TRIANGULAR-LIKE NUMBERS THAT ARE TRIANGULAR

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Abstract. A balancing-like sequence is a recurrence sequence satisfying the recurrence relation $x_{n+1} = Ax_n - x_{n-1}$ with initial terms $x_0 = 0$ and $x_1 = 1$ and $A > 2$ is a positive integer. For any given A, the nth triangular-like number is defined as $\tau_n(A) = \frac{x_n \cdot x_{n+1}}{A}$. All the triangular-like numbers corresponding to the balancing-like sequence with $A = 4$ are triangular numbers. However, no other balancing-like sequence enjoys this property.

1. Introduction

A balancing number B is a natural number such that the equation $1 + 2 + \cdots + (B - 1) = (B + 1) + \cdots + (B + R)$ holds for some natural number R, called the balancer corresponding to B [2]. If B is a balancing number, then $8B^2 + 1$ is a perfect square and its positive square root is called a Lucas-balancing number. The nth balancing number is denoted by B_n, and the balancing sequence satisfies the binary recurrence $B_{n+1} = 6B_n - B_{n-1}$ with initial terms $B_0 = 0$ and $B_1 = 1$. Similarly, the nth Lucas-balancing number is denoted by C_n, and the Lucas-balancing sequence satisfies $C_{n+1} = 6C_n - C_{n-1}$ with $C_0 = 1$ and $C_1 = 3$ [14].

Panda and Ray [9], with a minor modification to the definition of balancing numbers, introduced cobalancing numbers. They call a natural number b, a cobalancing number if $1 + 2 + \cdots + b = (b + 1) + \cdots + (b + r)$ for some natural number r, which they call the cobalancer corresponding to b. They proved that a nonnegative integer b is a cobalancing number if and only if $8b^2 + 8b + 1$ is a perfect square. The nth cobalancing number is denoted by b_n and the cobalancing sequence satisfies the binary recurrence $b_{n+1} = 6b_n - b_{n-1} + 2$ with initial terms $b_0 = b_1 = 0$.

The concept of balancing numbers was further generalized by Panda and Panda [7] with the introduction of almost balancing numbers. They call a natural number n an almost balancing number if $|\{1 + 2 + \cdots + (n - 1)\} - \{(n + 1) + \cdots + (n + r)\}| = 1$ for some natural number r. They proved that n is an almost balancing number if either $8n^2 + 9$ or $8n^2 - 7$ is a perfect square. If $8n^2 + 9$ is a perfect square, then they call n an almost balancing number of the first kind, and if $8n^2 - 7$ is a perfect square, then they call n an almost balancing number of the second kind. They further proved that the almost balancing numbers of the first kind are three times the respective balancing numbers, whereas the almost balancing numbers of the second kind are partitioned into two classes, and the numbers in these two classes are respectively $B_{n+1} - 2B_n$ and $2B_{n+1} - B_n,\ n = 0, 1, 2, \ldots$

Several generalizations and variations of balancing numbers are available in the literature [1, 3, 4, 9, 5, 7, 12, 13, 16]. Panda and Rout [11] generalized balancing numbers by replacing 6, appearing on the right side of their recurrence relation, by an arbitrary positive integer $A > 2$. For $A > 2$, they studied recurrence sequences $\{x_n\}$ defined by $x_{n+1} = Ax_n - x_{n-1}$ with initial terms $x_0 = 0$ and $x_1 = 1$, which are subsequently known as balancing-like sequences [15]. In this paper, we denote this sequence by $BL(A, -1)$. Panda and Rout [11] proved that if x is a balancing-like number with respect to a given A, then $Dx^2 + 1$, where $D = \frac{A^2-4}{4}$, is a perfect rational square (a perfect integral square only if A is even) and call its square root a

Lucas-balancing-like number. The Lucas-balancing-like sequence corresponding to $BL(A,-1)$ is denoted by $\{y_n\}$ and it satisfies a recurrence relation identical with that of balancing-like numbers with initial terms $y_0 = 1$ and $y_1 = \frac{A}{2}$.

