SUMS OF SQUARES OF TETRANACCI NUMBERS: A GENERATING FUNCTION APPROACH

HELMUT PRODINGER AND SARAH J. SELKIRK

Abstract. It is demonstrated how an explicit expression of the (partial) sum of Tetranacci numbers can be found and proved using generating functions and the Hadamard product. We also provide a Binet-type formula for generalized Fibonacci numbers, by explicitly factoring the denominator of their generating functions.

1. Introduction

Tetranacci numbers u_n (OEIS: A000078, [2]) are defined either by the recursion

$$u_{n+4} = u_{n+3} + u_{n+2} + u_{n+1} + u_n, \qquad \text{where} \qquad u_0 = 0,\ u_1 = 1,\ u_2 = 1,\ u_3 = 2,$$

or via the generating function

$$\sum_{n\ge 0} u_n z^n = \frac{z}{1-z-z^2-z^3-z^4}.$$

A typical result in the recent paper [5] is the evaluation

$$\sum_{0\le k\le n} u_k^2 = \frac{1}{3} + u_n u_{n+1} - \frac{1}{3}(u_{n+1}-u_{n-1})^2 + \frac{1}{3}u_n u_{n-2} + \frac{1}{3}u_{n-2}u_{n-3},$$

for which a (long) proof by induction had been given.

The present note sheds some light on how to use generating functions to prove such a result and also how to find this (or an equivalent formula).

Furthermore, *all* the roots of the polynomial $1 - z - z^2 - \cdots - z^h$ are explicitly determined in terms of generalized binomial series. This leads to a Binet-type formula for generalized Fibonacci numbers, and as an example we provide the formula for Tetranacci numbers.

2. The Hadamard Product of Two Power Series

For two power series (generating functions) $f(z) = \sum_n a_n z^n$ and $g(z) = \sum_n b_n z^n$, the Hadamard product is defined as

$$\sum_{n\ge 0} a_n b_n z^n.$$

If $f(z)$ and $g(z)$ are rational, the resulting power series of their Hadamard product is again rational. There are computer algorithms to do this effectively, for instance Gfun [4], implemented in Maple.

We first provide a simple example of the Hadamard product of two generating functions: Let

$$f(z) = \sum_{n\ge 0} n2^n z^n = \frac{2z}{(1-2z)^2}, \qquad \text{and} \qquad g(z) = \sum_{n\ge 0} n3^n z^n = \frac{3z}{(1-3z)^2}.$$

By multiplying corresponding coefficients of $f(z)$ and $g(z)$, the Hadamard product is given by

$$\sum_{n\geq 0} n^2 6^n z^n = \frac{6z(1+6z)}{(1-6z)^2},$$

where the generating function is computed by Gfun.

To give an example in the context of Tetranacci numbers, the Hadamard product of the generating function $z/(1-z-z^2-z^3-z^4)$ with itself is given by

$$\sum_{n\geq 0} u_n^2 z^n = \frac{z-z^2-2\,z^3-2\,z^4-2\,z^5+z^6+z^7}{1-2\,z-4\,z^2-6\,z^3-12\,z^4+4\,z^5+6\,z^6+2\,z^8-z^{10}}.$$

By general principles, the generating function of the partial sums is then given as

$$\begin{aligned}\sum_{n\geq 0}\Big(\sum_{0\leq k\leq n} u_k^2\Big) z^n &= \frac{1}{1-z}\cdot\frac{z-z^2-2\,z^3-2\,z^4-2\,z^5+z^6+z^7}{1-2\,z-4\,z^2-6\,z^3-12\,z^4+4\,z^5+6\,z^6+2\,z^8-z^{10}}\\ &= \frac{z}{3(1-z)}+\frac{2\,z+z^2-z^3-z^4+5\,z^5+4\,z^6+z^7+z^8-z^9-z^{10}}{3(1-2\,z-4\,z^2-6\,z^3-12\,z^4+4\,z^5+6\,z^6+2\,z^8-z^{10})}.\end{aligned} \tag{2.1}$$

