THE PARAMETRIC PASCAL RHOMBUS

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Abstract. In this note, we first show a connection between the complete central coefficients of the Pascal rhombus and the row sums of the left-bounded rhombus. Then, we introduce the concept of parametric Pascal rhombus and left-bounded parametric rhombus, and give the Riordan array characterizations for them. Moreover, we use the reflection principle to prove a relationship between the entries of the parametric Pascal rhombus and the left-bounded parametric rhombus. By applying this result, we show that the complete central coefficients of the parametric Pascal rhombus are the same as the row sums of the left-bounded parametric rhombus. Finally, we present several examples to illustrate that numerus combinatorial sequences appear in parametric Pascal rhombus and left-bounded parametric rhombus.

1. Introduction

The Pascal rhombus, introduced in 1997 by Klostermeyer, et al. [8] as a generalization of the Pascal triangle, is an infinite array $\mathcal{R} = (r_{i,j})_{i\in\mathbb{N}, j\in\mathbb{Z}}$, where $r_{i,j}$ is defined by

$$\begin{cases} r_{i,j} &= r_{i-1,j-1} + r_{i-1,j} + r_{i-1,j+1} + r_{i-2,j},\ i \geq 2,\ j \in \mathbb{Z}, \\ r_{0,0} &= r_{1,-1} = r_{1,0} = r_{1,1} = 1, r_{0,j} = 0\,(j \neq 0),\ r_{1,j} = 0\,(j \neq -1, 0, 1). \end{cases} \tag{1.1}$$

The left-bounded rhombus $S = (s_{i,j})_{i,j\in\mathbb{N}}$ is an infinite lower triangular matrix, where $s_{i,j}$ is defined by the analogue rules

$$\begin{cases} s_{i,j} &= s_{i-1,j-1} + s_{i-1,j} + s_{i-1,j+1} + s_{i-2,j},\ i \geq 2,\ 0 \leq j \leq i, \\ s_{0,0} &= s_{1,0} = s_{1,1} = 1, s_{i,-1} = 0\,(i \geq 0),\ r_{i,j} = 0\,(i < j). \end{cases} \tag{1.2}$$

The first lines of the Pascal rhombus are given on the left of Figure 1, and the first lines of the left-bounded rhombus are given on the right.

						1						
					1	1	1					
				1	2	4	2	1				
			1	3	8	9	8	3	1			
		1	4	13	22	29	22	13	4	1		
	1	5	19	42	72	82	72	42	19	5	1	
1	6	26	70	146	218	255	218	146	70	26	6	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

1						
1	1					
3	2	1				
6	7	3	1			
16	18	12	4	1		
40	53	37	18	5	1	
109	148	120	64	25	6	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Figure 1. Pascal rhombus and left-bounded rhombus

Recently, Ramírez found a closed expression for the entries of the Pascal rhombus in [14], and Yang, et al. [20] established the connection between the Pascal rhombus and the Riordan array. For more results on the Pascal rhombus, the reader may consult [1, 7, 11, 18].

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The central coefficients of the Pascal rhombus are the entries, $(r_{n,0})_{n\geq 0}$, that form the central symmetric axis of the Pascal rhombus. This sequence appears in the OEIS [12] as sequence $A059345$. We call the sum of column 0 and column 1 of the Pascal rhombus the complete central coefficients, and denote it by $r(n)$, i.e., $r(n) = r_{n,0} + r_{n,1}$, $n \geq 0$.

In March 2019, the second author observed, empirically, a connection between $S(n) = \sum_{k=0}^{n} s_{n,k}$, the nth row sum of the left-bounded rhombus, and the complete central coefficients [3, 4], $r(n) = r_{n,0} + r_{n,1}$, that is $S(n) = r(n)$, as illustrated in Figure 1.

