FIBONACCI GRAPHS

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Abstract. By considering the Fibonacci numbers combinatorially, as counting the number of tilings of a strip of blocks with squares and dominoes, we introduce a graph that represents the sequence of Fibonacci numbers. Additionally, we consider individual graphs representing each Fibonacci number. Finally, we consider the graphic structure of these Fibonacci graphs and show how certain graphic properties relate to some well-known identities of the Fibonacci numbers.

1. Introduction

Having been studied for centuries, the Fibonacci numbers are one of the better-known integer sequences. Countless books and articles have been published about this number sequence and its many different generalizations, [4, 1, 5] for example. In an attempt to understand the many different properties this sequence possesses, people have studied them using a variety of approaches and techniques, including matrix methods, analysis, and combinatorial arguments. Despite this long history, the idea of using a graph to represent the sequence has not been explored.

We attempt to rectify this situation by introducing an infinite graph that represents the entire Fibonacci sequence. In addition, we introduce a collection of graphs, where each graph represents a Fibonacci number. After introducing these graphs, and showing how they relate to the Fibonacci numbers, we next consider the relationship between properties of the graphs and the corresponding properties of the Fibonacci sequence. Although we do not use these graphs to obtain any additional identities for the Fibonacci sequence, we hope that they can be used in the future to determine new identities, or new combinatorial proofs for already known identities.

2. The Fibonacci Graphs

In addition to the recursive definition of the Fibonacci numbers

$$F_0 = 0,\ F_1 = 1,\ \text{and } F_n = F_{n-1} + F_{n-2} \text{ for } n > 1,$$

there are a variety of additional equivalent definitions for the Fibonacci numbers. One such definition of interest will be the combinatorial definition involving coverings by squares and dominoes.

Let f_n be the number of ways to tile a strip of n blocks with squares and dominoes. It is a standard exercise to show that $f_n = F_{n+1}$. For example, for $n = 5$, the set of all tilings of a 5-strip is $\{s^5, s^3d, s^2ds, sds^2, ds^3, sd^2, dsd, d^2s\}$, where s denotes a square tile and d denotes a domino. Notice that there are eight such tilings and $F_6 = 8$. See [2] for explanations regarding this particular definition of f_n.

To construct the infinite graph Fib, we will use the collection of all such tilings as the set of vertices. We let $\mathcal{W}$ denote the set of all words, including the empty word (1), in symbols s and d:

$$\mathcal{W} = \{1, s, d, s^2, s^3, sd, ds, d^2, s^4, s^2d, sds, ds^2, \ldots\}.$$

Considering the elements of $\mathcal{W}$ as tilings of strips of blocks, it is natural for each s in the word to contribute one to the length and each d to contribute two to the length of the word. With this in mind, we define:

Definition 2.1. *Given $w \in \mathcal{W}$, the* **length** *of w, $|w|$, is defined to be $|w| = n_s(w) + 2n_d(w)$, where $n_s(w)$ is the number of times s appears in w and $n_d(w)$ is the number of times d appears in w.*

Additionally, there are some natural "moves" to translate one word in $\mathcal{W}$ into another word in $\mathcal{W}$:

Definition 2.2. *Given $w, w' \in \mathcal{W}$, we will say that w and w' are* **adjacent** *if and only if one of the following holds:*

(1) $w = w_1 s d w_2$ *and* $w' = w_1 d s w_2$ *with* $w_1, w_2 \in \mathcal{W}$ *(adjacent s and d switch order).*
(2) $w = w_1 d w_2$ *and* $w' = w_1 s^2 w_2$ *with* $w_1, w_2 \in \mathcal{W}$ *(a d is converted to s^2).*
(3) $|w| < |w'|$ *and* $w' = w w_1$ *for some* $w_1 \in \mathcal{W}$ *(w is a prefix of w').*
(4) $|w| < |w'|$, $w = w_1 s$ *and* $w' = w_1 d w_2$ *for some* $w_1, w_2 \in \mathcal{W}$ *(an s at the end of a word is switched to a d and is the prefix of the longer word).*

With this definition, we define the graph Fib as follows:

Definition 2.3. *The graph* Fib *consists of the vertex set $V = \mathcal{W}$ and edge set E, where two words in $\mathcal{W}$ will share an edge if they are adjacent.*

If e is an edge between u and v and $|u| = |v|$, then we will call e a *horizontal edge*. If e is an edge between u and v and $|u| < |v|$, then we will call e a *vertical edge*. The beginning of Fib is shown in *Figure 1* with horizontal edges shown as solid edges and vertical edges shown as dashed edges.

