ON THE x-COORDINATES OF PELL EQUATIONS THAT ARE PRODUCTS OF TWO LUCAS NUMBERS

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Abstract. Let $\{L_n\}_{n\geq 0}$ be the sequence of Lucas numbers given by $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. In this paper, for an integer $d \geq 2$ that is square-free, we show that there is at most one value of the positive integer x participating in the Pell equation $x^2 - dy^2 = \pm 1$, which is a product of two Lucas numbers, with a few exceptions that we completely characterize.

1. Introduction

Let $\{L_n\}_{n\geq 0}$ be the sequence of Lucas numbers given by $L_0 = 2$, $L_1 = 1$, and

$$L_{n+2} = L_{n+1} + L_n$$

for all $n \geq 0$. This is sequence A000032 on the On-Line Encyclopedia of Integer Sequences (OEIS). The first few terms of this sequence are

$$\{L_n\}_{n\geq 0} \quad = \quad 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, \ldots.$$

Putting $(\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$ for the roots of the characteristic equation $r^2 - r - 1 = 0$ of the Lucas sequence, the Binet formula for its general terms is given by

$$L_n = \alpha^n + \beta^n, \qquad \text{for all} \quad n \geq 0. \tag{1.1}$$

Furthermore, we can prove by induction that the inequality

$$\alpha^{n-1} \leq L_n \leq \alpha^{n+2}, \tag{1.2}$$

holds for all $n \geq 0$.

Let $d \geq 2$ be a positive integer that is not a perfect square. It is well-known that the Pell equation

$$x^2 - dy^2 = \pm 1 \tag{1.3}$$

has infinitely many positive integer solutions (x, y). Letting (x_1, y_1) be the smallest positive solution, all solutions are of the form (x_k, y_k) for some positive integer k, where

$$x_k + y_k\sqrt{d} = (x_1 + y_1\sqrt{d})^k \qquad \text{for all} \quad k \geq 1. \tag{1.4}$$

Furthermore, the sequence $\{x_k\}_{k\geq 1}$ is binary recurrent. The following formula

$$x_k = \frac{(x_1 + y_1\sqrt{d})^k + (x_1 - y_1\sqrt{d})^k}{2},$$

holds for all positive integers k.

Kafle, et al. [11] considered the Diophantine equation

$$x_n = F_\ell F_m, \tag{1.5}$$

where $\{F_m\}_{m\geqslant 0}$ is the sequence of Fibonacci numbers given by $F_0 = 0$, $F_1 = 1$, and $F_{m+2} = F_{m+1} + F_m$ for all $m \geqslant 0$. They proved that equation (1.5) has at most one solution n in positive integers except for $d = 2, 3, 5$, for which case equation (1.5) has the solutions $x_1 = 1$ and $x_2 = 3$, $x_1 = 2$ and $x_2 = 26$, $x_1 = 2$ and $x_2 = 9$, respectively.

There are many other researchers who have studied related problems involving the intersection sequence $\{x_n\}_{n\geq 1}$ with linear recurrence sequences of interest. For example, see [4, 7, 8, 9, 12, 13, 14, 16, 17, 19].

2. Main Result

In this paper, we study a similar problem to that of Kafle, et al. [11], but with the Lucas numbers instead of the Fibonacci numbers. That is, we show that there is at most one value of the positive integer x participating in (1.3), which is a product of two Lucas numbers, with a few exceptions that we completely characterize. This can be interpreted as solving the Diophantine equation

$$x_k = L_n L_m, \tag{2.1}$$

in nonnegative integers (k, n, m) with $k \geq 1$ and $0 \leq m \leq n$.