In this note, we first prove this connection by using generating functions. Then, we introduce the concept of parametric Pascal rhombus and left-bounded parametric rhombus, and give the Riordan array characterizations for them, and show that the complete central coefficients of the parametric Pascal rhombus are the same as the row sums of the left-bounded parametric rhombus. Finally, we present several examples to illustrate numerous combinatorial sequences appearing in parametric Pascal rhombus and left-bounded parametric rhombus.

Here, we briefly recall the concept of Riordan matrix. A Riordan array, originally introduced by Shapiro, et al. [16], is defined in terms of generating functions of its columns. An infinite lower triangular matrix $D = (d_{n,k})_{n,k\geq 0}$ is a Riordan array, if there exist generating functions $g(t)$ and $f(t)$ such that

$$d_{n,k} = [t^n]g(t)f(t)^k, \; n, k \in \mathbb{N}, \tag{1.3}$$

where $g(t)$ and $f(t)$ satisfy the conditions $g(0) = 1$, $f(0) = 0$, and $f'(0) \neq 0$. The Riordan array corresponding to the pair $g(t)$, $f(t)$ is denoted by $(g(t), f(t))$, whose kth column has the generating function $g(t)f(t)^k$.

The set of all Riordan arrays forms a group under ordinary row-by-column product with the multiplication identity $(1, t)$. The product of two Riordan arrays is given by

$$(d(t), h(t))(g(t), f(t)) = (d(t)g(h(t)), f(h(t))), \tag{1.4}$$

and the inverse of $(d(t), h(t))$ is the Riordan array

$$(d(t), h(t))^{-1} = (1/d(\bar{h}(t)), \bar{h}(t)), \tag{1.5}$$

where $\bar{h}(t)$ is compositional inverse of $h(t)$, i.e., $h(\bar{h}(t)) = \bar{h}(h(t)) = t$. Some of the main results on the Riordan group and its applications can be found in [10, 15, 21].

If $(b_n)_{b\in\mathbb{N}}$ is any sequence having $b(t) = \sum_{n=0}^{\infty} b_n t^n$ as its generating function, then for every Riordan array $(d(t), h(t)) = (g_{n,k})_{n,k\in\mathbb{N}}$,

$$\sum_{k=0}^{n} g_{n,k} b_k = [t^n]d(t)b(h(t)). \tag{1.6}$$

This is called the fundamental theorem of Riordan arrays [9, 16] and is rewritten as

$$(d(t), h(t))b(t) = d(t)b(h(t)). \tag{1.7}$$

In particular, the generating function of the row sums of the Riordan array $(d(t), h(t))$ is

$$S(t) = \frac{d(t)}{1 - h(t)}. \tag{1.8}$$

A characterization of Riordan arrays was established by Merlini, et al. [10] as follows.

Lemma 1.1. *A lower triangular array $(g_{n,k})_{n,k\in\mathbb{N}}$ is a Riordan array if and only if there exists another array $(\alpha_{i,j})_{i,j\in\mathbb{N}}$, with $\alpha_{0,0} \neq 0$, and s sequences $\{\rho_j^{[i]}\}_{j\in\mathbb{N}}$, $i = 1, 2, \ldots, s$, such that*

$$g_{n+1,k+1} = \sum_{i\geq 0}\sum_{j\geq 0} \alpha_{i,j} g_{n-i,k+j} + \sum_{i=1}^{s}\sum_{j\geq 0} \rho_j^{[i]} g_{n+i,k+i+j+1}. \tag{1.9}$$

The array $(\alpha_{i,j})_{i,j\in\mathbb{N}}$ in the previous lemma is called the A-matrix of the Riordan array $(d(t), h(t)) = (g_{n,k})_{n,k\in\mathbb{N}}$. If $\Phi^{[i]}(t)$ denotes the generating function of the ith row of the A-matrix and $\Psi^{[i]}(t)$ is the generating function for the sequence $\{\rho_j^{[i]}\}_{j\in\mathbb{N}}$, then $h(t)$ is determined by [10]:

$$h(t) = \sum_{i\geq 0} t^{i+1}\Phi^{[i]}(h(t)) + \sum_{i=1}^{s} t^{1-i}h(t)^{i+1}\Psi^{[i]}(h(t)). \tag{1.10}$$