For a simpler expression, we can compute the Hadamard product involving coefficients u_n and u_{n+2},

$$\sum_{n\geq 0} u_n u_{n+2} z^n = \frac{2z}{1-2\,z-4\,z^2-6\,z^3-12\,z^4+4\,z^5+6\,z^6+2\,z^8-z^{10}},$$

and by letting $t_n = \frac{1}{2}u_n u_{n+2}$, we can compare this with the generating function in (2.1) to find that for $n \geq 1$,

$$\sum_{0\leq k\leq n} u_k^2 = \frac{1}{3}+\frac{1}{3}(2t_n+t_{n-1}-t_{n-2}-t_{n-3}+5t_{n-4}+4t_{n-5}+t_{n-6}+t_{n-7}-t_{n-8}-t_{n-9}).$$

This is an equivalent formula to the one obtained in [5]. Note that $t_n = 0$ for negative indices. That these two formulas are indeed equivalent can be checked by a computer, and all the generating functions

$$\sum_{n\geq 0} u_{n-i}u_{n-j}z^n,$$

for fixed integers i, j, can be effectively computed via the Hadamard product algorithm implemented in Gfun.

3. Higher Order Fibonacci-type Recursions

To show how the generating function machinery works on similar but more involved sums, define

$$\sum_{n\geq 0} u_n z^n = \frac{z}{1-z-z^2-z^3-z^4-z^5}.$$

Again computing the Hadamard product of this generating function with itself, we find

$$\begin{aligned}\sum_{n\geq 0}\Big(\sum_{0\leq k\leq n} u_k^2\Big) z^n &= \frac{3z}{8(1-z)}\\ &+\frac{-5\,z-3\,z^2+z^3+4\,z^4+2\,z^5-34\,z^6-30\,z^7-20\,z^8-20\,z^9-16\,z^{10}+6\,z^{11}+6\,z^{12}+3\,z^{13}+3\,z^{14}+3\,z^{15}}{8(1-2z-4z^2-7z^3-14z^4-28z^5+4z^6+6z^7+4z^9+10z^{10}-z^{12}-z^{15})}.\end{aligned}$$

With a shift in coefficients, as we did in the previous example, we compute that

$$\sum_{n\geq 0} u_n u_{n+3} z^n = \frac{4z}{1-2z-4z^2-7z^3-14z^4-28z^5+4z^6+6z^7+4z^9+10z^{10}-z^{12}-z^{15}}.$$

We let $t_n = \frac{1}{4}u_n u_{n+3}$ and then express the sum in question as follows:

$$\sum_{0\leq k\leq n} u_k^2 = \frac{3}{8} + \frac{1}{8}(-5t_n - 3t_{n-1} + t_{n-2} + 4t_{n-3} + \cdots + 3t_{n-13} + 3t_{n-14}).$$

This process can be generalized to any higher order Fibonacci-type recursion, such as

$$\sum_{n\geq 0} u_n z^n = \frac{z}{1-z-z^2-z^3-z^4-z^5-z^6},$$

and other identities and related expressions can also be computed, but this process is too long to be displayed here.

4. Higher Order Fibonacci-type Numbers

The identities discussed in previous sections can be proved via higher level computations involving algorithms to compute Hadamard products. In this section, we discuss the roots of the polynomial $1-z-\cdots-z^h$, from which we can obtain an explicit expression for coefficients. This allows us (in principle) to verify identities involving generalized Fibonacci numbers on the level of coefficients.