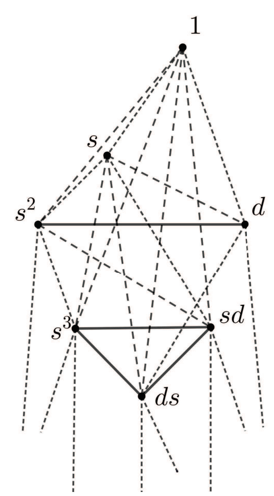


FIGURE 1. Beginning of Fib

We assume that the reader has an understanding of basic graph theory concepts. We recommend [3] by Chartrand and Lesniak for the basic definitions of graph theory. One notation that we will repeatedly use is that of an induced subgraph.

Definition 2.4. *If G is a graph and U is a subset of the vertices of G, then $G[U]$ will denote the subgraph of G induced by U.*

Certain induced subgraphs of Fib will be useful to consider, in particular, the subgraphs that represent each individual Fibonacci number. We will let $\mathcal{W}_n$ denote the set of all words of length n. We will denote the subgraph of Fib induced by the words of length n by $\text{Fib}_n = \text{Fib}[\mathcal{W}_n]$. Some examples of Fib_n for various values on n are shown in *Figure 2*. It will also be convenient to consider Fib restricted to words of a few different lengths. For this reason, we will use $\text{Fib}_{m,n}$ to denoted the graph $\text{Fib}[\mathcal{W}_m \cup \mathcal{W}_n]$. Finally, we will denote the subgraph of Fib induced by words of length at most n by $\text{Fib}_{[n]}$.

3. Properties of the Fibonacci Graphs

We first consider the number of vertices in each Fib_n. Clearly, $|V(\text{Fib}_0)| = 1$ and $|V(\text{Fib}_1)| = 1$. For $n \geq 2$, we partition $\mathcal{W}_n$ into two subsets: $V_1 = \{ws \mid w \in \mathcal{W}_{n-1}\}$ and $V_2 = \{wd \mid w \in \mathcal{W}_{n-2}\}$. Clearly, $V(\text{Fib}_n) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. Therefore, $|V(\text{Fib}_n)| = |V_1| + |V_2|$. Using an inductive argument, it is also clear that $|V_1| = f_{n-1}$ and $|V_2| = f_{n-2}$. Thus, $|V(\text{Fib}_n)| = f_n$. This justifies the name of the graphs Fib and Fib_n.

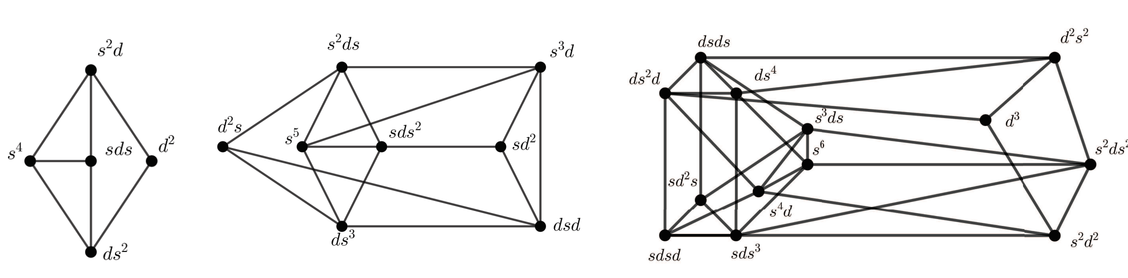
We next consider some additional properties of the Fibonacci graphs Fib and Fib_n. Most of these properties reflect a similar property for the Fibonacci numbers. The combinatorial proof of these properties for the Fibonacci numbers is the idea that underlies the proof for these graphic properties. The combinatorial proof for these properties of Fibonacci numbers can be found in [2]. Where possible, we will indicate the identity number from [2] that corresponds to the graphic property. Before considering these properties, we need a few additional graph theoretical definitions.

It is obvious that $\text{Fib}_n[V_1] \cong \text{Fib}_{n-1}$ and $\text{Fib}_n[V_2] \cong \text{Fib}_{n-2}$. Additionally, we need the following two definitions.

Definition 3.1. *Suppose G and H are graphs with vertex sets V and U, respectively. We define* **the product** $G \square H$ *to be the graph with vertex set $V \times U$ and has (v, u) adjacent to (v', u') if and only if $v = v'$ and u adjacent to u' in H or $u = u'$ and v adjacent to v' in G.*

Definition 3.2. *Suppose $G, H_1, H_2, \ldots, H_n$ are graphs. We will say that G has* **partition type** $\langle H_1, H_2, \ldots, H_n \rangle$*, $G \sim \langle H_1, H_2, \ldots, H_n \rangle$, if there exists a partition $(U_1, U_2, \ldots, U_n)$ of the vertex set of G such that for all $i = 1, \ldots, n$, $G[U_i] \cong H_i$.*

