# ALTERNATING OFFERS BARGAINING AND THE GOLDEN RATIO

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Abstract. I augment Rubinstein's alternating offers bargaining game by adding to it a preliminary stage that is played to determine the identity of the first proposer. When the players become infinitely patient, the golden ratio emerges as a key component in the game's equilibrium.

## 1. Introduction

Rubinstein's [7] bargaining game is the central piece in the game theoretic literature on strategic bargaining. In this game, two players alternate roles as proposer and responder, and they exchange proposals on the division of a unit surplus (a dollar, a pie) until a proposal is accepted.[1] That is, player 1 makes an offer regarding how to split the surplus, to which player 2 responds by acceptance or rejection; if he accepts, then the offer is implemented and the game ends, and if he rejects, then there is a one-period delay followed by player 2 making an offer, to which player 1 needs to respond; the roles keep on alternating in this fashion as long as no agreement is reached. Suppose that each player discounts the future by a factor $\delta \in (0,1)$; that is, receiving a "pie slice" of size $x$ after $t$ periods of bargaining has the same utility as receiving $\delta^{t-1}x$ right away. Rubinstein showed that this game has a unique (subgame perfect) equilibrium; in this equilibrium there is an immediate agreement on the division $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$, where the larger share goes to the first proposer. However, who is the first proposer — and hence, who enjoys the upper hand — is not part of the analysis: it is assumed exogenously that the first proposer is "player 1."

Several papers [2,3,6] have addressed this issue in the following way: instead of considering the exogenous sequence of proposers $(1,2,1,2,1,\ldots)$, they assumed that in every period (prior to which no agreement has been reached) the players play a periodic game to determine that period's proposer. The advantage of this approach is that it does not impose a first proposer, but lets this role be determined endogenously, in equilibrium. The disadvantage is that the alternating offers structure is lost — any sequence of proposer identities can result via some play of the game — and this is a shortcoming because alternating offers is an intuitive model of back-and-forth negotiations. Below, I modify Rubinstein's game to determine endogenously *only* the first proposer, not the proposer in every subsequent period. This achieves the goal of not imposing a prespecified player with an upper hand, while keeping the alternating offers feature intact. The game I propose has a unique symmetric stationary equilibrium; the golden ratio appears naturally in this equilibrium. The golden ratio also appears in several earlier works on game theoretic bargaining. To make the comparison with my model more effective, I will discuss these works in the Conclusion section, after my model and result have been presented.

[1] If no proposal is ever accepted, the game results in perpetual disagreement.

## 2. The Game

Each of two players announces, simultaneously, one of two messages, *me* or *you*; a player is allowed to chose his message at random, namely play a mixed strategy. If one player says *me* and the other says *you* then the former becomes the first proposer in Rubinstein's game; in this case, the players play the (unique subgame perfect) equilibrium of Rubinstein's game, which means that they agree to the split $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$, where the larger share goes to the proposer. If the players announce the same message, then no one obtains the position of the proposer, and there is a one-period delay after which the game re-starts. The interpretation is that saying *me* is asking (or demanding) to be the first proposer, whereas saying *you* means that a player agrees that his opponent be the first proposer, if he so demands; there has to be agreement on who proposes first, or else the game re-starts (at the cost of one-period delay). Denote this game by $G$.

The game $G$ has multiple equilibria. For example, the strategy under which player $i$ always announces *me* and $j$ announces *you* is such an equilibrium, for either $(i, j) = (1, 2)$ or $(i, j) = (2, 1)$. However, I am interested in stationary (history-independent) equilibria that are symmetric. This is the subject of the next section.

## 3. The Result

Fix a symmetric stationary equilibrium, and let $V$ denote its value—namely, a player's equilibrium expected utility from starting to play this game (by stationarity, the payoff is the same after every history). Given this equilibrium and its value $V$, the proposer-selection phase is described by the following game table:

| Player 1\ Player 2 | *me* | *you* |
|---|---|---|
| *me* | $\delta V, \delta V$ | $\frac{1}{1+\delta}, \frac{\delta}{1+\delta}$ |
| *you* | $\frac{\delta}{1+\delta}, \frac{1}{1+\delta}$ | $\delta V, \delta V$ |