If column 0 of the Riordan array $(d(t), h(t)) = (g_{n,k})_{n,k\in\mathbb{N}}$ has the following linear relation,

$$g_{n+1,0} = \sum_{i\geq 0}\sum_{j\geq 0}\beta_{i,j}g_{n-i,j} + \sum_{i=1}^{s}\sum_{j\geq 0}\eta_j^{[i]}g_{n+i,i+j},\ n\geq 0, \tag{1.11}$$

then the function $d(t)$ is given by the following formula:

$$d(t) = \frac{g_{0,0}}{1-\sum_{i\geq 0}t^{i+1}R^{[i]}(h(t)) - t\sum_{i=1}^{s}t^{1-i}h(t)^{i}S^{[i]}(h(t))}, \tag{1.12}$$

where $R^{[i]}(t) = \sum_{j\geq 0}\beta_{i,j}t^j$, $i = 0, 1, \ldots$, and $S^{[i]}(t) = \sum_{j\geq 0}\eta_j^{[i]}t^j$, $i = 0, 1, \ldots, s$.

From [20], the right half of the Pascal rhombus and the left-bounded rhombus can be represented as Riordan arrays.

Lemma 1.2. ([20]). *Let $R = (r_{n,k})_{n,k\in\mathbb{N}}$ denote the right half of the Pascal rhombus. Then,*

$$R = \left(\frac{1}{\sqrt{(1-t-t^2)^2-4t^2}}, \frac{1-t-t^2-\sqrt{(1-t-t^2)^2-4t^2}}{2t}\right).$$

Lemma 1.3. ([20]). *The left-bounded rhombus $S = (s_{n,k})_{n,k\in\mathbb{N}}$ is the Riordan array*

$$S = \left(\frac{1-t-t^2-\sqrt{(1-t-t^2)^2-4t^2}}{2t^2}, \frac{1-t-t^2-\sqrt{(1-t-t^2)^2-4t^2}}{2t}\right).$$

Theorem 1.4. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n} s_{n,k} = r_{n,0} + r_{n,1},$$

i.e., the nth row sum of the left-bounded rhombus is equal to the nth complete central coefficient of the Pascal rhombus.

Proof. From (1.7), the generating function of the row sums of $S = (s_{n,k})_{n,k\in\mathbb{N}}$ is $S(t) = \frac{g(t)}{1-tg(t)}$, where $g(t) = \frac{1-t-t^2-\sqrt{(1-t-t^2)^2-4t^2}}{2t^2}$. Simplifying gives $S(t) = \frac{2}{1-3t-t^2+\sqrt{(1-t-t^2)^2-4t^2}}$.

On the other hand, the sum of the generating functions of the first two columns of $R = (r_{n,k})_{n,k\in\mathbb{N}}$ is

$$\frac{1}{\sqrt{(1-t-t^2)^2-4t^2}}\left(1+\frac{1-t-t^2-\sqrt{(1-t-t^2)^2-4t^2}}{2t}\right)$$
$$= \frac{2}{1-3t-t^2+\sqrt{(1-t-t^2)^2-4t^2}}.$$

Therefore, it follows that $r_{n,0} + r_{n,1} = \sum_{k=0}^{n} s_{n,k}$ for all $n \geq 0$. □

2. The Parametric Pascal Rhombus

In this section, we develop a type of parametric Pascal rhombuses. Let a, b, and c be nonnegative integers. We define parametric Pascal rhombus as an infinite array $\mathcal{R}(a,b,c) = (r_{i,j})$, where $i \in \mathbb{N}$ and $j \in \mathbb{Z}$, and $r_{i,j}$ satisfies

$$\begin{cases} r_{i,j} &= br_{i-1,j-1} + ar_{i-1,j} + br_{i-1,j+1} + cr_{i-2,j},\ i \geq 2,\ j \in \mathbb{Z}, \\ r_{0,0} &= 1, r_{1,0} = a, r_{1,-1} = r_{1,1} = b, r_{0,j} = 0\,(j \neq 0),\ r_{1,j} = 0\,(j \neq -1,0,1). \end{cases} \tag{2.1}$$

