INDEPENDENCE POLYNOMIALS OF FIBONACCI TREES ARE LOG-CONCAVE

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Abstract. We use a factorization of the independence polynomials of Fibonacci trees and the properties of a relation between polynomials called partial synchronization to prove that these independence polynomials are log-concave.

1. Introduction

The Fibonacci number of a graph is defined as the number of its independent sets [13]. This is a generalization of Fibonacci numbers in the sense that the usual Fibonacci numbers are recovered as the Fibonacci numbers of the path graphs. A further generalization is the independence polynomial of a graph [5] (or Fibonacci polynomial [6]), which is the polynomial with coefficients given by the number of independent sets of each size. The Fibonacci number of the graph is just the evaluation of the independence polynomial of that graph at $x = 1$.

The most famous conjecture about independence polynomials, made by Alavi, et al. [1], is that each tree must have a *unimodal* independence polynomial, where a real polynomial $p = \sum_{i=0}^{t} a_i x^i$ is called *unimodal* if there exists $1 \leq m \leq t$ such that $a_0 \leq a_1 \leq \cdots \leq a_m \geq a_{m+1} \geq \cdots \geq a_t$. To verify this conjecture, many authors have considered particular classes of trees in the hope that these could uncover the behavior of the general case. In this paper, we will consider the class of Fibonacci trees.

A concept related to unimodality is *log-concavity*. A real polynomial $p = \sum_{i=0}^{t} a_i x^i$ is called *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for $i = 1, \ldots, t-1$. Because a log-concave polynomial with nonnegative coefficients and without internal zeros is unimodal, a common technique is to prove log-concavity instead of directly proving unimodality. In this paper, we confirm the conjecture of Alavi, et al. for Fibonacci trees by means of the log-concavity concept and using the relation between polynomials called *partial synchronization* [9]. More specifically, our main result is that the independence polynomials of the Fibonacci trees are log-concave.

The Fibonacci trees $\{T_n\}_{n\geq 0}$ are rooted graphs defined recursively as follows: $T_0 = K_1$ and $T_1 = K_1$ with roots $r_0 \in V(T_0)$ and $r_1 \in V(T_1)$. For $n \geq 2$, the tree T_n is defined as the disjoint union of T_{n-1} and T_{n-2}, together with a new vertex r_n connected to the roots r_{n-1} and r_{n-2} [7, 8, 4, 10]. However, other types of Fibonacci trees appear in the literature with different initial conditions: $T_0 = K_1$ and $T_1 = K_2$ [3, 15]. Bencs [3] proved a stronger result than ours for this second type of Fibonacci tree: namely, that their independence polynomials have only real roots (recall that polynomials with only real roots are log-concave because of Newton's theorem [14]). In spite of that, our Fibonacci trees have nonreal complex roots (Corollary 4.2), thus we have to use related, but different techniques than Bencs.

In the following, we will work with the first mentioned version of Fibonacci trees, unless otherwise stated. Changes in the initial conditions can be made to apply our results to the second version of Fibonacci trees. The details are presented in Section 5.

2. Partial Synchronization Properties

Let $\mathbb{R}^{\geq 0}[x]$ be the semiring of polynomials with nonnegative real coefficients in variable x. The *partial synchronization* [9] concept is defined over this semiring. All our proofs are based on this concept.

Definition 2.1. *Let $p = \sum_i a_i x^i$ and $q = \sum_i b_i x^i$ be a pair of polynomials with nonnegative coefficients. The polynomials p and q are partially synchronized, written $p \sim_{\mathrm{p}} q$, if*

$$a_{m+1}b_{n-1} + a_{n-1}b_{m+1} \leq a_m b_n + a_n b_m \tag{2.1}$$

for all $m \geq n$.

An *internal zero* of a polynomial with real coefficients $p = \sum_i a_i x^i$ is a coefficient a_j such that there exist m and n with $m < j < n$, $a_m a_n \neq 0$, and $a_j = 0$.

Let $\mathcal{L}$ denote the set of log-concave polynomials in $\mathbb{R}^{\geq 0}[x]$ that do not have internal zeros; let $\mathcal{L}^*$ stand for $\mathcal{L} - \{0\}$.