In a recent paper, Panda and Pradhan [8] introduced triangular-like numbers corresponding to a given balancing-like sequence as the product of any two consecutive terms of the sequence divided by A. The nth triangular-like number corresponding to the sequence $BL(A,-1)$ is $\frac{x_n \cdot x_{n+1}}{A}$. It is easy to see that the sequence of natural numbers satisfies the binary recurrence $x_{n+1} = 2x_n - x_{n-1}$, $x_0 = 0$, and $x_1 = 1$, hence the balancing-like sequences generalize the sequence of natural numbers. The definition of nth triangular-like number is the same as that of nth triangular number when $A = 2$. In this paper, we explore all the balancing-like sequences with the property that all their triangular-like numbers are triangular numbers.

It is well-known that a natural number T is a triangular number if $8T + 1$ is a perfect square. Panda and Pradhan [8] proved that a natural number τ is a triangular-like number corresponding to $BL(A,-1)$ if both $(A^2+2A)\tau+1$ and $(A^2-2A)\tau+1$ are perfect squares. If $A = 2$, observe that the second condition becomes trivial. It can be verified that if $A > 2$, then fulfillment of merely one of the two conditions is not sufficient for τ to become a triangular-like number of $BL(A,-1)$

2. Preliminaries

In this section, we recall certain results on balancing sequence, cobalancing sequence, almost balancing sequence, balancing-like sequence, Pell sequence, and associated Pell sequence. These results are essential for the development of the main results of this paper. We shall keep referring back to this section whenever necessary with or without further reference.

The Pell sequence is defined by means of a second order recurrence relation $P_{n+1} = 2P_n + P_{n-1}$, with initial terms $P_0 = 0$ and $P_1 = 1$, whereas the associated Pell sequence is defined recursively as $Q_{n+1} = 2Q_n + Q_{n-1}$, with initial terms $Q_0 = 1$ and $Q_1 = 1$.

Lemma 2.1. *For any natural number n, the following hold:*

(a) $b_{2n} = P_{2n}Q_{2n-1}$ *and* $b_{2n+1} = P_{2n}Q_{2n+1}$.
(b) $P_{2n} = 2B_n$ *and* $Q_{2n+1} = B_n + B_{n+1}$.
(c) $x_{n-1} \cdot x_{n+1} = x_n^2 - 1$.
(d) $x_1 + x_3 + \cdots + x_{2n-1} = x_n^2$.
(e) $x_2 + x_4 + \cdots + x_{2n} = x_n \cdot x_{n+1}$.
(f) $x_1 + x_2 + \cdots + x_{2n-1} = x_n \cdot (x_n + x_{n-1})$.
(g) $x_1 + x_2 + \cdots + x_{2n} = x_n \cdot (x_n + x_{n+1})$.

The proofs of (a) and (b) are available in [10]. For the proofs of (c), (d), and (e), see [6]. (f) and (g) are direct consequences of (d) and (e), respectively.

3. Main Results

For a fixed positive integer $A > 2$, we denote the nth triangular-like number $\frac{x_n \cdot x_{n+1}}{A}$ of this sequence by $\tau_n(A)$. Also, we denote the nth triangular number $\frac{n(n+1)}{2}$ by T_n. The main objective of this section is to prove that $BL(4,-1)$ is the only balancing-like sequence such that all its triangular-like numbers are triangular numbers.

For an arbitrary positive integer $A > 2$, the terms of $BL(A,-1)$ are $x_1 = 1$, $x_2 = A$, $x_3 = A^2 - 1$, $x_4 = A^3 - 2A$, ... and accordingly, the triangular-like numbers of this sequence are $\tau_1(A) = 1$, $\tau_2(A) = A^2 - 1$, $\tau_3(A) = (A^2-2)(A^2-1)$, ... The first triangular-like number

$\tau_1(A) = 1$ is also a triangular number for each A. Next, we discuss the condition under which the second triangular-like number $\tau_2(A)$ is a triangular number.