Theorem 2.1. *For each square-free integer $d \geq 2$, there is at most one integer k such that the equation (2.1) holds, except for $d \in \{2, 3, 5, 15, 17, 35\}$ for which $x_1 = 1$, $x_2 = 3$, $x_3 = 7$, $x_9 = 1393$ (for $d = 2$), $x_1 = 2$, $x_2 = 7$ (for $d = 3$), $x_1 = 2$, $x_2 = 9$ (for $d = 5$), $x_1 = 4$, $x_5 = 15124$ (for $d = 15$), $x_1 = 4$, $x_2 = 33$ (for $d = 17$), and $x_1 = 6$, $x_3 = 846$ (for $d = 35$).*

3. Preliminary Results

3.1. **Notations and terminology from algebraic number theory.** We begin by recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (x - \eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. Then, the *logarithmic height* of η is given by

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\eta^{(i)}|, 1\}\right)\right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|, q\}$. The following are some of the properties of the logarithmic height function $h(\cdot)$, which will be used in the next sections of this paper without reference:

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2, \\ h(\eta\gamma^{\pm 1}) &\leq h(\eta) + h(\gamma), \\ h(\eta^s) &= |s| h(\eta) \qquad (s \in \mathbb{Z}). \end{aligned} \tag{3.1}$$

3.2. **Linear Forms in Logarithms.** To prove our main result Theorem 2.1, we need to use several times a Baker–type lower bound for a nonzero linear form in logarithms of algebraic numbers. There are many such in the literature like that of Baker and Wüstholz from [2]. We start by recalling the result of Bugeaud, Mignotte, and Siksek ([5], Theorem 9.4, pp. 989), which is a modified version of the result of Matveev [18], which is one of our main tools in this paper.

Theorem 3.1. *Let $\gamma_1, \ldots, \gamma_t$ be positive real numbers in a number field $\mathbb{K} \subseteq \mathbb{R}$ of degree D, $b_1, \ldots, b_t$ be nonzero integers, and assume that*

$$\Lambda = \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1, \tag{3.2}$$

is nonzero. Then,

$$\log|\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t,$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{Dh(\gamma_i), |\log\gamma_i|, 0.16\}, \qquad \text{for all} \qquad i = 1, \ldots, t.$$

When $t = 2$ and γ_1 and γ_2 are positive and multiplicatively independent, we can use a result of Laurent, Mignotte, and Nesterenko [15]. Namely, in this case, let B_1 and B_2 be real numbers larger than 1 such that

$$\log B_i \geq \max\left\{h(\gamma_i), \frac{|\log\gamma_i|}{D}, \frac{1}{D}\right\}, \qquad \text{for} \quad i = 1, 2,$$

and put

$$b' = \frac{|b_1|}{D\log B_2} + \frac{|b_2|}{D\log B_1}.$$

Put

$$\Gamma = b_1\log\gamma_1 + b_2\log\gamma_2. \tag{3.3}$$

We note that $\Gamma \neq 0$ because γ_1 and γ_2 are multiplicatively independent. The following result is Corollary 2 in [15].

Theorem 3.2. *With the above notations, assuming that γ_1, γ_2 are positive and multiplicatively independent, then*

$$\log|\Gamma| > -24.34D^4\left(\max\left\{\log b' + 0.14, \frac{21}{D}, \frac{1}{2}\right\}\right)^2 \log B_1 \log B_2. \tag{3.4}$$

Note that with Γ given by (3.3), we have $e^\Gamma - 1 = \Lambda$, where Λ is given by (3.2) in case $t = 2$, which explains the connection between Theorem 3.1 and Theorem 3.2.

3.3. **Reduction Procedure.** During the calculations, we get upper bounds on our variables that are too large; thus, we need to reduce them. To do so, we use some results from the theory of continued fractions.

For the treatment of linear forms homogeneous in two integer variables, we use the well-known classic result in the theory of Diophantine approximation.

Lemma 3.3. *Let τ be an irrational number, $\frac{p_0}{q_0}$, $\frac{p_1}{q_1}$, $\frac{p_2}{q_2}$, $\ldots$ be all the convergents of the continued fraction of τ, and M be a positive integer. Let N be a nonnegative integer such that $q_N > M$. Then, putting $a(M) = \max\{a_i : i = 0, 1, 2, \ldots, N\}$, the inequality*

$$\left|\tau - \frac{r}{s}\right| > \frac{1}{(a(M)+2)s^2},$$

holds for all pairs (r, s) of positive integers with $0 < s < M$.

