3.3 has additional consequences. To this end, we need the following result.

Lemma 3.5. *Let k be an integer ≥ 4. Then, $t_{n+2^k} \equiv t_n \pmod 8$.*

Proof.

$$\begin{aligned}
2\left(t_{n+2^k} - t_n\right) &= \left(n + 2^k\right)\left(n + 2^k + 1\right) - n(n + 1) \\
&= 2^{k+1} n + 2^k (2^k + 1) \\
t_{n+2^k} - t_n &= 2^k n + 2^{k-1}(2^k + 1) \\
&\equiv 0 \pmod 8.
\end{aligned}$$

□

Corollary 3.6. *The sequence $\{C_{t_{t_n}} (\text{mod } 10)\}$ is periodic with period 16.*

Proof. Because $t_{n+16} - t_n = 16n + 8 \cdot 17 \equiv 0 \pmod 8$, $\{t_n \pmod 8\}$ is periodic with period 16. So, by Theorem 3.3, $C_{t_{t_{n+16}}} \equiv C_{t_{t_n}} (\text{mod } 10)$; thus $\{C_{t_{t_n}} (\text{mod } 10)\}_{n \ge 1}$ is periodic. When $C_n = J_n$, the first repeating block is 1113531135311155 and when $C_n = j_n$, the corresponding block is 1517775577715177. Because both consist of 16 digits, the period of the sequence is 16. □

Using a similar technique, it follows that the sequence $\{C_{t_{t_{t_n}}} \pmod{10}\}$ is periodic with period 32: $C_{t_{t_{t_{n+32}}}} \equiv C_{t_{t_{t_n}}} \pmod{10}$. More generally, the sequence $\{C_\gamma \pmod{10}\}$ is periodic with period 2^k, where $\gamma = t_{t_{\cdot_{\cdot_{\cdot_{t_{n+2^k}}}}}}$ and $k \geq 3$.

Next, we examine the periodicity of the sequence $\{C_{2t_n} \pmod{10}\}$.

Theorem 3.7. *The sequence $\{C_{2t_n} \pmod{10}\}$ is periodic with period 4. The repeating block is*

$$\begin{cases} 1155 & \text{if } C_n = J_n \\ 5577 & \text{otherwise.} \end{cases}$$

Proof. Let $C_n = J_n$ and $n = 4k + r$, where $0 \leq r \leq 3$. Then,

$$\begin{aligned} 3J_{2t_n} &\equiv 6 \cdot 2^{r(r+1)} - 1 \pmod{10} \\ J_{2t_n} &\equiv 2 \cdot 2^{r(r+1)} - 7 \pmod{10}. \end{aligned}$$

When $r = 1$, $J_{2t_n} \equiv 8 - 7 \equiv 1 \pmod{10}$; when $r = 2$, $J_{2t_n} \equiv 2 \cdot 2^6 - 7 \equiv 1 \pmod{10}$; and when $r = 3 \equiv -1 \pmod 4$ and when $r = 0$, $J_{2t_n} \equiv 2 - 7 \equiv 5 \pmod{10}$.

The given result now follows.

The case $C_n = j_n$ follows similarly. $\square$

For example, $J_{2t_5} \equiv 1 \equiv J_{2t_6} \pmod{10}$ and $J_{2t_7} \equiv 5 \equiv J_{2t_8} \pmod{10}$; and $j_{2t_5} \equiv 5 \equiv j_{2t_6} \pmod{10}$ and $j_{2t_7} \equiv 7 \equiv j_{2t_8} \pmod{10}$.

Next, we explore the periodicity of the sequence $\{C_{4t_n} \pmod{10}\}$.

Theorem 3.8.

$$C_{4t_n} \equiv \begin{cases} 5 \pmod{10} & \text{if } C_n = J_n \\ 7 \pmod{10} & \text{otherwise.} \end{cases}$$

Proof. The proofs follow from the facts that $3J_{4t_n} \equiv 6^{n(n+1)/2} - 1 \pmod{10}$ and $j_{4t_n} \equiv 6^{n(n+1)/2} + 1 \pmod{10}$. $\square$

For example, $J_{4t_5} \equiv 5 \equiv J_{4t_6} \pmod{10}$ and $j_{4t_5} \equiv 7 \equiv j_{4t_6} \pmod{10}$.