The left-bounded parametric rhombus $\mathcal{S}(a,b,c) = (s_{i,j})$, where $i, j \in \mathbb{N}$, and $s_{i,j}$ satisfies

$$\begin{cases} s_{i,j} &= bs_{i-1,j-1} + as_{i-1,j} + bs_{i-1,j+1} + cs_{i-2,j},\ i \geq 2,\ 0 \leq j \leq i, \\ s_{0,0} &= 1, s_{1,0} = a, s_{1,1} = b, s_{i,-1} = 0\,(i \geq 0),\ r_{i,j} = 0\,(i < j). \end{cases} \tag{2.2}$$

The first few rows of the parametric Pascal rhombus are given in Figure 2, and the first few rows of the left-bounded parametric rhombus are given in Figure 3.

$$\begin{array}{lllllllll}
 & & & & 1 & & & & \\
 & & & b & a & b & & & \\
 & & b^2 & 2ab & a^2+2b^2+c & 2ab & b^2 & & \\
 & b^3 & 3ab^2 & b(3a^2+3b^2+2c) & a^3+6ab^2+2ac & b(3a^2+3b^2+2c) & 3ab^2 & b^3 & \\
b^4 & 4ab^3 & b^2(6a^2+4b^2+3c) & 2ab(2a^2+6b^2+3c) & a^4+12a^2b^2+6b^4+3a^2c+6b^2c+c^2 & 2ab(2a^2+6b^2+3c) & b^2(6a^2+4b^2+3c) & 4ab^3 & b^4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

Figure 2. Parametric Pascal rhombus

$$\begin{array}{lllll}
1 & & & & \\
a & b & & & \\
a^2+b^2+c & 2ab & b^2 & & \\
a(a^2+3b^2+2c) & b(3a^2+2b^2+2c) & 3ab^2 & b^3 & \\
a^4+6a^2b^2+3a^2c+2b^4+3b^2c+c^2 & 2ab(2a^2+4b^2+3c) & 3b^2(2a^2+b^2+c) & 4ab^3 & b^4 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

Figure 3. Left-bounded parametric rhombus

We will show that the right half of the parametric Pascal rhombus and the left-bounded parametric rhombus can be represented as Riordan arrays.

Theorem 2.1. *Let $\mathcal{R}(a,b,c) = (r_{n,k})_{n,k\in\mathbb{N}}$ denote the right half of the parametric Pascal rhombus. Then,*

$$\mathcal{R}(a,b,c) = \left(\frac{1}{\sqrt{(1-at-ct^2)^2 - 4b^2t^2}}, \frac{1-at-ct^2-\sqrt{(1-at-ct^2)^2-4b^2t^2}}{2bt} \right).$$

Proof. It follows from (2.1) and Lemma 1.1 that $\mathcal{R}(a,b,c) = (r_{n,k})_{n,k\in\mathbb{N}}$ is a Riordan array $(d(t), h(t))$ with the A-matrix

$$A = \begin{pmatrix} b & a & b \\ 0 & c & 0 \end{pmatrix}.$$

Now, we can directly use (1.10) to obtain the function $h(t)$. Because $\Phi^{[0]}(t) = b + at + bt^2$, $\Phi^{[1]}(t) = ct$, $\Phi^{[i]}(t) = 0$ for $i \geq 2$, and $\Psi^{[i]}(t) = 0$ for $i \geq 1$, $h(t)$ is the solution to the equation

$$h(t) = t(b + ah(t) + bh(t)^2) + ct^2 h(t),$$

from which it follows that $h(t) = \frac{1-at-ct^2-\sqrt{(1-at-ct^2)^2-4b^2t^2}}{2bt}$.