For a nonhomogeneous linear form in two integer variables, we use a slight variation of a result by Dujella and Pethő (see [10], Lemma 5a). For a real number X, we write $||X|| = \min\{|X - n| : n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 3.4. *Let M be a positive integer, $\frac{p}{q}$ be a convergent of the continued fraction of the irrational number τ such that $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Furthermore, let $\varepsilon = ||\mu q|| - M||\tau q||$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v, and w with

$$u \le M \quad \text{and} \quad w \ge \frac{\log(Aq/\varepsilon)}{\log B}.$$

At various occasions, we need to find a lower bound for linear forms in logarithms with bounded integer coefficients in three and four variables. In this case, we use the LLL algorithm that we describe below. Let $\tau_1, \tau_2, \ldots, \tau_t \in \mathbb{R}$ and the linear form

$$x_1\tau_1 + x_2\tau_2 + \cdots + x_t\tau_t \quad \text{with} \quad |x_i| \le X_i. \tag{3.5}$$

We put $X = \max\{X_i\}$ and $C > (tX)^t$ and consider the integer lattice Ω generated by

$$\mathbf{b}_j = \mathbf{e}_j + \lfloor C\tau_j \rceil \quad \text{for} \quad 1 \le j \le t-1 \quad \text{and} \quad \mathbf{b}_t = \lfloor C\tau_t \rceil \mathbf{e}_t,$$

where C is a sufficiently large positive constant.

Lemma 3.5. *Let $X_1, X_2, \ldots, X_t$ be positive integers such that $X = \max\{X_i\}$ and $C > (tX)^t$ is a fixed sufficiently large constant. With the above notation on the lattice Ω, we consider a reduced base $\{\boldsymbol{b}_i\}$ to Ω and its associated Gram-Schmidt orthogonalization base $\{\boldsymbol{b}_i^*\}$. We set*

$$c_1 = \max_{1 \le i \le t} \frac{||\boldsymbol{b}_1||}{||\boldsymbol{b}_i^*||}, \quad \theta = \frac{||\boldsymbol{b}_1||}{c_1}, \quad Q = \sum_{i=1}^{t-1} X_i^2, \quad \text{and} \quad R = \left(1 + \sum_{i=1}^{t} X_i\right)/2.$$

If the integers x_i are such that $|x_i| \le X_i$, for $1 \le i \le t$ and $\theta^2 \ge Q + R^2$, then we have

$$\left|\sum_{i=1}^{t} x_i\tau_i\right| \ge \frac{\sqrt{\theta^2 - Q} - R}{C}.$$

For the proof and further details, we refer the reader to the book of Cohen. (Proposition 2.3.20 in [6], pp. 58–63).

3.4. **Pell Equations and Dickson Polynomials.** Here we give some relations about Pell equations and Dickson polynomials that will be useful in the next section of this paper.

Let $d \geq 2$ be a square-free integer. We put $\delta = x_1 + \sqrt{x_1^2 - \epsilon}$ for the smallest positive integer x_1 such that

$$x_1^2 - dy_1^2 = \epsilon, \qquad \epsilon \in \{\pm 1\}$$

for some positive integer y_1. Then,

$$x_k + y_k\sqrt{d} = \delta^k \qquad \text{and} \qquad x_k - y_k\sqrt{d} = \eta^k, \qquad \text{where} \qquad \eta = \epsilon\delta^{-1}.$$

From the above, we get

$$2x_k = \delta^k + (\epsilon\delta^{-1})^k \qquad \text{for all} \qquad k \geq 1. \tag{3.6}$$

There is a formula expressing $2x_k$ in terms of $2x_1$ by means of the Dickson polynomial $D_k(2x_1, \epsilon)$, where

$$D_k(x, y) = \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{k}{k-i}\binom{k-i}{i}(-y)^i x^{k-2i}.$$

These polynomials appear naturally in many number theory problems and results, for example see a result of Bilu and Tichy [3] concerning polynomials $f(X), g(X) \in \mathbb{Z}[X]$ such that the Diophantine equation $f(x) = g(y)$ has infinitely many integer solutions (x, y).