Theorem 3.1. *$\tau_2(A)$ is a triangular number if and only if A is an almost balancing number of the second kind. In particular, $A = B_{n+1} - 2B_n$ or $A = 2B_{n+1} - B_n$ for some nonnegative integer n.*

Proof. $\tau_2(A) = A^2 - 1$ is a triangular number if and only if $8(A^2 - 1) + 1 = 8A^2 - 7$ is a perfect square and hence, A is an almost balancing number of the second kind ([7], p. 150). Thus, $A = B_{n+1} - 2B_n$ or $A = 2B_{n+1} - B_n$ for some nonnegative integer n. □

Because our focus is on balancing-like sequences with the property that all their triangular-like numbers are triangular numbers, it is natural to examine what makes $\tau_3(A)$ a triangular number. The following theorem establishes the association of such A with cobalancing numbers.

Theorem 3.2. *$\tau_3(A)$ is a triangular number if and only if $8A^4 - 24A^2 + 17$ is a perfect square or equivalently, $A^2 - 2$ is a cobalancing number.*

Proof. A natural number N is a triangular number if and only if $8N + 1$ is a perfect square and consequently, $\tau_3(A)$ is a triangular number if and only if

$$8\tau_3(A) + 1 = 8(A^2 - 2)(A^2 - 1) + 1 = 8A^4 - 24A^2 + 17$$

is a perfect square. A nonnegative integer b is a cobalancing number if and only if $8b^2 + 8b + 1$ is a perfect square [9]. Because $8A^4 - 24A^2 + 17 = 8(A^2 - 2)^2 + 8(A^2 - 2) + 1$, the conclusion of the theorem follows. □

One can check that $(A, B) = (1, 1), (2, 7), (4, 41)$ satisfies the Diophantine equation $8A^4 - 24A^2 + 17 = B^2$. We tried to solve $8A^4 - 24A^2 + 17 = B^2$, but did not succeed. We did not find any more solutions in the range $1 \leq A \leq 10^6$. Because we restrict A to be greater than 2, the only feasible solution among the three available solutions is $(A, B) = (4, 41)$, which indicates that if $A = 4$, then $\tau_3(A)$ is triangular. However, we are interested in balancing-like sequences with the property that all their triangular-like numbers are triangular numbers. Hence, it is a better idea to look for those A for which both $\tau_2(A)$ and $\tau_3(A)$ are triangular numbers. The following theorem characterizes such A.

Theorem 3.3. *For $A > 2$, both $\tau_2(A)$ and $\tau_3(A)$ are triangular numbers if and only if $A = 4$.*

Proof. It is easy to see that if $A = 4$, then $\tau_2(A)$ and $\tau_3(A)$ are triangular numbers. Conversely, if $\tau_2(A)$ is a triangular number, then, in view of Theorem 3.1, A is an almost balancing number of the second kind and which are of the form $B_{n+1} - 2B_n$, $2B_{n+1} - B_n$, for $n = 0, 1, 2, \ldots$. For $n = 1, 2, \ldots$, let

$$A_{2n-1} = B_n - 2B_{n-1}, \ A_{2n} = 2B_n - B_{n-1}.$$

We need to look for triangular numbers in the sequence $\{\tau_3(A_n)\}$. The requirement $A > 2$ disqualifies $A_1 = 1$ and $A_2 = 2$ from our search. $A_3 = 4$ and also $\tau_3(4) = (4^2 - 2)(4^2 - 1) = 210$ is a triangular number. However, $A_4 = 11$ and $\tau_3(11) = (11^2 - 2)(11^2 - 1) = 14280$ is not a triangular number.

In view of Theorem 3.2, $\tau_3(A)$ is a triangular number if and only if $A^2 - 2$ is a cobalancing number, such as $A^2 - 2 = b$ or $A^2 = b + 2$. We will show that if $n > 3$ (and hence $A_n > 4$), then

$$A_{n-1}^2 < b_n + 2 < A_n^2, \tag{3.1}$$

where b_n is the nth cobalancing number. We divide the proof into two parts. First, we show that

$$A_{2n-1}^2 < b_{2n} + 2 < A_{2n}^2 \tag{3.2}$$

for $n \geq 2$. By virtue of Lemma 2.1(a),

$$b_{2n} = P_{2n}Q_{2n-1} = 2B_n(B_n + B_{n-1}) = 2B_n^2 + 2B_nB_{n-1},$$

and consequently,

$$A_{2n}^2 - (b_{2n} + 2) = 2B_n^2 + B_{n-1}^2 - 6B_nB_{n-1} - 2 = 2B_n(B_n - 3B_{n-1}) + B_{n-1}^2 - 2.$$