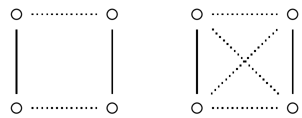
As the partition type of a graph has not been previously defined, we give an example of this concept:



(a) Fib_4 (b) Fib_5 (c) Fib_6

Figure 2. Fib_n

Example 3.3. *Both C_4 and K_4 have partition type $\langle P_2, P_2 \rangle$:*



3.1. **Fibonacci Identities.** Many of the structural properties of Fib_n and the other various Fibonacci graphs arise from identities involving the Fibonacci numbers f_n. The combinatorial proofs of these identities are often the basis for the proofs of the structural properties. To emphasize this relationship, we note the Fibonacci identities that we will mimic to determine the properties of the Fibonacci graphs. Each of these identities – with either their proofs or hints regarding their proofs – can be found in [2].

We now mention each identity, with its identity number from [2].

Proposition 3.4 (Identity 3)**.** *For $m, n \geq 0$, $f_{m+n} = f_m f_n + f_{m-1} f_{n-1}$.*

Proposition 3.5 (Identity 13)**.** *For $n \geq 0$, $f_n^2 + f_{n+1}^2 = f_{2n+2}$.*

Proposition 3.6 (Identity 1)**.** *For $n \geq 0$, $f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.*

Proposition 3.7 (Identity 2)**.** *For $n \geq 0$, $f_0 + f_2 + f_4 + \cdots + f_{2n} = f_{2n+1}$.*

Proposition 3.8 (Identity 7)**.** *For $n \geq 1$, $3f_n = f_{n+2} + f_{n-2}$.*

Proposition 3.9 (Identity 6)**.** *For $n \geq 0$, $f_{2n-1} = \sum_{k=1}^{n} \binom{n}{k} f_{k-1}$.*

Proposition 3.10 (Identity 9)**.** *For $n \geq 0$, $\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}$.*

Proposition 3.11 (Identity 15)**.** *For $n \geq 0$, $f_{2n+2} = f_{n+1} f_{n+2} - f_{n-1} f_n$.*

We now consider properties of Fib, Fib_n, and related graphs. We begin with properties of Fib_n.

3.2. **Properties of** Fib_n**.** The first thing we consider is the number of edges contained in Fib_n.

If we let E_n be the number of edges in Fib_n, then $E_0 = 0$, $E_1 = 0$, and $E_2 = 1$. Considering V_1 and V_2 as before, $E_n = E_{n-1} + E_{n-2} + B$, where B is the number of edges between V_1 and V_2. *Figure 3* shows an example of Fib_5 with the edges in B indicated with dashed lines.

Suppose $u \in V_1$. Then, $u = w_1 s^2$ or $u = w_1 ds$. If $u = w_1 s^2$, then there is an edge between u and $w_1 d \in V_2$. Similarly, if $u = w_1 ds$, then there is an edge between u and $w_1 sd \in V_2$. Thus, each vertex $u \in V_1$ has exactly one edge to a vertex in V_2. Therefore, $B = |V_1| = F_{n-1}$.

By combining these facts, with standard solution methods for solving recursive equations, it is straightforward to prove the following results about the number of edges in Fib_n:

Theorem 3.12. *Suppose E_n is the number of edges in Fib_n with $n \geq 0$. Then,*

(1) *$E_0 = 0$, $E_1 = 0$, and for $n \geq 2$, $E_n = E_{n-1} + E_{n-2} + f_{n-1}$.*

(2)
$$E_n = \sum_{i=0}^{n-2} f_i \cdot f_{n-1-i}.$$

(3) *If $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, then*
$$E_n = \frac{(n(5+\sqrt{5}) - 6)\alpha^n - (n(5-\sqrt{5}) - 6)\beta^n}{10\sqrt{5}}.$$

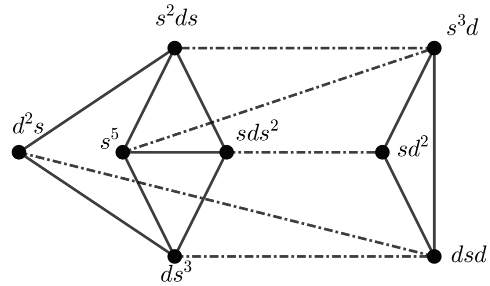


FIGURE 3. Fib_5 with edges in B indicated

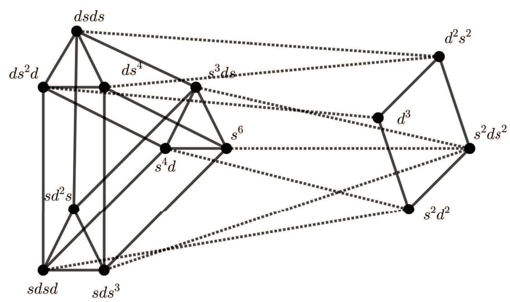


FIGURE 4. Fib_6 has partition type $\langle \mathrm{Fib}_3 \,\Box\, \mathrm{Fib}_3, \mathrm{Fib}_2 \,\Box\, \mathrm{Fib}_2 \rangle$