Example 3.6. (i) $k = 2$. *We have*

$$2x_2 = \sum_{i=0}^{1} \frac{2}{2-i}\binom{2-i}{i}(-\epsilon)^i(2x_1)^{2-2i} = 4x_1^2 - 2\epsilon, \quad \textit{so} \quad x_2 = 2x_1^2 - \epsilon.$$

(ii) $k = 3$. *We have*

$$2x_3 = \sum_{i=0}^{1} \frac{3}{3-i}\binom{3-i}{i}(-\epsilon)^i(2x_1)^{3-2i} = (2x_1)^3 - 3\epsilon(2x_1), \quad \textit{so} \quad x_3 = 4x_1^3 - 3\epsilon x_1.$$

4. Bounding the Variables

We assume that (x_1, y_1) is the smallest positive solution of the Pell equation (1.3). As in Subsection 3.4, we set

$$x_1^2 - dy_1^2 = \epsilon, \qquad \epsilon \in \{\pm 1\},$$

and put

$$\delta = x_1 + \sqrt{d}y_1 \qquad \text{and} \qquad \eta = x_1 - \sqrt{d}y_1 = \epsilon\delta^{-1}.$$

From (1.4), we get

$$x_k = \frac{1}{2}\left(\delta^k + \eta^k\right). \tag{4.1}$$

Since $\delta \geq 1 + \sqrt{2} > \alpha^{3/2}$, it follows that the estimate

$$\frac{\delta^k}{\alpha^2} \leq x_k < \frac{\delta^k}{\alpha} \quad \text{holds for all} \quad k \geq 1. \tag{4.2}$$

We let $(k, n, m) = (k_i, n_i, m_i)$ for $i = 1, 2$ be the solutions of (2.1). By (1.2) and (4.2), we get

$$\alpha^{n+m-2} \leq L_n L_m = x_k < \frac{\delta^k}{\alpha} \quad \text{and} \quad \frac{\delta^k}{\alpha^2} \leq x_k = L_n L_m \leq \alpha^{n+m+4}, \tag{4.3}$$

so

$$kc_1 \log\delta - 6 < n + m < kc_1 \log\delta + 1 \quad \text{where} \quad c_1 = \frac{1}{\log\alpha}. \tag{4.4}$$

To fix ideas, we assume that

$$n \geq m \qquad \text{and} \qquad k_1 < k_2.$$

We also put

$$m_3 = \min\{m_1, m_2\}, \quad m_4 = \max\{m_1, m_2\}, \quad n_3 = \min\{n_1, n_2\}, \quad n_4 = \max\{n_1, n_2\}.$$

Using the inequality (4.4) together with $\delta \geq 1 + \sqrt{2} = \alpha^{3/2}$ (so, $c_1 \log \delta > 3/2$), gives us that

$$\frac{3}{2} k_2 < k_2 c_1 \log \delta < 2n_2 + 6 \leq 2n_4 + 6,$$

so

$$k_1 < k_2 < \frac{4}{3} n_4 + 4. \tag{4.5}$$

Thus, it is enough to find an upper bound on n_4. Substituting (1.1) and (4.1) in (2.1) we get

$$\frac{1}{2}(\delta^k + \eta^k) = (\alpha^n + \beta^m)(\alpha^m + \beta^m). \tag{4.6}$$