Because $B_n = 6B_{n-1} - B_{n-2} > 5B_{n-1}$, it follows that

$$A_{2n}^2 - (b_{2n} + 2) > 4B_nB_{n-1} + B_{n-1}^2 - 2 > 0 \tag{3.3}$$

for $n \geq 2$. Furthermore,

$$(b_{2n} + 2) - A_{2n-1}^2 = B_n^2 - 4B_{n-1}^2 + 6B_nB_{n-1} + 2 > 21B_{n-1}^2 + 6B_nB_{n-1} + 2 > 0 \tag{3.4}$$

for $n \geq 1$. Now (3.2) follows from (3.3) and (3.4).

We next show that

$$A_{2n}^2 < b_{2n+1} + 2 < A_{2n+1}^2 \tag{3.5}$$

for $n \geq 2$. In view of Lemma 2.1(a) and the recurrence relation for balancing numbers,

$$b_{2n+1} = P_{2n}Q_{2n+1} = 2B_n(B_n + B_{n+1}) = 2B_n^2 + 2B_nB_{n+1} = 14B_n^2 - 2B_nB_{n-1},$$

and this implies

$$A_{2n+1}^2 - (b_{2n+1} + 2) = (4B_n - B_{n-1})^2 - (b_{2n+1} + 2) = 2B_n(B_n - 3B_{n-1}) + B_{n-1}^2 - 2 > 0 \tag{3.6}$$

for $n \geq 2$. Lastly,

$$(b_{2n+1} + 2) - A_{2n}^2 = 10B_n^2 - B_{n-1}^2 + 2B_nB_{n-1} + 2 > 0 \tag{3.7}$$

for $n \geq 1$ and (3.5) follows from (3.6) and (3.7). From (3.2) and (3.5), it follows that $b_n + 2$ is not a perfect square and hence, $\tau_3(A_n)$ is also not a triangular number if $n > 3$; the only value of n for which $\tau_3(A_n)$ is a triangular number is $n = 3$, in which case $A_n = 4$. This completes the proof. □

The conclusion of Theorem 3.3 is that if $A > 4$, then $\tau_2(A)$ and $\tau_3(A)$ cannot be simultaneously triangular numbers. This suggests that if $A > 4$, then all the triangular-like numbers corresponding to $BL(A, -1)$ cannot be triangular numbers. However, the case $A = 4$ is an exception as will be seen in the following theorem.

Theorem 3.4. *All the triangular-like numbers corresponding to $BL(4, -1)$ are triangular numbers.*

Proof. It is well-known that a natural number T is a triangular number if and only if $8T + 1$ is a perfect square. The nth triangular-like number of the sequence $BL(4, -1)$ is $\tau_n(4) = \frac{x_n \cdot x_{n+1}}{4}$. Using the Lemma 2.1 (c) and (e), we get

$$\begin{aligned} 8\tau_n(4) + 1 &= 2x_n \cdot x_{n+1} + 1 = 2x_n \cdot x_{n+1} + x_n^2 - x_{n-1} \cdot x_{n+1} \\ &= 2x_n \cdot x_{n+1} + x_n^2 - (4x_n - x_{n+1})x_{n+1} = (x_{n+1} - x_n)^2. \end{aligned}$$

Because n is an arbitrary positive integer, it follows that $\tau_n(4)$ is a triangular number for each positive integer n. □

In the previous theorem, we proved that $\tau_n(4)$ is a triangular number for each positive integer n. But, it is interesting to see which triangular number $\tau_n(4)$ represents. The following theorem answers this question.