Theorem 3.8 has interesting consequences. It follows from the summation formula [6]

$$\sum_{k=1}^{n} 4^{(n-k)(n+k+1)} J_{4k^2}^2 J_{4k} = J_{4t_n}^2 \tag{3.3}$$

that

$$\sum_{k=1}^{n} 4^{(n-k)(n+k+1)} J_{4k^2}^2 J_{4k} \equiv 5 \pmod{10}.$$

As an example,

$$\begin{aligned} \sum_{k=1}^{n} 4^{(n-k)(n+k+1)} J_{4k^2}^2 J_{4k} &= 4^{18} J_4^2 + 4^{14} J_{16} J_8 + 4^8 J_{36} j_{12} + j_{64} J_{16} \\ &= 6 \cdot 5 + 6 \cdot 5 \cdot 5 + 6 \cdot 5 \cdot 5 + 5 \cdot 5 \pmod{10} \\ &\equiv 5 \pmod{10} \\ &\equiv J_{4t_4}^2 \pmod{10}. \end{aligned}$$

The summation formula (3.3) has a semi-counterpart for the Jacobsthal-Lucas subfamily [6]:

$$9\sum_{k=1}^{n} 4^{(n-k)(n+k+1)} J_{4k^2}^2 J_{4k} = j_{4t_n}^2 - 4\cdot 4^{2t_n}. \tag{3.4}$$

Then also,

$$\begin{aligned}\sum_{k=1}^{n} 4^{(n-k)(n+k+1)} J_{4k^2}^2 J_{4k} &\equiv -9 + 4\cdot 6 \pmod{10}\\ &\equiv 5 \pmod{10}.\end{aligned}$$

It follows from formulas (3.3) and (3.4) that [6]

$$9J_{4t_n}^2 = j_{4t_n}^2 - 4\cdot 4^{2t_n}. \tag{3.5}$$

For example, $j_{4t_2}^2 - 4\cdot 4^{2t_2} = 16,769,025 = 9J_{4t_2}^2$.

Formula (3.5) has a byproduct:

$$J_{4t_n}^2 + j_{4t_n}^2 \equiv 4 \pmod{10}.$$

Theorem 3.1 indeed confirms this congruence.

As an example, $J_{4t_4}^2 + j_{4t_4}^2 = 366,503,875,925^2 + 1,099,511,627,777^2 \equiv 4 \pmod{10}$.

We now study the periodicity of the sequence $\{C_{t_{n^2}} \pmod{10}\}$ using modulus 4.

4. Periodicity of the Sequence $\{C_{t_{n^2}} (\text{mod } 10)\}$

Let $n = 4k + r$, where $0 \le r \le 3$. Then,

$$\begin{aligned}J_{t_{n^2}} &\equiv 2\cdot 2^{r^2(r^2+1)/2} - 7(-1)^{r^2(r^2+1)/2} \pmod{10};\\ j_{t_{n^2}} &\equiv 6\cdot 2^{r^2(r^2+1)/2} + (-1)^{r^2(r^2+1)/2} \pmod{10}.\end{aligned}$$

Using these congruences, we can find the periodicity of the sequence and the repeating block, as stated in the following theorem. In the interest of brevity, we omit the details.

Theorem 4.1. *The sequence $\{C_{t_{n^2}} (\text{mod } 10)\}$ is periodic with period 4. The repeating block is*

$$\begin{cases} 1115 & \text{if } C_n = J_n\\ 1517 & \text{otherwise.}\end{cases}$$

For example, $J_{t_{4^2}} \equiv 2 - 7 \equiv 5 \pmod{10}$ and $j_{t_{4^2}} \equiv 6 + 1 \equiv 7 \pmod{10}$. Theorem 3.1 confirms these values, as expected.

We now study the Jacobsthal sequences with nested triangular subscripts.

5. Periodicity of the Sequence $\{C_{t_{t_n}} (\text{mod } 10)\}$

Theorem 2.1, coupled with Theorem 3.1, can be employed to study the periodicity of the sequence $\{C_{t_{t_n}} (\text{mod } 10)\}$. Recall that the sequences $\{C_n \pmod{10}\}$ and $\{C_{t_n} (\text{mod } 10)\}$ are both periodic with periods 4 and 8, respectively. So, we can interpret Theorem 3.1 as follows: By replacing the n in $A = \{C_n \pmod{10}\}$ with t_n, we obtain the subsequence $B = \{C_{t_n} \pmod{10}\}$ with period twice that of A, namely 8.

Using this argument, it follows that the period of the sequence $C = \{C_{t_{t_n}} (\text{mod } 10)\}$ is twice that of B, namely, 16. Computationally, we can confirm that the repeating block is 1113531135311155 if $C_n = J_n$, and 1517775577715177 if $C_n = j_n$.

Thus, we have the following result; see Figure 2.