This can be regrouped as

$$\delta^k 2^{-1} \alpha^{-n-m} - 1 = -2^{-1} \eta^k \alpha^{-n-m} + (\beta\alpha^{-1})^n + (\beta\alpha^{-1})^m + (\beta\alpha^{-1})^{n+m}.$$

Since $\beta = -\alpha^{-1}$, $\eta = \varepsilon\delta^{-1}$ and using $\delta^k \geq \alpha^{n+m-1}$ (by (4.3)), we get

$$\begin{aligned} \left|\delta^k 2^{-1} \alpha^{-n-m} - 1\right| &\leq \frac{1}{2\delta^k \alpha^{n+m}} + \frac{1}{\alpha^{2n}} + \frac{1}{\alpha^{2m}} + \frac{1}{\alpha^{2(n+m)}} \\ &\leq \frac{\alpha}{2\alpha^{2(n+m)}} + \frac{3}{\alpha^{2m}} \quad < \quad \frac{6}{\alpha^{2m}}. \end{aligned}$$

In the above, we have also used $n \geq m$ and $(1/2)\alpha + 3 < 6$. Hence,

$$\left|\delta^k 2^{-1} \alpha^{-n-m} - 1\right| < \frac{6}{\alpha^{2m}}. \tag{4.7}$$

We let $\Lambda_1 = \delta^k 2^{-1} \alpha^{-n-m} - 1$. We put

$$\Gamma_1 = k \log \delta - \log 2 - (n+m) \log \alpha. \tag{4.8}$$

Note that $e^{\Gamma_1} - 1 = \Lambda_1$. If $m > 100$, then $\frac{6}{\alpha^{2m}} < \frac{1}{2}$. Since $|e^{\Gamma_1} - 1| < 1/2$, it follows that

$$|\Gamma_1| < 2|e^{\Gamma_1} - 1| < \frac{12}{\alpha^{2m}}. \tag{4.9}$$

By recalling that $(k, n, m) = (k_i, n_i, m_i)$ for $i = 1, 2$, we get that

$$|k_i \log \delta - \log 2 - (n_i + m_i) \log \alpha| < \frac{12}{\alpha^{2m_i}} \tag{4.10}$$

holds for both $i = 1, 2$ provided $m_3 > 100$.

We apply Theorem 3.1 on the left side of (4.7). First, we need to check that $\Lambda_1 \neq 0$. If it were, then $\delta^k \alpha^{-n-m} = 2$. However, this is impossible because $\delta^k \alpha^{-n-m}$ is a unit, whereas 2 is not. Thus, $\Lambda_1 \neq 0$, and we can apply Theorem 3.1. We take the data

$$t = 3, \quad \gamma_1 = \delta, \quad \gamma_2 = 2, \quad \gamma_3 = \alpha, \quad b_1 = k, \quad b_2 = -1, \quad b_3 = -n-m.$$

We take $\mathbb{K} = \mathbb{Q}(\sqrt{d}, \alpha)$, which has degree $D \leq 4$ (it could be that $d = 5$ in which case $D = 2$; otherwise, $D = 4$). Since $\delta \geq 1 + \sqrt{2} > \alpha$, the second inequality in (4.4) tells us that $k < n + m$, so we take $B = 2n$. We have $h(\gamma_1) = h(\delta) = \frac{1}{2} \log \delta$, $h(\gamma_2) = h(2) = \log 2$, and

$h(\gamma_3) = h(\alpha) = \frac{1}{2}\log\alpha$. Thus, we can take $A_1 = 2\log\delta$, $A_2 = 4\log 2$, and $A_3 = 2\log\alpha$. Now, Theorem 3.1 tells us that

$$\begin{aligned}\log|\Lambda_1| &> -1.4\times 30^6\times 3^{4.5}\times 4^2(1+\log 4)(1+\log(2n))(2\log\delta)(4\log 2)(2\log\alpha)\\ &> -2.92\times 10^{13}\log\delta(1+\log(2n)).\end{aligned}$$