From the work counting the number of edges in Fib_n, we have already proven the following:

Lemma 3.13. *For $n > 0$, $\mathrm{Fib}_n \sim \langle \mathrm{Fib}_{n-1}, \mathrm{Fib}_{n-2} \rangle$.*

Next, we consider some additional properties of the Fibonacci graphs Fib_n. As appropriate, we will indicate the Proposition that is the basis of each property after the statement of each lemma.

Lemma 3.14. *For $n > 0$ and $0 < k < n$, $\mathrm{Fib}_n \sim \langle \mathrm{Fib}_k \,\Box\, \mathrm{Fib}_{n-k}, \mathrm{Fib}_{k-1} \,\Box\, \mathrm{Fib}_{n-k-1} \rangle$. [Proposition 3.4]*

Proof. We partition $\mathcal{W}_n$ into two subsets:

$$A = \{w \in \mathcal{W}_n \mid w = w_1 \cdot w_2 \text{ with } w_1 \in \mathcal{W}_k \text{ and } w_2 \in \mathcal{W}_{n-k}\}$$
$$B = \{w \in \mathcal{W}_n \mid w = w_1 \cdot d \cdot w_2 \text{ with } w_1 \in \mathcal{W}_{k-1} \text{ and } w_2 \in \mathcal{W}_{n-k-1}\}$$

Clearly, $\mathcal{W}_n = A \cup B$. The result will be proven if $\mathrm{Fib}_n[A] \cong \mathrm{Fib}_k \,\Box\, \mathrm{Fib}_{n-k}$ and if $\mathrm{Fib}_n[B] \cong \mathrm{Fib}_{k-1} \,\Box\, \mathrm{Fib}_{n-k-1}$.

Suppose $w, v \in A$ share an edge. If $w = w_1 \cdot w_2$ and $v = v_1 \cdot v_2$, then this implies that either $w_1 = v_1$ and w_2 and v_2 are adjacent or $w_2 = v_2$ and w_1 and v_1 are adjacent. This implies that $\mathrm{Fib}_n[A] \cong \mathrm{Fib}_k \,\Box\, \mathrm{Fib}_{n-k}$. Similarly, $\mathrm{Fib}_n[B] \cong \mathrm{Fib}_{k-1} \,\Box\, \mathrm{Fib}_{n-k-1}$. Thus, $\mathrm{Fib}_n \sim \langle \mathrm{Fib}_k \,\Box\, \mathrm{Fib}_{n-k}, \mathrm{Fib}_{k-1} \,\Box\, \mathrm{Fib}_{n-k-1} \rangle$. □

Figure 4 shows Fib_6 with the given partition type exhibited. As a quick application of Lemma 3.14, if the length of the word is $2n + 2$ and $k = n + 1$, then we obtain the following corollary:

Corollary 3.15. *For $n \geq 0$,* $\mathrm{Fib}_{2n+2} \sim \langle \mathrm{Fib}_n \,\Box\, \mathrm{Fib}_n, \mathrm{Fib}_{n+1} \,\Box\, \mathrm{Fib}_{n+1} \rangle$. *[Proposition 3.5]*

Next, we consider the partition type of Fib_{n+2} for $n \geq 0$.

Proposition 3.16. *For $n \geq 0$, $Fib_{n+2} \sim \langle K_1, \mathrm{Fib}_0, \mathrm{Fib}_1, \mathrm{Fib}_2, \ldots, \mathrm{Fib}_n \rangle$. [Proposition 3.6]*

Proof. We partition $\mathcal{W}_{n+2}$ into subsets depending on the location of the final d. We let $A_{-1} = \{s^{n+2}\}$ and $A_i = \{wds^{n-i} \mid w \in \mathcal{W}_i\}$ for $i = 0, 1, \ldots, n$. Clearly, $\mathcal{W}_{n+2} = \bigcup_{i=-1}^{n} A_i$. It is clear that $\mathrm{Fib}[A_{-1}] \cong K_1$. Similarly, by the definition of A_i for $0 \leq i \leq n$, $\mathrm{Fib}[A_i] \cong \mathrm{Fib}[\mathcal{W}_i] \cong \mathrm{Fib}_i$. Combining these facts, $\mathrm{Fib}_n \sim \langle K_1, \mathrm{Fib}_0, \mathrm{Fib}_1, \ldots, \mathrm{Fib}_n \rangle$. □

From the last property of this section, we consider the partition type of Fib_n for odd n.