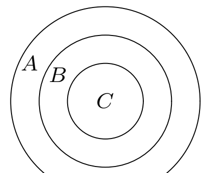

Figure 2: Sets of sequences A, B, and C

Theorem 5.1. *The sequence* $\{C_{t_{t_n}} (\text{mod } 10)\}$ *is periodic with period 16. The repeating block is*

$$\begin{cases} 1113531135311155 & \text{if } C_n = J_n \\ 1517775577715177 & \text{otherwise.} \end{cases}$$

For example, $3J_{t_{t_{10}}} = 2^{1540} - 1 \equiv 5 \pmod{10}$ and hence, $J_{t_{t_{10}}} \equiv 5 \pmod{10}$; and $3J_{t_{t_{13}}} = 2^{4186} - 1 \equiv 3 \pmod{10}$; so $J_{t_{t_{13}}} \equiv 1 \pmod{10}$.

Next, we investigate a related subsequence with a unique residue in both cases.

5.1. **Related Sequence** $\{C_{8t_{t_n}} (\text{mod } 10)\}$**.** Notice that $n^2(n+1)^2 + 2n(n+1) \equiv 0 \pmod 8$; so $n^2(n+1)^2 + 2n(n+1) = 8m$ for some integer $m \geq 1$. Then, $3J_{8t_{t_n}} = 2^{8m} - 1 \equiv 5 \pmod{10}$ and hence, $J_{8t_{t_n}} \equiv 5 \pmod{10}$; and $j_{8t_{t_n}} = 2^{8m} + 1 \equiv 7 \pmod{10}$.

Thus,

$$C_{8t_{t_n}} \equiv \begin{cases} 5 \pmod{10} & \text{if } C_n = J_n \\ 7 \pmod{10} & \text{otherwise.} \end{cases}$$

For example, $3J_{8t_{t_{11}}} = 2^{8\cdot 2211} - 1 \equiv 5 \pmod{10}$ and hence, $J_{8t_{t_{11}}} \equiv 5 \pmod{10}$. In addition, $j_{8t_{t_{11}}} = 2^{8\cdot 2211} + 1 \equiv 7 \pmod{10}$.

Next, we explore the periodicity of the sequence $\{C_{C_n}\}$.

6. Periodicity of the Sequence $\{C_{C_n} (\text{mod } 10)\}$

It follows by induction that $2^{4^n} \equiv 6 \pmod{10}$ when $n \geq 1$. We will employ this result in our exploration.

Part 1. Consider J_{J_n}. Clearly, $J_{J_1} = 1$; so we let $n \geq 2$. Because J_n is odd, then $3J_{J_n} - 1 = 2^{J_n} = 2^{[2^n - (-1)^n]/3}$. Letting $A = 3J_{J_n} - 1$, this implies $A^3 = 2^{2^n - (-1)^n}$; notice that A is even.

Case 1. Let $n = 2k$, where $k \geq 1$. Then $A^3 = 2^{2^{2k} - 1} = 2^{4^k - 1}$; so $2A^3 = 2^{4^k} \equiv 6 \pmod{10}$. Because A is even, this implies $A^3 \equiv 8 \pmod{10}$ and hence, $A \equiv 2 \pmod{10}$. This yields $J_{J_n} \equiv 1 \pmod{10}$.

Case 2. Let $n = 2k+1$. Then, $A^3 = 2^{2^{2k+1}+1} = 2\left(2^{4^k}\right)^2 \equiv 2 \pmod{10}$; so $A \equiv 8 \pmod{10}$. This implies $J_{J_n} \equiv 3 \pmod{10}$.

Thus,

$$J_{J_n} \equiv \begin{cases} 1 \pmod{10} & \text{if } n > 1 \text{ and is even} \\ 3 \pmod{10} & \text{if } n > 1 \text{ and is odd,} \end{cases}$$

where $J_{J_1} = 1$.

Part 2. Now consider J_{j_n}. Because $J_{j_1} = 1$, we let $n \geq 2$.

Case 1. Let $n = 2k$, where $k \geq 1$. Then, $3J_{j_n} = 2^{2^{2k}+1} + 1 = 2 \cdot 2^{4^k} + 1 = 2 \cdot 6 + 1 \equiv 3 \pmod{10}$; so, $J_{j_n} \equiv 1 \pmod{10}$.

Case 2. Let $n = 2k+1$. Let $B = 3J_{j_n}$; then, $2(B-1) = 2^{2 \cdot 4^k} \equiv 6 \pmod{10}$. So, $B - 1 \equiv 3$ or 8 modulo 10, and hence, $B \equiv 4$ or 9 modulo 10. Because B is odd, this forces $B \equiv 9 \pmod{10}$. Consequently, $3J_{j_n} \equiv 9 \pmod{10}$ and hence, $J_{j_n} \equiv 3 \pmod{10}$.

Thus,

$$J_{j_n} \equiv \begin{cases} 1 \pmod{10} & \text{if } n > 1 \text{ and is even} \\ 3 \pmod{10} & \text{if } n > 1 \text{ and is odd,} \end{cases}$$

where $J_{j_1} = 1$.