By comparing the above inequality with (4.7), we get

$$2m\log\alpha - \log 6 < 2.92\times 10^{13}\log\delta(1+\log(2n)). \tag{4.11}$$

Thus,

$$m < 6.06\times 10^{13}\log\delta(1+\log(2n)). \tag{4.12}$$

Because $\delta^k < \alpha^{n+m+6}$, we get that

$$k\log\delta < (n+m+6)\log\alpha \le (2n+6)\log\alpha, \tag{4.13}$$

which with the estimate (4.12) gives

$$km < 5.84\times 10^{13}n(1+\log(2n)). \tag{4.14}$$

We have just proved the following lemma, which will be important later.

Lemma 4.1. *If $x_k = L_nL_m$ and $n \ge m$, then*

$$m < 6.06\times 10^{13}\log\delta(1+\log(2n)),\quad km < 5.84\times 10^{13}n(1+\log(2n)),\quad k\log\delta < 4n\log\alpha.$$

Note that we did not assume that $m_3 > 100$ for Lemma 4.1 because we have worked with the inequality (4.7) and not (4.9). We again assume that $m_3 > 100$. Then, the two inequalities (4.10) hold. We eliminate the term involving $\log\delta$ by multiplying the inequality for $i = 1$ with k_2 and the one for $i = 2$ with k_1, subtract them, and apply the triangle inequality as follows

$$\begin{aligned}&|(k_2-k_1)\log 2 - (k_2(n_1+m_1) - k_1(n_2+m_2))\log\alpha|\\ &= |k_2(k_1\log\delta - \log 2 - (n_1+m_1)\log\alpha) - k_1(k_2\log\delta - \log 2 - (n_2+m_2)\log\alpha)|\\ &\le k_2\,|k_1\log\delta - \log 2 - (n_1+m_1)\log\alpha| + k_1\,|k_2\log\delta - \log 2 - (n_2+m_2)\log\alpha|\\ &\le \frac{12k_2}{\alpha^{2m_1}} + \frac{12k_1}{\alpha^{2k_2}} \ < \ \frac{24k_2}{\alpha^{2m_3}}.\end{aligned}$$

Thus,

$$|\Gamma_2| = |(k_2-k_1)\log 2 - (k_2(n_1+m_1) - k_1(n_2+m_2))\log\alpha| < \frac{24k_2}{\alpha^{2m_3}}. \tag{4.15}$$

We are now set to apply Theorem 3.2 with the data

$$t = 2,\quad \gamma_1 = 2,\quad \gamma_2 = \alpha,\quad b_1 = k_2 - k_1,\quad b_2 = k_2(n_1+m_1) - k_1(n_2+m_2).$$

That $\gamma_1 = 2$ and $\gamma_2 = \alpha$ are multiplicatively independent follows because α is a unit whereas 2 is not. We observe that $k_2 - k_1 < k_2$, whereas by the absolute value of the inequality in (4.15), we have

$$|k_2(n_1+m_1) - k_1(n_2+m_2)| \le (k_2-k_1)\frac{\log 2}{\log\alpha} + \frac{24k_2}{\alpha^{2m_3}\log\alpha} < 2k_2,$$

because $m_3 > 10$. We have that $\mathbb{K} = \mathbb{Q}(\alpha)$, which has $D = 2$. So, we can take

$$\log B_1 = \max\left\{h(\gamma_1), \frac{|\log\gamma_1|}{2}, \frac{1}{2}\right\} = \log 2,$$

and

$$\log B_2 = \max\left\{h(\gamma_2), \frac{|\log\gamma_2|}{2}, \frac{1}{2}\right\} = \frac{1}{2}.$$

Thus,

$$b' = \frac{|k_2 - k_1|}{2\log B_2} + \frac{|k_2(n_1+m_1) - k_1(n_2+m_2)|}{2\log B_1} \le k_2 + \frac{k_2}{\log 2} < 3k_2.$$