Theorem 3.5. *The nth triangular-like number corresponding to $BL(4,-1)$ is the $(\sum_{i=1}^{n} x_i)$th triangular number. In other words, $\tau_n(4) = \frac{(\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} x_i+1)}{2}$.*

Proof. We divide the proof into two parts. First, let n be even, for example $n = 2k$. Then,

$$\tau_n(4) = \tau_{2k}(4) = \frac{x_{2k} \cdot x_{2k+1}}{4}.$$

By virtue of Lemma 2.1 (d) and (e), $x_{2k} = x_k(x_{k+1} - x_{k-1})$ and $x_{2k+1} = x_{k+1}^2 - x_k^2$. Hence, we can write $\tau_{2k}(4)$ as

$$\tau_{2k}(4) = \frac{x_k(x_{k+1} - x_{k-1})(x_{k+1}^2 - x_k^2)}{4}$$

and using the recurrence relation $x_{k+1} = 4x_k - x_{k-1}$ and Lemma 2.1 (c), we get

$$\begin{aligned}\tau_{2k}(4) = \frac{x_k(2x_k - x_{k-1})(x_{k+1}^2 - x_k^2)}{2} &= \frac{x_k(x_{k+1} + x_k)(2x_k x_{k+1} - 3x_k^2 + x_k x_{k-1} + 1)}{2}\\ &= \frac{x_k(x_{k+1} + x_k)(x_k x_{k+1} + x_k^2 + 1)}{2} = T_{u_k},\end{aligned}$$

where $u_k = x_k(x_{k+1} + x_k)$. If n is odd, for example $n = 2k - 1$, then

$$\tau_n(4) = \tau_{2k-1}(4) = \frac{x_{2k-1} \cdot x_{2k}}{4}.$$

Once again using Lemma 2.1 (d) and (e), we can write $\tau_{2k-1}(4)$ as

$$\tau_{2k-1}(4) = \frac{x_k(x_{k+1} - x_{k-1})(x_k^2 - x_{k-1}^2)}{4}.$$

Now, using the recurrence relation $x_{k+1} = 4x_k - x_{k-1}$ and Lemma 2.1 (c), we get

$$\begin{aligned}\tau_{2k-1}(4) &= \frac{x_k(2x_k - x_{k-1})(x_k^2 - x_{k-1}^2)}{2} = \frac{x_k(x_k + x_{k-1})(2x_k^2 - 3x_k x_{k-1} + x_{k-1}^2)}{2}\\ &= \frac{x_k(x_{k+1} + x_k)(2x_k^2 - x_{k-1}x_{k+1} + x_k x_{k-1})}{2}\\ &= \frac{x_k(x_k + x_{k-1})(x_k x_{k-1} + x_k^2 + 1)}{2} = T_{v_k},\end{aligned}$$

where $v_k = x_k(x_k + x_{k-1})$. By Lemma 2.1 (g), $u_k = \sum_{i=1}^{2k} x_i$, and by 2.1 (f), $v_k = \sum_{i=1}^{2k-1} x_i$; the conclusion of the theorem follows. □

In the last theorem, we showed that all the triangular-like numbers corresponding to $BL(4,-1)$ are triangular numbers. Instead of calculating these triangular numbers using the terms of the sequence $BL(4,-1)$, it is possible to calculate them by means of a recurrence relation for the indices of these triangular numbers. In view of ([8],Theorem 3.1), these indices determine a sequence that is nothing but the sequence of cobalancing-like numbers corresponding to $BL(4,-1)$ divided by 2.

Theorem 3.6. *If $\tau_n(4) = T_{w_n}$ for $n = 0, 1, 2, \ldots$, then the sequence $\{w_n\}$ satisfies the nonhomogeneous binary recurrence $w_{n+1} = 4w_n - w_{n-1} + 1$ with initial terms $w_0 = 0$, and $w_1 = 1$.*

Proof. In view of Theorem 3.5, $w_n = \sum_{i=1}^{n} x_i$. Thus, $w_0 = 0$ and $w_1 = 1$, and

$$4w_n - w_{n-1} + 1 = 4\sum_{i=1}^{n} x_i - \sum_{i=1}^{n-1} x_i + 1 = \sum_{i=1}^{n}(4x_i - x_{i-1}) + 1 = 1 + \sum_{i=2}^{n+1} x_i = \sum_{i=1}^{n+1} x_i = w_{n+1}.$$

This completes the proof. □

The sequence of triangular numbers can be calculated in two ways. The nth triangular number is defined as the product of the nth and $(n+1)$st natural numbers divided by the second natural number, and this has motivated the authors in [8] to define the triangular-like numbers corresponding to $BL(A, -1)$. Coincidentally, the nth triangular number is also equal to the sum of the first n natural numbers. But, in the case of $BL(A, -1)$, the sum of the first n terms is not equal to the nth triangular-like number $\tau_n(A)$. The following theorem tells us about the sequence, the sum of whose first n terms is equal to $\tau_n(A)$.