Column 0 of the Riordan array $(r_{n,k})_{n,k\in\mathbb{N}} = (d(t), h(t))$ satisfies

$$r_{i+1,0} = ar_{i,0} + 2br_{i,1} + cr_{i-1,0}.$$

Hence from (1.12), the function $d(t)$ is given by

$$d(t) = \frac{1}{1 - t(a + 2bh(t)) - ct^2} = \frac{1}{\sqrt{(1 - at - ct^2)^2 - 4b^2t^2}}.$$

□

Theorem 2.2. *The left-bounded parametric rhombus $\mathcal{S}(a, b, c) = (s_{n,k})_{n,k\in\mathbb{N}}$ is the Riordan array*

$$\mathcal{S}(a, b, c) = \left(\frac{1 - at - ct^2 - \sqrt{(1 - at - ct^2)^2 - 4b^2t^2}}{2b^2t^2}, \frac{1 - at - ct^2 - \sqrt{(1 - at - ct^2)^2 - 4b^2t^2}}{2bt}\right).$$

Proof. The proof is similar to that of Theorem 2.1, and is omitted. □

A Motzkin path of length n is a lattice path from $(0, 0)$ to $(n, 0)$ consisting of up steps $U = (1, 1)$, horizontal steps $H_1 = (1, 0)$, and down steps $D = (1, -1)$ that never goes below the x-axis. The number of Motzkin paths of length n is the nth Motzkin number M_n, and the Motzkin numbers form the sequence A001006 in [12]. Many other examples of bijections between Motzkin paths and others combinatorial objects can be found in [2, 6, 17]. A grand Motzkin path of length n is a Motzkin path without the condition of never passing below the x-axis. The number of grand Motzkin paths of length n is the nth central trinomial coefficient; they form sequence A002426 in [12].

A generalized grand Motzkin path of length n is a lattice path from $(0, 0)$ to $(n, 0)$ with up steps $U = (1, 1)$, horizontal steps $H_1 = (1, 0)$, down steps $D = (1, -1)$, and double horizontal steps $H_2 = (2, 0)$. We weight the steps by assigning b to each up step U, a to each horizontal step H_1, b to each down step D, and c to each double horizontal step H_2. The weight of a path P, denoted by $|P|$, is the product of the weights of its steps, and the weight of a set of paths S, denoted by $|S|$, is the sum of the weights of the paths in S.

The set of all partial generalized grand Motzkin paths ending at (i, j) is denoted by $\mathcal{R}_{i,j}$. Then, $|\mathcal{R}_{n,0}|$ is the number of all generalized grand Motzkin path of length n.

In Figure 4, we give an illustration of the dependence of $r_{i+1,j+1}$ from the other elements in the array $(r_{i,j})_{i,j\in\mathbb{N}}$, so that $r_{i,j}$ satisfies the recurrence relation and the boundary conditions of (1.2). Therefore, we obtain the following theorem.

Theorem 2.3. *The number of the generalized grand Motzkin paths ending at (i, j) is equal to the entry $r_{i,j}$ in the parametric Pascal rhombus, i.e., $r_{i,j} = |\mathcal{R}_{i,j}|$, where $i, j \in \mathbb{N}$.*

A generalized Motzkin path is a generalized grand Motzkin path that never goes below the x-axis. A partial generalized Motzkin path, also called a generalized Motzkin path ending at (i, j), is defined as an initial segment of a generalized Motzkin path with terminal point (i, j). Let $\mathcal{S}_{i,j}$ be the set of all partial generalized Motzkin paths ending at (i, j), where $\mathcal{S}_{0,0} = \{\varepsilon\}$

and ε is the empty path. Then, $|\mathcal{S}_{n,0}|$ is the number of all generalized Motzkin path of length n. By considering the recurrence relations between the numbers $|\mathcal{S}_{i,j}|$, we obtain the following result.