Hu, et al. [9] originally defined the partial synchronization relation on the set $\mathcal{L}$. However, it is convenient to extend this definition to the whole semiring $\mathbb{R}^{\geq 0}[x]$. It is clear that $\sim_{\mathrm{p}}$ is symmetric on $\mathbb{R}^{\geq 0}[x]$ and reflexive on $\mathcal{L}$.

Straightforward calculations show that the partial synchronization relation is preserved by the sum of polynomials, in the following sense.

Theorem 2.2. *Let $p, q, r \in \mathbb{R}^{\geq 0}[x]$. If $p \sim_{\mathrm{p}} r$ and $q \sim_{\mathrm{p}} r$, then $p + q \sim_{\mathrm{p}} r$.*

A proof of Theorem 2.2 is given in the proof of Theorem 3.6 in [9]. In addition, [9] also proved the following two theorems.

Theorem 2.3. *Let $p, q, r \in \mathcal{L}^*$. If $p \sim_{\mathrm{p}} q$, then $pr \sim_{\mathrm{p}} qr$.*

Theorem 2.4. *If $p, q \in \mathcal{L}$ and $p \sim_{\mathrm{p}} q$, then $p + q \in \mathcal{L}$.*

The following lemma will be used in the proof of Lemma 4.3.

Lemma 2.5. *If $p \in \mathcal{L}^*$, then $p \sim_{\mathrm{p}} xp$.*

Proof. Because p has no internal zeros, we can write $p = \sum_{i=h}^{t} a_i x^i$, where each $a_i > 0$ for $h \leq i \leq t$. Now, for $q = xp$ in Definition 2.1, condition (2.1) becomes

$$a_{m+1}a_{n-2} \leq a_n a_{m-1}. \tag{2.2}$$

Therefore, we must prove (2.2) for all $m \geq n$. Let us assume that $m \geq n$. We have several cases: a) $h > n-2$, b) $h \leq n-2$ and $m \leq t$, and c) $h \leq n-2$ and $m > t$.

Case a): We have $a_{n-2} = 0$. Thus, (2.2) trivially holds.

Case b): We have $t \geq m \geq m-1 \geq \cdots \geq n > h$. Then, $t > m-2 \geq m-3 \geq \cdots \geq n-2 > h$. Thus, we can write $a_{m+1}/a_m \leq a_m/a_{m-1}$ and $a_m/a_{m-1} \leq a_{m-1}/a_{m-2}$ because p is log-concave. Next, by multiplying such inequalities side by side, we get $a_{m+1}/a_{m-1} \leq a_m/a_{m-2}$. Similarly,

$$\frac{a_m}{a_{m-2}} \leq \frac{a_{m-1}}{a_{m-3}} \leq \cdots \leq \frac{a_n}{a_{n-2}}.$$

Then, by transitivity, $a_{m+1}/a_{m-1} \leq a_n/a_{n-2}$, which is equivalent to (2.2).

Case c): In this case, $a_{m+1} = 0$. Thus, (2.2) trivially holds. $\square$

3. Other Properties of Polynomials

The classic result of the product of two log-concave polynomials also being log-concave will be useful for us [12].

Theorem 3.1. *If $p, q \in \mathcal{L}$, then $pq \in \mathcal{L}$.*

For a given graph G, we use $I(G;x)$ to denote the independence polynomial of G. The following lemma helps calculate the independence polynomial of an arbitrary graph [2, 5].