Proposition 3.17. *For $n \geq 0$,* $\mathrm{Fib}_{2n+1} \sim \langle K_1, \mathrm{Fib}_2, \mathrm{Fib}_4, \ldots, \mathrm{Fib}_{2n} \rangle$. *[Propsition 3.7]*

Proof. Suppose $n \geq 0$. Since $2n+1$ is odd, every word in $\mathcal{W}_{2n+1}$ must contain at least one s. We partition $\mathcal{W}_{2n+1}$ into subsets depending on the location of the last s that occurs in the word. The subword before this occurrence of s must have an even length, and the subword after this occurrence of s must consist of a power of d. We let $B_{2i} = \{wsd^{n-i} \mid w \in \mathcal{W}_{2i}\}$ for $i = 0, 1, 2, \ldots, n$. Clearly, $\mathrm{Fib}_{2n+1} \sim \langle \mathrm{Fib}[B_0], \mathrm{Fib}[B_2], \ldots, \mathrm{Fib}[B_{2n}] \rangle$. Additionally, $\mathrm{Fib}[B_{2i}] \cong \mathrm{Fib}[\mathcal{W}_{2i}] \cong \mathrm{Fib}_{2i}$, thus proving the result. □

3.3. **Properties of** Fib**.** We now consider properties of the graph Fib. We start by considering an infinite graph that occurs as an induced subgraph of Fib.

Lemma 3.18. Fib *contains an infinite number of copies of* Fib *as induced subgraphs.*

Proof. Suppose $w \in \mathcal{W}$. Define $\mathcal{V}_w = \{ww' \mid w' \in \mathcal{W}\}$. Consider the induced subgraph $\mathrm{Fib}[\mathcal{V}_w]$. Clearly, $\mathrm{Fib}[\mathcal{V}_w] \cong \mathrm{Fib}$. Therefore, because $\mathcal{W}$ is infinite, Fib contains an infinite number of induced subgraphs isomorphic to Fib itself. □

Lemma 3.19. $\mathrm{Fib}_n \,\Box\, \mathrm{Fib}$ *is an induced subgraph of* Fib *for every $n \geq 0$.*

Proof. Let $n > 0$ be given. Similar to the proof of Lemma 3.14, we define $A = \{w \in \mathcal{W} \mid w = w_1 \cdot w_2$ with $w_1 \in \mathcal{W}_n\}$. We show that $\mathrm{Fib}[A] \cong \mathrm{Fib}_n \,\Box\, \mathrm{Fib}$.

Suppose $v, w \in \mathrm{Fib}[A]$ with v and w adjacent. Since $v = v1v2$, $v_1 \in \mathcal{W}_n$ and $w = w_1w_2$, $w_1 \in \mathcal{W}_n$, either v_1 is adjacent to w_1 in $\mathcal{W}_n$ with $v_2 = w_2$, or $v_1 = w_1$ and v_2 is adjacent to w_2 in $\mathcal{W}$. Thus, $\mathrm{Fib}[A] \cong \mathrm{Fib}_n \,\Box\, \mathrm{Fib}$. □

Proposition 3.20. *For every $n \geq 0$,* $\mathrm{Fib} \sim \langle \mathrm{Fib}_{[n-1]}, \mathrm{Fib}_n \,\Box\, \mathrm{Fib}, \mathrm{Fib}_{n-1} \,\Box\, \mathrm{Fib} \rangle$.

Proof. Let $n \geq 0$. We partition $\mathcal{W}$ into 3 sets:

$$\begin{aligned} A &= \{w \in \mathcal{W} \mid \ |w| \leq n-1\} \\ B &= \{w \in \mathcal{W} \mid w = w_1 \cdot w_2 \text{ with } w_1 \in \mathcal{W}_n\} \\ C &= \{w \in \mathcal{W} \mid w = w_1 d w_2 \text{ with } w_1 \in \mathcal{W}_{n-1}\}. \end{aligned}$$

Using the previous proof, $\mathrm{Fib}[N] \cong \mathrm{Fib}_n \,\Box\, \mathrm{Fib}$, $\mathrm{Fib}[C] \cong \mathrm{Fib}_{n-1} \,\Box\, \mathrm{Fib}$, and $\mathrm{Fib}[A] \cong \mathrm{Fib}_{[n-1}$ by definition. Thus, $\mathrm{Fib} \sim \langle \mathrm{Fib}_{[n-1]}, \mathrm{Fib}_n \,\Box\, \mathrm{Fib}, \mathrm{Fib}_{n-1} \,\Box\, \mathrm{Fib} \rangle$. □

3.4. **Properties of** $\mathrm{Fib}_{m,n}$**.** Next, we consider the structure of Fib over a few levels together. We begin by considering when $n - m = 4$.