Theorem 3.7. *If $A > 2$ is any given positive integer, then the nth triangular-like number corresponding to the sequence $BL(A, -1)$ is equal to the sum of the first n terms of the sequence $BL(A^2 - 2, -1)$.*

Proof. By a proof similar to Theorem 3.6, it is easy to see that the partial sum sequence $\{s_n\}$ of $BL(A^2 - 2, -1)$ satisfies the recurrence relation

$$s_{n+1} = (A^2 - 2)s_n - s_{n-1} + 1;\ s_0 = 0,\ s_1 = 1. \tag{3.8}$$

Now, going back to the sequence $BL(A, -1)$,

$$(A^2 - 2)\tau_n(A) - \tau_{n-1}(A) + 1 = \frac{(A^2 - 2)x_n \cdot x_{n+1} - x_{n-1} \cdot x_n + A}{A}.$$

Using the recurrence relation $x_{n+1} = Ax_n - x_{n-1}$ and Lemma 2.1 (c), we get

$$\begin{aligned}(A^2 - 2)\tau_n(A) - \tau_{n-1}(A) + 1 &= \frac{A(x_{n+1} + x_{n-1})x_{n+1} - 2x_n \cdot x_{n+1} - (Ax_n - x_{n+1})x_n + A}{A} \\ &= \frac{Ax_{n+1}^2 - x_n \cdot x_{n+1}}{A} = \frac{x_{n+1} \cdot x_{n+2}}{A} = \tau_{n+1}(A).\end{aligned} \tag{3.9}$$

For each A, $\tau_0(A) = 0$ and $\tau_1(A) = 1$. It is clear from (3.8) and (3.9) that the sequences $\{s_n\}$ and $\{\tau_n(A)\}$ satisfy identical recurrence relations with identical initial terms. Hence, the two sequences are identical and the proof ends. □

4. Conclusion

In Theorem 3.3, we proved that for $A > 2$, the second and third triangular-like numbers $\tau_2(A)$ and $\tau_3(A)$ are triangular if and only if $A = 4$. To prove this, we first noticed that $\tau_2(A)$ is a triangular number if and only if A is an almost balancing number of the second kind. In this process, we do not take into consideration the numbers 1 and 2 that are the first two almost balancing numbers of the second kind because of the restriction $A > 2$. If $A = 2$, then the sequence generated by the recurrence relation $x_{n+1} = Ax_n - x_{n-1}$; $x_0 = 0$, $x_1 = 1$ is the sequence of natural numbers where the concept of triangular-like numbers and triangular numbers coincide. Further, if $A = 1$, then the recurrence relation $x_{n+1} = Ax_n - x_{n-1}$; $x_0 = 0$, $x_1 = 1$ generates the sequence $1, 1, 0, -1, -1, 0, \ldots$ and each triangular-like number of this sequence (if we extend the definition to this sequence) is 1 or 0. But, 1 or 0 are triangular numbers.

Among the values of $A(> 2)$ that make $\tau_2(A)$ a triangular number, we notice that $A = 4$ is the only one for which $\tau_3(A)$ is also triangular. In this process, we proved that $A = 1, 2, 4$, are the only positive integral values of A such that both $8A^2 - 7$ and $8A^4 - 24A^2 + 17$ are perfect squares. After verifying several special cases, we believe that the following is true.

Conjecture 4.1. *The only solutions of the Diophantine equation* $8x^4 - 24x^2 + 17 = y^2$ *in positive integers are* $(x, y) = (1, 1), (2, 7), (4, 41)$.

We also believe that the following two conjectures are also true.

Conjecture 4.2. *There does not exist any cobalancing number* b *other than* $b_2 = 2$ *and* $b_3 = 14$ *such that* $b + 2$ *is a perfect square.*

Conjecture 4.3. *There does not exist any positive cobalancing number* b *other than* $b_2 = 2$ *and* $b_3 = 14$ *such that* $b + 1$ *is a triangular number.*

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