Lemma 3.2. *If G, G_1, and G_2 are arbitrary simple graphs and v is an arbitrary vertex of G, then*

$$I(G;x) = I(G-v;x) + xI(G-N[v];x) \tag{3.1}$$

and

$$I(G_1 \sqcup G_2;x) = I(G_1;x)I(G_2;x), \tag{3.2}$$

where $\sqcup$ stands for the disjoint union operator, and $N[v]$ is the closed neighborhood of v, i.e.,

$$N[v] = \{v\} \cup \{w \in V(G) : w \text{ is neighbor of } v\}.$$

4. Fibonacci Trees and Their Independence Polynomials

Let $p_n = I(T_n;x)$ be the independence polynomial of the Fibonacci tree T_n, $n \geq 0$. From Lemma 3.2 with $v = r_n$, the natural root of Fibonacci tree T_n, it follows that

$$p_n = p_{n-2}(p_{n-1} + x\,p_{n-3}^2 p_{n-4}), \quad n \geq 4. \tag{4.1}$$

The initial conditions are the following:

$$\begin{aligned} p_0 &= x+1 \\ p_1 &= x+1 \\ p_2 &= x^2+3x+1 \\ p_3 &= 2x^3+6x^2+5x+1. \end{aligned}$$

Also, for any $n \geq 0$, define a graph G_n with vertex set $\{0,\ldots,n\}$ and edges

$$E_n = \{\{i,j\} \,|\, 0 < |i-j| \leq 2\} \backslash \{\{0,1\}\}.$$

Let $\widehat{p}_n = I(G_n;x)$ be the independence polynomial of the graph G_n, $n \geq 0$. The following is a factorization of the polynomials p_n, which can be deduced from [3] or can be proved directly by mathematical induction. We show the latter.

Theorem 4.1. *For any integer $n \geq 1$,*

$$p_n = \left(\frac{\widehat{p}_1}{\widehat{p}_0}\right)^{f_{n-1}} \prod_{i=2}^{n} \widehat{p}_i^{\,f_{n-i}}, \tag{4.2}$$

where the sequence $\{f_n\}$ is a variant of the Fibonacci numbers defined by $f_0 = 1$, $f_1 = 0$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

Proof. For $n = 1$, we have $\widehat{p}_1 = (x+1)^2$ and $\widehat{p}_0 = x+1$, so $p_1 = x+1 = (\widehat{p}_1/\widehat{p}_0)^{f_0}$. For $n = 2$, $p_2 = \widehat{p}_2$ because T_2 is isomorphic to G_2. For $n = 3$, straightforward calculations show that $p_3 = (x+1)(2x^2+4x+1)$, whereas $(\widehat{p}_1/\widehat{p}_0)^{f_2} = x+1$ and $\widehat{p}_3^{f_0} = 2x^2+4x+1$. Thus, (4.2) holds for $n = 3$. For $n = 4$, from (4.1) we get

$$p_4 = p_2(p_3 + xp_1^2 p_0).$$

After substituting the previous values for p_n, $n = 1, 2, 3$ we get

$$p_4 = \widehat{p}_2 \left(\frac{\widehat{p}_1}{\widehat{p}_0} \widehat{p}_3 + x \left(\frac{\widehat{p}_1}{\widehat{p}_0} \right)^2 p_0 \right) = \widehat{p}_2(x+1)(\widehat{p}_3 + x(x+1)^2).$$

Because $\widehat{p}_4 = \widehat{p}_3 + x(x+1)^2$ (from (3.1) with $G = G_4$ and $v = 4$), we obtain $p_4 = (x+1)\widehat{p}_2\widehat{p}_4$, which is the formula (4.2) for $n = 4$.

Now, the induction step follows from the recurrence (4.1) and because

$$\widehat{p}_n = \widehat{p}_{n-1} + x\widehat{p}_{n-3} \tag{4.3}$$

for $n \geq 3$, which can be deduced from (3.1) with $G = G_n$ and $v = n$. □

The Fibonacci trees of [3] have independence polynomials with only real roots. Our Fibonacci trees have independence polynomials with nonreal complex roots.

Corollary 4.2. *For $n \geq 6$, the independence polynomial of T_n has nonreal complex roots.*

Proof. For $n \geq 6$, the factorization (4.2) tells us that p_n has a factor $\widehat{p}_4 = x^3 + 4x^2 + 5x + 1$. Standard calculus techniques show that such a factor has just one real root. □

Not just the roots of $\widehat{p}_4$ have nonreal complex roots of p_n. Computer calculations [11] show that $\widehat{p}_{12} = 11\,x^5 + 56\,x^4 + 93\,x^3 + 56\,x^2 + 13\,x + 1$ also has nonreal complex roots. Nevertheless, we can show that all the polynomials $\widehat{p}_n$ are log-concave with the help of the partial synchronization relationship.