Theorem 2.4. *The number of the generalized Motzkin paths ending at (i, j) is equal to the entry $s_{i,j}$ in the left-bounded parametric rhombus, i.e., $s_{i,j} = |\mathcal{S}_{i,j}|$, where $i, j \in \mathbb{N}$.*

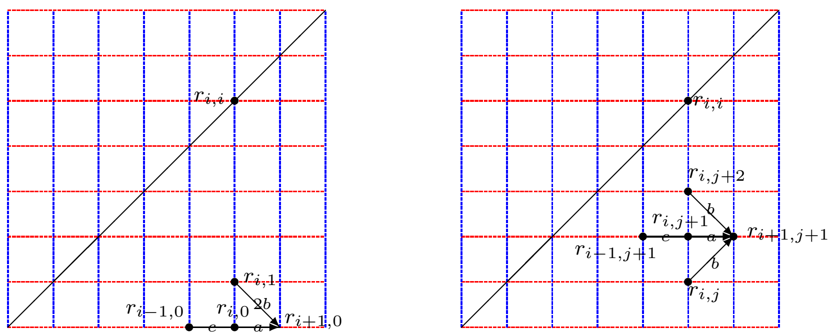

Figure 4: The recursion of the partial generalized Motzkin paths

Theorem 2.5. *For any integers $i \geq j \geq 0$, we have*

$$s_{i,j} = r_{i,j} - r_{i,j+2}, \tag{2.3}$$

i.e., the (i, j)-entry of the left-bounded parametric rhombus $R(a, b, c)$ is equal to the difference between the (i, j)-entry and $(i, j + 2)$-entry of the parametric Pascal rhombus $S(a, b, c)$.

Proof. To prove this theorem, we must establish a bijection between $\mathcal{R}_{i,j} - \mathcal{S}_{i,j}$ and $\mathcal{R}_{i,j+2}$, i.e., the paths from $(0, 0)$ to (i, j), which cross the x-axis are in bijection with paths from $(0, 0)$ to $(i, j + 2)$. Obviously, there is a bijection between $\mathcal{R}_{i,j+2}$ and $\mathcal{R}_{i,-j-2}$; it is sufficient to establish a bijection between $\mathcal{R}_{i,j} - \mathcal{S}_{i,j}$ and $\mathcal{R}_{i,-j-2}$.

The claimed bijection is established as follows. Consider a path $P \in \mathcal{R}_{i,j} - \mathcal{S}_{i,j}$, which is from $(0, 0)$ to (i, j) crossing the line $y = 0$. See Figure 5 for an example. Then, P must meet the line $y = -1$.

Among all the meeting points of P and $y = -1$, choose the right-most one. Denote this point by Q. Now reflect the portion of P from Q to (i, j) about the line $y = -1$, leaving the portion from $(0, 0)$ to Q invariant. Thus, we obtain a new path P' from $(0, 0)$ to $(i, -j - 2)$.

To construct the reverse mapping, we only have to observe that any path from $(0, 0)$ to $(i, -j - 2)$ must meet $y = -1$ because $(0, 0)$ and $(i, -j - 2)$ lie on different sides of $y = -1$. Again we choose the right-most meeting point, denote it by Q, and reflect the portion from Q to $(i, -j - 2)$ about the line $y = -1$, thus obtaining a path from $(0, 0)$ to (i, j) that meets the line $y = -1$, or, equivalently, crosses the line $y = 0$. □

Theorem 2.6. *The nth row sum of the left-bounded parametric rhombus $S(a, b, c)$ is equal to the sum of first two entries of nth row of right half of the parametric Pascal rhombus $R(a, b, c)$, i.e.,*

$$\sum_{k=0}^{n} s_{n,k} = r_{n,0} + r_{n,1}.$$

Proof. It follows from (2.3) that $\sum_{k=0}^{n} s_{n,k} = \sum_{k=0}^{n}(r_{n,k} - r_{n,k+2}) = r_{n,0} + r_{n,1}$. □