Proposition 3.21. *For $n \geq 2$,* $\mathrm{Fib}_{n-2,n+2} \sim \langle \mathrm{Fib}_n, \mathrm{Fib}_n, \mathrm{Fib}_n \rangle$. *[Proposition 3.8]*

Proof. We partition $\mathcal{W}_{n-2} \cup \mathcal{W}_{n+2}$ into three subsets:

$$\begin{aligned} V_1 &= \{w \in \mathcal{W}_{n+2} \mid w = w_1 s^2\} \\ V_2 &= \{w \in \mathcal{W}_{n+2} \mid w = w_1 d\} \\ V_3 &= \{w \in \mathcal{W}_{n+2} \mid w = w_1 ds\} \cup \mathcal{W}_{n-2} \end{aligned}$$

Clearly, $\mathrm{Fib}[V_1] \cong \mathrm{Fib}_n$ and $\mathrm{Fib}[V_2] \cong \mathrm{Fib}_n$. All that remains is to show that $\mathrm{Fib}[V_3] \cong \mathrm{Fib}_n$. Clearly, $\mathrm{Fib}[V_3]$ has f_n vertices. We associate each element of V_3 with a word of length n and then we will show that two words are adjacent in $\mathrm{Fib}_{n-2,n+2}[V_3]$ if and only if their corresponding words are adjacent in Fib_n. We associate $w \in \mathcal{W}_{n-2}$ with $wd \in \mathcal{W}_n$ and $wds \in \mathcal{W}_{n+2}$ with $ws \in \mathcal{W}_n$.

$$\begin{array}{ccc} w_1 \in \mathcal{W}_{n-2} & \mathrm{Fib}[V_3] & w_2 ds \in \mathcal{W}_{n+2} \\ \updownarrow & & \updownarrow \\ w_1 d \in \mathcal{W}_n & \mathrm{Fib}_n & w_2 s \in \mathcal{W}_n \end{array}$$

We consider the nonobvious cases.

Assume that $w_1 d$ is adjacent to $w_2 s$ in Fib_n. This implies that either $w_2 = w_1 s$ or that $w_1 = w_1' s$ and $w_2 = w_1' d$. For each case, we have the following diagrams:

$$\begin{array}{ccccccc} w_1 & \cdots\cdots & w_1 sds & \mathrm{Fib}[V_3] & w_1' s & \cdots\cdots & w_1' d^2 s \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ w_1 d & \sim\sim & w_1 s^2 & \mathrm{Fib}_n & w_1' sd & \sim\sim & w_1' ds \end{array}$$

In either of these cases, from the definition of Fib, w_1 is adjacent to $w_2 ds$.

Now, assume that w_1 is adjacent to $w_2 ds$ in $\mathrm{Fib}[V_3]$. Since $|w_1| = n-2$ and $|w_2 ds| = n+2$, this implies that $|w_2| = |w_1| + 1$. We consider the cases. From the definition of Fib, either $w_2 = w_1 s$ or $w_1 = w_1' s$ and $w_2 = w_1' d$. In either case, we have the following diagrams:

$$\begin{array}{ccccccc} w_1 & \sim\sim & w_1 sds & \mathrm{Fib}[V_3] & w_1' s & \sim\sim & w_1' d^2 s \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ w_1 d & \cdots\cdots & w_1 s^2 & \mathrm{Fib}_n & w_1' sd & \cdots\cdots & w_1' ds \end{array}$$

In both of these cases, $w_1 d$ is adjacent to $w_2 s$ in Fib_n. Thus, $\mathrm{Fib}[V_3] \cong \mathrm{Fib}_n$. Therefore, $\mathrm{Fib}_{n-2,n+2} \sim \langle \mathrm{Fib}_n, \mathrm{Fib}_n, \mathrm{Fib}_n \rangle$. □

Finally, we consider the structure of $\mathrm{Fib}_{m,n}$ with $n - m = 3$.

Proposition 3.22. *For $n \geq 2$, $\mathrm{Fib}_{n-2,n+1} \sim \langle \mathrm{Fib}_n, \mathrm{Fib}_n \rangle$. [Proposition 3.9]*

Proof. As above, we partition $\mathcal{W}_{n-2} \cup \mathcal{W}_{n+1}$ into two subsets. Let $V_1 = \{ws \mid ws \in \mathcal{W}_{n+1}\}$ and $V_2 = \{wd \mid wd \in \mathcal{W}_{n+1}\} \cup \mathcal{W}_{n-2}$. It is clear that $\mathrm{Fib}[V_1] \cong \mathrm{Fib}_n$, so all that remains is to show that $\mathrm{Fib}[V_2] \cong \mathrm{Fib}_n$ as well. As before, we will define an association between each element in V_2 and a word in $\mathcal{W}_n$. We associate $w_1 \in \mathcal{W}_{n-2}$ with $w_1 d \in \mathcal{W}_n$ and $w_2 d \in \mathcal{W}_{n+1}$ with $w_2 s \in \mathcal{W}_n$.