Now, Theorem 3.2 tells us that with

$$\Gamma_2 = (k_2 - k_1)\log 2 - (k_2(n_1+m_1) - k_1(n_2+m_2))\log\alpha,$$

we have

$$\log|\Gamma_2| > -24.34 \times 2^4 \left(\max\{\log(3k_2) + 0.14, 10.5\}\right)^2 \cdot (2\log 2)\cdot(1/2).$$

Thus,

$$\log|\Gamma_2| > -270\left(\max\{\log(3k_2) + 0.14, 10.5\}\right)^2.$$

By comparing the above inequality with (4.15), we get

$$2m_3\log\alpha - \log(24k_2) < 270\left(\max\{\log(3k_2) + 0.14, 10.5\}\right)^2.$$

If $k_2 \le 10523$, then $\log(3k_2) + 0.14 < 10.5$. Thus, the last inequality above gives

$$2m_3\log\alpha < 270 \times 10.5^2 + \log(24 \times 10523),$$

giving $m_3 < 30942$ in this case. Otherwise, $k_2 > 10523$, and we get

$$2m_3\log\alpha < 272(1+\log k_2)^2 + \log(24k_2) < 280(1+\log k_2)^2,$$

which gives

$$m_3 < 160(1+\log k_2)^2.$$

We have just proved the following lemma.

Lemma 4.2. *If $m_3 > 100$, then either*

(i) $k_2 \le 10523$ *and* $m_3 < 30942$, *or*
(ii) $k_2 > 10523$, *in which case* $m_3 < 160(1+\log k_2)^2$.

Now suppose that some m is fixed in (2.1), or at least we have some good upper bounds on it. We rewrite (2.1) using (1.1) and (4.1) as

$$\frac{1}{2}(\delta^k + \eta^k) = L_m(\alpha^n + \beta^n),$$

so

$$\delta^k (2L_m)^{-1}\alpha^{-n} - 1 = -\frac{1}{2L_m}\eta^k\alpha^{-n} + (\beta\alpha^{-1})^n.$$

Since $m \ge 1$, $\beta = -\alpha^{-1}$, $\eta = \varepsilon\delta^{-1}$, and $\delta^k > \alpha^{n+m-1}$, we get

$$\begin{aligned}\left|\delta^k (2L_m)^{-1}\alpha^{-n} - 1\right| &\le \frac{1}{2L_m\delta^k\alpha^n} + \frac{1}{\alpha^{2n}} \le \frac{\alpha}{\alpha^{2(n+m)}} + \frac{1}{\alpha^{2n}}\\ &\le \frac{\alpha+1}{\alpha^{2n}} < \frac{6}{\alpha^{2n}},\end{aligned}$$

where we used $n \ge m \ge 0$ and $\alpha + 1 < 6$. Hence,

$$|\Lambda_3| = \left|\delta^k (2L_m)^{-1}\alpha^{-n} - 1\right| < \frac{6}{\alpha^{2n}}. \tag{4.16}$$

We assume that $n_3 > 100$. In particular, $\frac{6}{\alpha^{2n}} < \frac{1}{2}$ for $n \in \{n_1, n_2\}$, so we get by the previous argument that

$$|\Gamma_3| = |k \log \delta - \log(2L_m) - n \log \alpha| < \frac{12}{\alpha^{2n}}. \tag{4.17}$$

We are now set to apply Theorem 3.1 on the left side of (4.16) with the data

$$t = 3, \quad \gamma_1 = \delta, \quad \gamma_2 = 2L_m, \quad \gamma_3 = \alpha, \quad b_1 = k, \quad b_2 = -1, \quad b_3 = -n.$$