Let $p$ be the probability that a player attaches to *me*; it is straightforward that $p \in (0, 1)$, which means that a player is indifferent between the two messages. Therefore, the value, which is obtained by announcing any of the messages, satisfies the following equations:

$$V = p\delta V + \frac{1-p}{1+\delta}, \tag{3.1}$$

and

$$V = \frac{p\delta}{1+\delta} + (1-p)\delta V. \tag{3.2}$$

Summing them up gives $2V = \delta V + \frac{1-p+p\delta}{1+\delta}$, hence

$$V = \frac{1-p+p\delta}{(2-\delta)(1+\delta)}. \tag{3.3}$$

On the other hand, rearranging (3.1) gives:

$$V = \frac{1-p}{(1-p\delta)(1+\delta)}. \tag{3.4}$$

Equating the two expressions for $V$ in (3.3) and (3.4) results in the quadratic equation:

$$\delta p^2 + p - 1 = 0,$$

whose unique solution in $(0,1)$ is $p(\delta) \equiv \frac{-1+\sqrt{1+4\delta}}{2\delta}$. It follows, in particular, that the game has a unique symmetric stationary equilibrium. Note that $p(\delta) \to \varphi - 1 = \varphi^{-1}$ as $\delta \to 1$, where $\varphi$ is the golden ratio.

This is summarized in the following proposition.

**Proposition 3.1.** *The game $G$ has a unique symmetric stationary equilibrium. In this equilibrium, the probability that a player asks to be the first proposer converges to the inverse of the golden ratio, $\varphi^{-1}$, as the players become infinitely patient, namely as $\delta \to 1$.*

**Remark.** Even though the punch line of the result is obtained in the limit, as $\delta \to 1$, to obtain the result, it is important that the derivation be carried out for $\delta$'s strictly smaller than 1, and only then limit-taking be applied. If one imposes $\delta = 1$ directly, then the analysis does not pin down a unique probability $p$. Specifically, when $\delta = 1$, any $p \in (0,1)$ is consistent with a game table that has all its payoff entries equal to $\frac{1}{2}$.

## 4. Conclusion

I have introduced a modification of Rubinstein's bargaining game, which aims at endogenezing the (superior) position of the first proposer. The game has a unique symmetric stationary equilibrium. As the players become infinitely patient, the equilibrium-probability that a player asks to be the first proposer converges to the inverse of the golden ratio.

The golden ratio appears in several earlier works on game theoretic bargaining. In [5], the alternating offers game is studied under different preferences: the players' preferences in that paper are history-dependent, a modeling assumption that is made to express reciprocity in bargaining. As the players become infinitely patient, the first responder's pie-share converges to the ratio of the proposer's share to the responder's share; that is, denoting the responder's share by $s$, the following equation is obtained in the limit, as the players become infinitely patient: $s = \frac{1-s}{s}$. Thus, $s = \varphi^{-1}$.[2] Interestingly, the probability of asking to be the first proposer in the "symmetrized version" of Rubinstein's game with standard preferences coincides with the pie-share of the responder in the "reciprocity version" of that game.

An earlier paper that studied the alternating offers game with history-dependent preferences, and in which the golden ratio appears, is [4]. The goal in that paper is to explain how delay and gradualism can come about in bargaining, even in environments with complete information. In that paper, a sufficient condition for delay in equilibrium is that the discount factor exceeds $\varphi^{-1}$.

Common features to [5] and the current paper are that the equilibrium involves an immediate agreement (i.e., no delay) and both results are obtained in the limit, as the players become infinitely patient; in [4], by contrast, the equilibrium involves delay and the result is obtained provided that the discount factor exceeds some cutoff, but there is no requirement that it converges to one. A common feature to [4] and the current paper is that the equilibrium division converges to $(\frac{1}{2}, \frac{1}{2})$ as $\delta \to 1$; in [5], by contrast, the division converges to $(1 - \varphi^{-1}, \varphi^{-1})$. A more recent bargaining paper involving the golden ratio is [1]. However, as opposed to [4], [5], and the current paper, [1] does not consider two-player alternating offers bargaining; instead, it considers games with at least four players that are similar to stopping games.

To summarize, I have studied a model that contributes to a small literature on bargaining games in the solution of which the golden ratio appears as a key element. Further investigation of how and when the golden ratio presents itself in other game theoretic models — specifically, in dynamic (or repeated) $2 \times 2$ games — is a topic for future research.

[2] Note that, as opposed to the original Rubinstein game, in [5] there is a first proposer disadvantage.

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