Due to (4.2), Theorem 3.1, and $\widehat{p}_1/\widehat{p}_0 = x + 1$, to prove that p_n is log-concave, it suffices to show that $\widehat{p}_i$ is log-concave for $i \geq 2$. This result follows from the following general lemma.

Lemma 4.3. *Let $\{q_n\}_{n\geq 0}$ be a sequence of polynomials in $\mathbb{R}^{\geq 0}[x]$ such that $q_n = q_{n-1} + xq_{n-3}$ for $n \geq 3$. If*

$$q_n \sim_{\mathrm{p}} xq_{n-2}, \; q_n \sim_{\mathrm{p}} xq_{n-1}, \; \textit{and } q_n \in \mathcal{L} \tag{4.4}$$

holds for $n = 2, 3, 4$, then (4.4) holds for all $n \geq 2$.

Proof. We proceed by mathematical induction on n. We only have to prove the induction step. Our induction hypothesis is

$$q_m \sim_{\mathrm{p}} xq_{m-2}, \; q_m \sim_{\mathrm{p}} xq_{m-1}, \text{ and } q_m \in \mathcal{L}$$

for $2 \leq m < n$. In addition, we assume that $n \geq 5$. Then, from $q_{n-1} \sim_{\mathrm{p}} q_{n-1}$ (since $q_{n-1} \in \mathcal{L}$) and $q_{n-1} \sim_{\mathrm{p}} xq_{n-3}$, we get

$$q_{n-1} \sim_{\mathrm{p}} q_{n-1} + xq_{n-3} = q_n \tag{4.5}$$

because of Theorem 2.2. Similar to (4.5), we get $q_{n-2} \sim_{\mathrm{p}} q_{n-1}$. This, with $q_{n-2} \sim_{\mathrm{p}} xq_{n-3}$ and Theorem 2.2, again, implies

$$q_{n-2} \sim_{\mathrm{p}} q_{n-1} + xq_{n-3} = q_n. \tag{4.6}$$

Similar to (4.6), we get $q_{n-3} \sim_{\mathrm{p}} q_{n-1}$. Then, $xq_{n-3} \sim_{\mathrm{p}} xq_{n-1}$ (Theorem 2.3). In addition, Lemma 2.5 ensures that $q_{n-1} \sim_{\mathrm{p}} xq_{n-1}$. It follows that $q_n = q_{n-1} + xq_{n-3} \sim_{\mathrm{p}} xq_{n-1}$ (Theorem 2.2).

Now, from $q_{n-1} \sim_{\mathrm{p}} xq_{n-2}$ and $q_{n-3} \sim_{\mathrm{p}} q_{n-2}$ (which is obtained in a similar way to (4.5)), we get $q_n = q_{n-1} + xq_{n-3} \sim_{\mathrm{p}} xq_{n-2}$ (Theorem 2.3 and Theorem 2.2).

It remains to prove that $q_n \in \mathcal{L}$. By the induction hypothesis, $q_{n-1} \sim_{\mathrm{p}} xq_{n-3}$. Then, $q_n = q_{n-1} + xq_{n-3} \in \mathcal{L}$ because of Theorem 2.4. □

Theorem 4.4. *For $n \geq 0$, the independence polynomial of the Fibonacci tree T_n is log-concave.*

Proof. For $n = 0, 1$, the result is obvious. For $n \geq 2$, it suffices to prove that each $\widehat{p}_i$, for $i \geq 2$, is log-concave, as explained above. Straightforward calculations show that

$$\widehat{p}_i \sim_{\mathrm{p}} x\widehat{p}_{i-2},\ \widehat{p}_i \sim_{\mathrm{p}} x\widehat{p}_{i-1},\ \text{and } \widehat{p}_i \in \mathcal{L}$$

for $i = 2, 3, 4$. Then, Lemma 4.3 shows that $\widehat{p}_i \in \mathcal{L}$ for all $i \geq 2$, because the polynomials $\widehat{p}_i$ satisfy the recurrence (4.3). □

5. Generalized Initial Conditions

In this section, we study the Fibonacci trees with generalized initial conditions $T_0 = G$, an arbitrary graph with root r_0, and $T_1 = H$, also an arbitrary graph with root r_1. We find conditions under which the independence polynomial of these generalized Fibonacci trees have log-concave polynomials.