$$\begin{array}{ccc} w_1 \in \mathcal{W}_{n-2} & \mathrm{Fib}[V_2] & w_2 d \in \mathcal{W}_{n+1} \\ \updownarrow & & \updownarrow \\ w_1 d \in \mathcal{W}_n & \mathrm{Fib}_n & w_2 s \in \mathcal{W}_n \end{array}$$

As in the previous proof, we need to show that w_1 and w_2d are adjacent in Fib if and only if w_1d and w_2s are adjacent in Fib_n. Again, we consider the nonobvious cases. Suppose first that w_1d and w_2s are adjacent in Fib_n. This can occur if $w_2 = w_1s$, or if $w_1 = w_1's$ and $w_2 = w_1'd$. Considering the diagrams:

$$\begin{array}{ccccccc} w_1 & \cdots\cdots & w_1sd & \qquad \mathrm{Fib}[V_2] \qquad & w_1's & \cdots\cdots & w_1'd^2 \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ w_1d & \sim\sim\sim & w_1s^2 & \qquad \mathrm{Fib}_n \qquad & w_1'sd & \sim\sim\sim & w_1'ds \end{array}$$

In either case, w_1 and w_2d are adjacent in $\mathrm{Fib}[V_2]$.

Now, suppose w_1 and w_2d are adjacent in $\mathrm{Fib}[V_2]$. This implies, since $|w_1| = |w_2d| - 3$, that $|w_1| = |w_2| - 1$. Either $w_2 = w_1s$ or $w_1 = w_1's$ and $w_2 = w_1'd$. For either case, we consider the diagrams:

$$\begin{array}{ccccccc} w_1 & \sim\sim\sim & w_1sd & \qquad \mathrm{Fib}[V_2] \qquad & w_1's & \sim\sim\sim & w_1'd^2 \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ w_1d & \cdots\cdots & w_1s^2 & \qquad \mathrm{Fib}_n \qquad & w_1'sd & \cdots\cdots & w_1'ds \end{array}$$

In either case, w_1d is adjacent to w_2s in Fib_n. Thus, $\mathrm{Fib}[V_2] \cong \mathrm{Fib}_n$. This implies that $\mathrm{Fib}_{n-2,n+1} \sim \langle \mathrm{Fib}_n, \mathrm{Fib}_n \rangle$. □

3.5. **Properties of** $\mathrm{Fib}_n \square \mathrm{Fib}_m$**.** We finish with the properties of the product of two Fibonacci graphs. For these products, some of the properties that hold for the Fibonacci sequence no longer hold for the graphs because the graphic structure is not preserved.

When considering the structure of $\mathrm{Fib}_n \square \mathrm{Fib}_m$, it will be helpful to introduce the idea of a common fault between two words $u \in \mathcal{W}_n$ and $v \in \mathcal{W}_m$.

Definition 3.23. *Given $u \in \mathcal{W}_n$ and $v \in \mathcal{W}_m$, u and v will be said to have a* **common fault at position** *i if $u = u_1u_2$ and $v = v_1v_2$ with $u_1, v_1 = \mathcal{W}_i$.*

Proposition 3.24. *For $n > 0$, $\mathrm{Fib}_n \square \mathrm{Fib}_{n+1} \sim \langle \mathrm{Fib}_0 \square \mathrm{Fib}_0, \mathrm{Fib}_1 \square \mathrm{Fib}_1, \ldots, \mathrm{Fib}_n \square \mathrm{Fib}_n \rangle$. [Proposition 3.10]*

Proof. Since for $w \in \mathcal{W}_n$ and $w' \in \mathcal{W}_{n+1}$, $w = 1w$ and $w' = 1w'$, they both have a common fault at position 0. Therefore, every pair of words shares at least one common fault. We group pairs of words depending on the location of the final common fault. For $k = 0, \ldots, n$, define

$$B_k = \{(w, w') \in \mathcal{W}_n \times \mathcal{W}_{n+1} \mid w \text{ and } w' \text{ have a final common fault at position } k\}.$$

Clearly, $\mathcal{W}_n \cup \mathcal{W}_{n+1} = \bigcup_{k=0}^{n} B_k$.