For generalized Fibonacci numbers defined by the usual initial values and the recursion

$$u_{n+h} = u_{n+h-1} + u_{n+h-2} + \cdots + u_n,$$

the corresponding generating function (and its simplification) is

$$\frac{z}{1-z-\cdots-z^h} = \frac{z}{1-z\cdot\dfrac{1-z^h}{1-z}} = \frac{z(1-z)}{1-2z+z^{h+1}}.$$

The dominant root of this rational function already occurs in the literature, see for example [3]. However, we can do better than that and describe *all* the roots of the denominator, obtaining in this way a Binet-type formula. We consider the generating function

$$\frac{1}{1-2z+z^{h+1}},$$

from which the original case can be obtained by simple shifts.

We determine the roots of the denominator in terms of generalized binomial series, going back to Lambert, and described in more detail in [1]. A generalized binomial series is defined as

$$\big(\mathcal{B}_t(x)\big)^r = \sum_{n\geq 0} \binom{tn+r}{n} \frac{r}{tn+r} x^n.$$

Given the expression $1-\frac{z}{u}+z^{h+1}$, let ζ be a primitive hth root of unity. Then, the $h+1$ roots can be expressed in terms of these generalized binomial series as

$$u\mathcal{B}_{h+1}(u^{h+1}) \qquad \text{and} \qquad \zeta^{-j}u^{-\frac{1}{h}}\mathcal{B}_{(h+1)/h}\big(\zeta^j u^{\frac{h+1}{h}}\big)^{-\frac{1}{h}} \quad \text{for } 0\leq j\leq h-1.$$

In our case, we are addressing the special case where $u=\frac{1}{2}$.

It is easy to verify (and the calculation for $u = \frac{1}{2}$ has appeared in [3]) that

$$1 - u\mathcal{B}_{h+1}(u^{h+1}) + u^{h+1}\mathcal{B}_{h+1}(u^{h+1})^{h+1} = 0.$$

Now, the other roots can be checked by considering first $j = 0$:

$$\begin{aligned}
&1 - u^{-(h+1)/h}\mathcal{B}_{(h+1)/h}\left(u^{\frac{h+1}{h}}\right)^{-\frac{1}{h}} + u^{-\frac{(h+1)}{h}}\mathcal{B}_{(h+1)/h}\left(u^{\frac{h+1}{h}}\right)^{-\frac{(h+1)}{h}} \\
&= 1 - \sum_{n\geq 0}\binom{\frac{(h+1)n-1}{h}}{n}\frac{-1}{(h+1)n-1}u^{\frac{(h+1)(n-1)}{h}} + \sum_{n\geq 0}\binom{\frac{(h+1)(n-1)}{h}}{n}\frac{-1}{n-1}u^{\frac{(h+1)(n-1)}{h}}.
\end{aligned}$$

Because

$$\binom{\frac{(h+1)n-1}{h}}{n}\frac{-1}{(h+1)n-1} = -\frac{\left(\frac{(h+1)(n-1)}{h}\right)\cdots\left(\frac{n-1+h}{h}\right)}{h\cdot n!},$$

and

$$\begin{aligned}
\binom{\frac{(h+1)(n-1)}{h}}{n}\frac{-1}{n-1} &= -\frac{\left(\frac{(h+1)(n-1)}{h}\right)\cdot\left(\frac{(h+1)(n-1)-h}{h}\right)\cdots\left(\frac{n-1+h}{h}\right)\cdot\left(\frac{n-1}{h}\right)}{(n-1)n!} \\
&= -\frac{\left(\frac{(h+1)(n-1)}{h}\right)\cdots\left(\frac{n-1+h}{h}\right)}{h\cdot n!},
\end{aligned}$$

the result follows. The roots for $j \neq 0$ follow from the substitution $u = 1\cdot u$, and with the power of $\frac{1}{h}$ playing a role at each u, we obtain all possible hth roots of unity.