For $n \geq 0$, define $\bar{G}_n$ as the graph obtained from G_n, G, and H by gluing the vertices 0 and 1 of G_n to the roots r_0 and r_1, respectively. Let $\bar{p}_n = I(\bar{G}_n; x)$ and $p_n = I(T_n; x)$ be the independence polynomials of $\bar{G}_n$ and T_n, respectively, for $n \geq 0$. We get a generalized version of the factorization (4.2).

Theorem 5.1. *For any integer $n \geq 1$,*

$$p_n = \left(\frac{\bar{p}_1}{\bar{p}_0}\right)^{f_{n-1}} \prod_{i=2}^{n} \bar{p}_i^{f_{n-i}}, \tag{5.1}$$

where the sequence $\{f_n\}$ is a variant of the Fibonacci numbers defined by $f_0 = 1$, $f_1 = 0$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

Proof. This proof goes along the same lines as Theorem 4.1. For $n = 1$, $p_1 = g = \bar{p}_1/\bar{p}_0$, since $\bar{p}_1 = gh$ and $\bar{p}_0 = h$. For $n = 2$, we have $p_2 = \bar{p}_2$ because T_2 is isomorphic to $\bar{G}_2$. For $n = 3$, from (3.1) with graph T_3 and v, the root vertex of T_3, we get $p_3 = h(\bar{p}_2 + xg\bar{h})$, where $g = I(G; x)$, $h = I(H; x)$, and $\bar{h} = I(H - N[r_1]; x)$. Also, from (3.1), for $\bar{G}_3$ and v the vertex 3, we obtain $\bar{p}_3 = \bar{p}_2 + xg\bar{h}$. Thus,

$$p_3 = h\bar{p}_3 = \left(\frac{\bar{p}_1}{\bar{p}_0}\right)\bar{p}_3.$$

In other words, (5.1) holds for $n = 3$. For $n = 4$, from (4.1) we get

$$p_4 = p_2(p_3 + xp_1^2 p_0).$$

After substituting the previous values for p_n, $n = 1, 2, 3$ we get

$$p_4 = \bar{p}_2\left(\frac{\bar{p}_1}{\bar{p}_0}\bar{p}_3 + x\left(\frac{\bar{p}_1}{\bar{p}_0}\right)^2 p_0\right) = h\bar{p}_2(\bar{p}_3 + xgh)$$

Because $\bar{p}_4 = \bar{p}_3 + xgh2$ (from (3.1) with $G = \bar{G}_4$ and $v = 4$), we obtain $p_4 = h\bar{p}_2\bar{p}_4$, which is the formula (4.2) for $n = 4$.

The induction step follows because the independence polynomials p_n satisfy the recurrence (4.1) and because $\bar{p}_n = \bar{p}_{n-1} + x\bar{p}_{n-3}$ for $n \geq 3$, just as in the proof of Theorem 4.1. □

As a consequence of factorization (5.1) and Theorem 3.1, to prove that p_n is log-concave, it suffices to show that the factors on the right side are log-concave. Note that $\bar{p}_1/\bar{p}_0 = I(H; x)$, and Lemma 4.3 gives us some conditions for the log-concavity of the factors $\bar{p}_i$. These remarks are summarized in the following theorem.

Theorem 5.2. *If $I(H;x) \in \mathcal{L}$ and*

$$\bar{p}_n \sim_{\mathrm{p}} x\bar{p}_{n-2}, \bar{p}_n \sim_{\mathrm{p}} x\bar{p}_{n-1}, \text{ and } \bar{p}_n \in \mathcal{L}$$

hold for $n = 2, 3, 4$, then for $n \geq 1$, the independence polynomial p_n of the Fibonacci tree T_n with generalized initial conditions is log-concave.

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