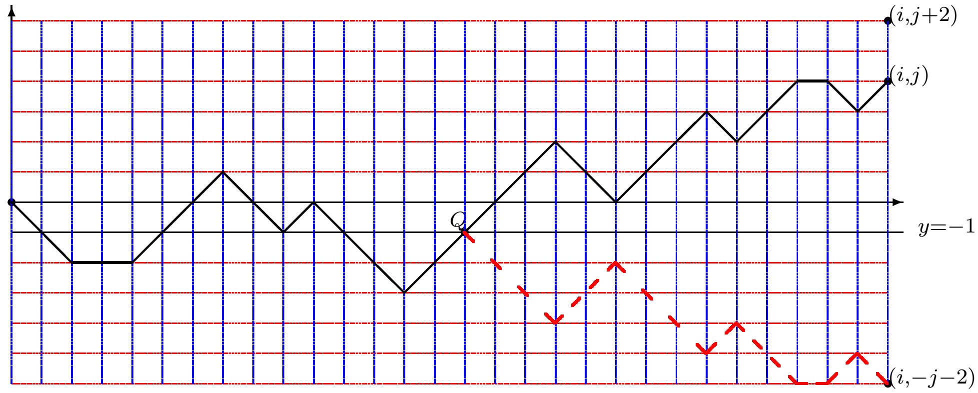

Figure 5: From a path in $\mathcal{R}_{29,4} - \mathcal{S}_{29,4}$ to a path in $\mathcal{R}_{29,-6}$

Theorem 2.7. *The generating function of the complete central coefficients of the parametric Pascal rhombus* $R(a, b, c)$ *is* $\frac{2}{1-(a+2b)t-ct^2+\sqrt{(1-at-ct^2)^2-4b^2t^2}}$.

Proof. It follows from Theorem 2.6 that the complete central coefficients of the parametric Pascal rhombus are equal to the row sums of the left-bounded parametric rhombus. Hence, using (1.8) and Theorem 2.2, we have the generating function

$$\frac{1-at-ct^2-\sqrt{(1-at-ct^2)^2-4b^2t^2}}{2b^2t^2}\cdot\frac{1}{1-\frac{1-at-ct^2-\sqrt{(1-at-ct^2)^2-4b^2t^2}}{2b^2t}}$$
$$=\frac{2}{1-(a+2b)t-ct^2+\sqrt{(1-at-ct^2)^2-4b^2t^2}}.$$

□

3. Some Special Examples

In this section, we will present several examples of the parametric Pascal rhombus and the left-bounded parametric rhombus. In each example, we will list the first lines of $R(a, b, c)$ and $S(a, b, c)$.

Example 3.1. If $a = c = 0$ and $b = 1$, then

$$R(0,1,0)=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 6 & 0 & 4 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 10 & 0 & 5 & 0 & 1 & 0 & \cdots \\ 20 & 0 & 15 & 0 & 6 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},\ S(0,1,0)=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 3 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 5 & 0 & 4 & 0 & 1 & 0 & \cdots \\ 5 & 0 & 9 & 0 & 5 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The central coefficients, complete central coefficients of $R(0, 1, 0)$, and the first column of $S(0, 1, 0)$ are sequence A126869, A001405, and A126120 in the OEIS [12], respectively.

Example 3.2. If $a = 0$ and $b = c = 1$, then

$$R(0,1,1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 5 & 0 & 1 & 0 & 0 & \cdots \\ 13 & 0 & 7 & 0 & 1 & 0 & \cdots \\ 0 & 25 & 0 & 9 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \; S(0,1,1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 4 & 0 & 1 & 0 & 0 & \cdots \\ 6 & 0 & 6 & 0 & 1 & 0 & \cdots \\ 0 & 16 & 0 & 8 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

The central coefficients, complete central coefficients of $R(0,1,0)$, and the first column of $S(0,1,0)$ are Central Delannoy numbers A001850, Larger Schröder numbers A006318, and A026003 in the OEIS [12], respectively.