From this, we can explicitly compute the coefficients of

$$\frac{1}{1-2z+z^{h+1}}.$$

For ease of notation, let $r_h = \frac{1}{2}\mathcal{B}_{h+1}\left(\frac{1}{2^{h+1}}\right)$, and for $0 \leq j \leq h-1$, let

$$r_j = \zeta^{-j}2^{\frac{1}{h}}\mathcal{B}_{(h+1)/h}\left(\zeta^j\left(\frac{1}{2}\right)^{\frac{h+1}{h}}\right)^{-\frac{1}{h}}.$$

Then, using partial fractions and these r_i values, we can compute that $[z^n]\frac{1}{1-2z+z^{h+1}}$ is equal to

$$[z^n]\frac{1}{(z-r_0)(z-r_1)\cdots(z-r_h)} = [z^n]\sum_{i=0}^{h}\frac{1}{(z-r_i)}\prod_{\substack{j=0\\j\neq i}}^{h}(r_i-r_j)^{-1} = -\sum_{i=0}^{h}\frac{1}{r_i^{n+1}}\prod_{\substack{j=0\\j\neq i}}^{h}(r_i-r_j)^{-1}.$$

Therefore, we have obtained a Binet-type formula for generalized Fibonacci numbers.

4.1. **A Formula for Tetranacci Numbers.** To provide a concrete example of how one would use these roots to compute generalized Fibonacci numbers, we provide the calculation of the formula for the case of the Tetranacci numbers.

Tetranacci numbers correspond to $h = 4$, so the five roots are (as calculated by a computer):

$$\begin{aligned}
r_4 &= \frac{1}{2}\mathcal{B}_5\Big(\frac{1}{2^5}\Big) = 0.518790063675884,\\
r_0 &= 2^{\frac{1}{4}}\mathcal{B}_{5/4}\Big(\Big(\frac{1}{2}\Big)^{\frac{5}{4}}\Big)^{-\frac{1}{4}} = 1,\\
r_1 &= -i2^{\frac{1}{4}}\mathcal{B}_{5/4}\Big(i\Big(\frac{1}{2}\Big)^{\frac{5}{4}}\Big)^{-\frac{1}{4}} = -0.114070631164587 - 1.21674600397435i,\\
r_2 &= -2^{\frac{1}{4}}\mathcal{B}_{5/4}\Big(-\Big(\frac{1}{2}\Big)^{\frac{5}{4}}\Big)^{-\frac{1}{4}} = -1.29064880134671,\\
r_3 &= i2^{\frac{1}{4}}\mathcal{B}_{5/4}\Big(-i\Big(\frac{1}{2}\Big)^{\frac{5}{4}}\Big)^{-\frac{1}{4}} = -0.114070631164587 + 1.21674600397435i.
\end{aligned}$$

Using these roots and the initial values, we can determine the values of A, B, C, D, and E in the expression

$$u_n = A \cdot r_4^{-n} + B \cdot r_0^{-n} + C \cdot r_1^{-n} + D \cdot r_2^{-n} + E \cdot r_3^{-n}.$$

Again using a computer, we find that these are given by

$$\begin{aligned}
A &= 0.293813062773642\\
B &= 0\\
C &= -0.0504502052166080 - 0.169681902881564i\\
D &= -0.192912652340427\\
E &= -0.0504502052166080 + 0.169681902881564i
\end{aligned}$$

Therefore, the nth Tetranacci number can be calculated via the formula:

$$\begin{aligned}
u_n = &\frac{0.293813062773642}{(0.518790063675884)^n} - \frac{0.0504502052166080 + 0.169681902881564i}{(-0.114070631164587 - 1.21674600397435i)^n}\\
&- \frac{0.192912652340427}{(-1.29064880134671)^n} + \frac{-0.0504502052166080 + 0.169681902881564i}{(-0.114070631164587 + 1.21674600397435i)^n}.
\end{aligned}$$

Analogous computations provide similar formulas for other generalized Fibonacci numbers.

References

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Department of Mathematics, University of Stellenbosch 7602, Stellenbosch, South Africa
Email address: hproding@sun.ac.za

Department of Mathematics, University of Stellenbosch 7602, Stellenbosch, South Africa
Email address: sjselkirk@sun.ac.za