Suppose $(w, w') \in B_k$. Without loss of generality, assume $n - k$ is even. This implies that $w = w_1d^{(n-k)/2}$ and $w' = w_1'sd^{(n-k)/2}$ for $w_1, w_1' \in \mathcal{W}_k$. Thus,

$$(\mathrm{Fib}_n \square \mathrm{Fib}_{n+1})[B_k] = \mathrm{Fib}_n[\mathcal{W}_k] \square \mathrm{Fib}_{n+1}[\mathcal{W}_k] \cong \mathrm{Fib}_k \square \mathrm{Fib}_k .$$

Combining these facts,

$$\mathrm{Fib}_n \square \mathrm{Fib}_{n+1} \sim \langle \mathrm{Fib}_0 \square \mathrm{Fib}_0, \mathrm{Fib}_1 \square \mathrm{Fib}_1, \ldots, \mathrm{Fib}_n \square \mathrm{Fib}_n \rangle.$$

□

Our final proposition does not translate exactly from the identity on which it is based:

Proposition 3.25. *For $n \geq 0$, $\mathrm{Fib}_{n+1} \square \mathrm{Fib}_{n+2} \sim$ $\langle \mathrm{Fib}_n \square \mathrm{Fib}_{n+2}, \mathrm{Fib}_{n-1} \square \mathrm{Fib}_{n+1}, \mathrm{Fib}_{n-1} \square \mathrm{Fib}_n \rangle$. [Proposition 3.11]*

Proof. To prove this proposition, we will partition $\mathcal{W}_{n+1} \times \mathcal{W}_{n+2}$ into three sets and show the appropriate isomorphisms for the three induced subgraphs. Define the sets A, B, and C as follows:

$$\begin{aligned} A &= \{(w,w') \in \mathcal{W}_{n+1} \times \mathcal{W}_{n+2} \mid w = w_1 s\} \\ B &= \{(w,w') \in \mathcal{W}_{n+1} \times \mathcal{W}_{n+2} \mid w = w_2 d \text{ and } w' = w_1' s\} \\ C &= \{(w,w') \in \mathcal{W}_{n+1} \times \mathcal{W}_{n+2} \mid w = w_2 d \text{ and } w' = w_2' d\} \end{aligned}$$

Clearly, $w_1 \in \mathcal{W}_n$, $w_2 \in \mathcal{W}_{n-1}$, $w_1' \in \mathcal{W}_{n+1}$, and $w_2' \in \mathcal{W}_n$. Thus,

$$\begin{aligned} (\mathrm{Fib}_{n+1} \,\square\, \mathrm{Fib}_{n+2})[A] &= \mathrm{Fib}_{n+1}[\mathcal{W}_n] \square \mathrm{Fib}_{n+2}[\mathcal{W}_{n+2}] \cong \mathrm{Fib}_n \,\square\, \mathrm{Fib}_{n+2} \\ (\mathrm{Fib}_{n+1} \,\square\, \mathrm{Fib}_{n+2})[B] &= \mathrm{Fib}_{n+1}[\mathcal{W}_{n-1}] \square \mathrm{Fib}_{n+2}[\mathcal{W}_{n+1}] \cong \mathrm{Fib}_{n-1} \,\square\, \mathrm{Fib}_{n+1} \\ (\mathrm{Fib}_{n+1} \,\square\, \mathrm{Fib}_{n+2})[C] &= \mathrm{Fib}_{n+1}[\mathcal{W}_{n-1}] \square \mathrm{Fib}_{n+2}[\mathcal{W}_n] \cong \mathrm{Fib}_{n-1} \,\square\, \mathrm{Fib}_n \end{aligned}$$

This implies that

$$\mathrm{Fib}_{n+1} \,\square\, \mathrm{Fib}_{n+2} \sim \langle \mathrm{Fib}_n \,\square\, \mathrm{Fib}_{n+2}, \mathrm{Fib}_{n-1} \,\square\, \mathrm{Fib}_{n+1}, \mathrm{Fib}_{n-1} \,\square\, \mathrm{Fib}_n \rangle.$$

□

Proposition 3.11 (Identity 15 from [2]) states that $f_{n+1} f_{n+2} = f_{2n+2} + f_{n-1} f_n$. However, when considering the structure of $G = (\mathrm{Fib}_{n+1} \,\square\, \mathrm{Fib}_{n+2})[A \cup B]$, there are the correct number of vertices, but there are some words that are adjacent in Fib_{2n+2} that are not adjacent in G. This shows that the structure of Fib is more restrictive than the corresponding identities for the Fibonacci sequence (a result that is not surprising).

By adding a graphic structure to the sequence of Fibonacci numbers, we have been able to apply known Fibonacci identities to develop structural properties of these graphs. There is potential for this relationship to go in the other direction. Hopefully, structure properties of Fib and Fib_n will lead to new identities for the Fibonacci sequence.

Additionally, this technique can be applied to other number sequences that have a combinatorial interpretation. Applying this approach and relating graphic properties to sequence identities for a variety of other number sequences is the topic of a future paper.

References

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