Example 3.3. If $a = 0$, $b = 1$, and $c = 2$, then

$$R(0,1,2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 4 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 7 & 0 & 1 & 0 & 0 & \cdots \\ 22 & 0 & 10 & 0 & 1 & 0 & \cdots \\ 0 & 46 & 0 & 13 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \; S(0,1,2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 6 & 0 & 1 & 0 & 0 & \cdots \\ 12 & 0 & 9 & 0 & 1 & 0 & \cdots \\ 0 & 33 & 0 & 12 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

The central coefficients of $R(0,1,2)$, and the first column of $S(0,1,2)$ are sequence A069835, and A047891 in the OEIS [12], respectively.

Example 3.4. If $a = 0$, $b = 1$, and $c = r$, then

$$R(0,1,r) = \left(\frac{1}{\sqrt{(1-rt^2)^2+4t^2}}, \frac{1-rt^2-\sqrt{(1-rt^2)^2+4t^2}}{2t} \right)$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ r+2 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2r+3 & 0 & 1 & 0 & 0 & 0 & \cdots \\ r^2+6r+6 & 0 & 3r+4 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 3r^2+12r+10 & 0 & 4r+5 & 0 & 1 & 0 & \cdots \\ r^3+12r^2+30r+20 & 0 & 6r^2+20r+15 & 0 & 5r+6 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$S(0,1,r) = \left(\frac{1-rt^2-\sqrt{(1-rt^2)^2+4t^2}}{2t^2}, \frac{1-rt^2-\sqrt{(1-rt^2)^2+4t^2}}{2t}\right)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ r+1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2r+2 & 0 & 1 & 0 & 0 & 0 & \cdots \\ r^2+3r+2 & 0 & 3r+3 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 3r^2+8r+5 & 0 & 4r+4 & 0 & 1 & 0 & \cdots \\ r^3+6r^2+10r+5 & 0 & 6r^2+15r+9 & 0 & 5r+5 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $R(0,1,r)$ is the aeration of the right half of the Pascal-like triangle $\left(\frac{1}{1-t}, \frac{t(1+rt)}{1-t}\right)$ [4, 19], and $S(0,1,r)$ is the matrix whose row sums are the complete central coefficients of $\left(\frac{1}{1-t}, \frac{t(1+rt)}{1-t}\right)$ [4].

Example 3.5. If $a = k$, $b = 1$, and $c = 0$, then

$$R(k,1,0) = \left(\frac{1}{\sqrt{(1-kt)^2-4t^2}}, \frac{1-kt-\sqrt{(1-kt)^2-4t^2}}{2t}\right) =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ k & 1 & 0 & 0 & 0 & 0 & \cdots \\ k^2+2 & 2k & 1 & 0 & 0 & 0 & \cdots \\ k^3+6k & 3k^2+3 & 3k & 1 & 0 & 0 & \cdots \\ k^4+12k^2+6 & 4k^3+12k & 6k^2+4 & 4k & 1 & 0 & \cdots \\ k^5+20k^3+30k & 5k^4+30k^2+10 & 10k^3+20k & 10k^2+5 & 5k & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$S(k,1,0) = \left(\frac{1-kt-\sqrt{(1-kt)^2-4t^2}}{2t^2}, \frac{1-kt-\sqrt{(1-kt)^2-4t^2}}{2t}\right) =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ k & 1 & 0 & 0 & 0 & 0 & \cdots \\ k^2+1 & 2k & 1 & 0 & 0 & 0 & \cdots \\ k^3+3k & 3k^2+2 & 3k & 1 & 0 & 0 & \cdots \\ k^4+6k^2+2 & 4k^3+8k & 6k^2+3 & 4k & 1 & 0 & \cdots \\ k^5+10k^3+10k & 5k^4+20k^2+5 & 10k^3+15k & 10k^2+4 & 5k & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $R(k,1,0)$ is an element of the hitting-time subgroup of Riordan group [13], and $S(k,1,0)$ is the k-Motzkin matrix [5].

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