Part 3. Consider j_{J_n}. Clearly, $j_{J_n} = 1$; so we let $n \geq 2$. Then,

$$\begin{aligned} (j_{J_n} + 1)^3 &= 2^{3J_n} \\ &= 2^{2^n - (-1)^n}. \end{aligned} \tag{6.1}$$

Case 1. Let $n = 2k$, where $k \geq 1$. Then,

$$\begin{aligned} (j_{J_n} + 1)^3 &= 2^{2^{2k}-1} \\ 2\left(j_{J_n} + 1\right)^3 &\equiv 6 \pmod{10}. \end{aligned}$$

Because j_n is odd, this implies $j_{J_n} + 1 \equiv 2 \pmod{10}$. So, $j_{J_n} \equiv 1 \pmod{10}$.

Case 2. Let $n = 2k+1$. It then follows from equation (6.1) that

$$\begin{aligned} (j_{J_n} + 1)^3 &= 2^{2^{2k+1}+1} \\ &= 2\left(2^{4^k}\right)^2 \\ &\equiv 2 \pmod{10}. \end{aligned}$$

Then, $j_{J_n} + 1 \equiv 8 \pmod{10}$; so, $j_{J_n} \equiv 7 \pmod{10}$.

Thus,

$$j_{J_n} \equiv \begin{cases} 1 \pmod{10} & \text{if } n > 1 \text{ and is even} \\ 7 \pmod{10} & \text{if } n > 1 \text{ and is odd,} \end{cases}$$

where $j_{J_1} = 1$.

Part 4. Now consider j_{j_n}. Because $j_{j_n} = 1$, we let $n \geq 2$. The odd parity of j_n yields

$$\begin{aligned} j_{j_n} &= 2^{j_n} + (-1)^{j_n} \\ &= 2^{2^n + (-1)^n} - 1. \end{aligned}$$

Case 1. Let $n = 2k$, where $k \geq 1$. Then,

$$\begin{aligned} j_{j_n} &= 2^{2^{2k}+1} - 1 \\ &= 2 \cdot 2^{4^k} - 1 \\ &\equiv 1 \pmod{10}. \end{aligned}$$

Case 2. Let $n = 2k+1$. Then,

$$\begin{aligned} j_{j_n} &= 2^{2^{2k+1}-1} - 1 \\ 2(j_{j_n} + 1) &\equiv 6 \pmod{10}. \end{aligned}$$

Because j_{j_n} is odd, this implies $j_{j_n} + 1 \equiv 8 \pmod{10}$; so, $j_{j_n} \equiv 7 \pmod{10}$. Thus,

$$j_{j_n} \equiv \begin{cases} 1 \pmod{10} & \text{if } n > 1 \text{ and is even} \\ 7 \pmod{10} & \text{if } n > 1 \text{ and is odd,} \end{cases}$$

where $j_{j_1} = 1$.

Combining the four parts, we have the following result.

Theorem 6.1.

$$J_{C_n} \equiv \begin{cases} 1 \pmod{10} & \text{if } n > 1 \text{ and is even} \\ 3 \pmod{10} & \text{if } n > 1 \text{ and is odd;} \end{cases}$$

$$j_{C_n} \equiv \begin{cases} 1 \pmod{10} & \text{if } n > 1 \text{ and is even} \\ 7 \pmod{10} & \text{if } n > 1 \text{ and is odd,} \end{cases}$$

where $C_{C_1} = 1$.

It now follows that the sequences $\{J_{C_n} \pmod{10}\}$ and $\{j_{C_n} \pmod{10}\}$ are both periodic with period 2, where $n \geq 2$. Their repeating blocks are 13 and 17, respectively.

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References

[1] A. F. Horadam, *Jacobsthal representation numbers*, The Fibonacci Quarterly, **34.1** (1996), 40–54.
[2] T. Koshy, *Elementary Number Theory with Applications*, 2nd ed., Academic Press, Burlington, Massachusetts, 2007.
[3] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Volume I, Second Edition, Wiley, Hoboken, New Jersey, 2018.
[4] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Volume II, Wiley, New York, 2019.
[5] T. Koshy, *Polynomial extensions of two Gibonacci delights and their graph-theoretic confirmations*, The Mathematical Scientist, **43** (2018), 96–108.
[6] T. Koshy, *Some extended Gibonacci polynomial sums with dividends*, The Fibonacci Quarterly, (to appear).
[7] J. Kramer and V. E. Hoggatt, Jr., *Special cases of Fibonacci periodicity*, The Fibonacci Quarterly, **10.5** (1972), 519–522.

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