First, we need to check that $\Lambda_3 = \delta^k(2L_m)^{-1}\alpha^{-n} - 1 \neq 0$. If not, then $\delta^k = 2L_m\alpha^m$. The left side belongs to the field $\mathbb{Q}(\sqrt{d})$ but is not rational, whereas the right side belongs to the field $\mathbb{Q}(\sqrt{5})$. This is not possible unless $d = 5$. In this last case, δ is a unit in $\mathbb{Q}(\sqrt{5})$ whereas $2L_m$ is not a unit in $\mathbb{Q}(\sqrt{5})$ because the norm of this first element is $4L_m^2 \neq \pm 1$. So, $\Lambda_3 \neq 0$. Thus, we can apply Theorem 3.1. We have the field $\mathbb{K} = \mathbb{Q}(\sqrt{d}, \sqrt{5})$, which has degree $D \leq 4$. We also have

$$\begin{aligned} h(\gamma_2) &= h(2L_m) = h(2) + h(L_m) \\ &\leq \log 2 + (m+1)\log\alpha < 2 + m\log\alpha \\ &\leq 2.92 \times 10^{13} \log\delta(1 + \log(2n)) \quad \text{by (4.12)}. \end{aligned}$$

So, we take

$$h(\gamma_1) = \frac{1}{2}\log\delta, \quad h(\gamma_2) = 2.92 \times 10^{13}\log\delta(1 + \log(2n)), \quad \text{and} \quad h(\gamma_3) = \frac{1}{2}\log\alpha.$$

Then,

$$A_1 = 2\log\delta, \quad A_2 = 1.18 \times 10^{14}\log\delta(1 + \log(2n)), \quad \text{and} \quad A_3 = 2\log\alpha.$$

Then, by Theorem 3.1, we get

$$\begin{aligned} \log|\Lambda_3| &> -1.4 \times 30^6 \times 3^{4.5} \times 4^2(1 + \log 4)(1 + \log n)(2\log\delta) \\ &\quad \times (1.18 \times 10^{14}\log\delta(1 + \log(2n)))(2\log\alpha) \\ &> -8.6 \times 10^{26}(1 + \log(2n))^2(\log\delta)^2\log\alpha. \end{aligned}$$

Comparing the above inequality with (4.16), we get

$$2n\log\alpha - \log 6 < 8.6 \times 10^{26}(1 + \log(2n))^2(\log\delta)^2\log\alpha,$$

which implies that

$$n < 4.3 \times 10^{26}(1 + \log(2n))^2(\log\delta)^2. \tag{4.18}$$

We state what we have proved.

Lemma 4.3. *If $x_k = L_nL_m$ with $n \geq m \geq 1$, then we have*

$$n < 4.3 \times 10^{26}(1 + \log(2n))^2(\log\delta)^2.$$

Note that we did not use the assumption that $m_3 > 100$ or that $n_3 > 100$ for Lemma 4.3 because we worked with the inequality (4.16), not with the inequality (4.17). We now assume that $n_3 > 100$ and, in particular, (4.17) holds for $(k, n, m) = (k_i, n_i, m_i)$ for both $i = 1, 2$. By the previous procedure, we also eliminate the term involving $\log\delta$ as follows:

$$|k_2\log(2L_{m_1}) - k_1\log(2L_{m_2}) - (k_2n_1 - k_1n_2)\log\alpha| < \frac{12k_2}{\alpha^{2n_1}} + \frac{12k_1}{\alpha^{2n_2}} < \frac{24k_2}{\alpha^{2n_3}}. \tag{4.19}$$

We assume that $\alpha^{2n_3} > 48k_2$. If we put

$$\Gamma_4 = k_2\log(2L_{m_1}) - k_1\log(2L_{m_2}) - (k_2n_1 - k_1n_2)\log\alpha,$$

we have that $|\Gamma_4| < 1/2$. Then, we get that

$$|\Lambda_4| = |e^{\Gamma_4} - 1| < 2|\Gamma_4| < \frac{48k_2}{\alpha^{2n_3}}. \tag{4.20}$$

We apply Theorem 3.1 to

$$\Lambda_4 = (2L_{m_1})^{k_2}(2L_{m_2})^{-k_1}\alpha^{-(k_2n_1-k_1n_2)} - 1.$$

First, we need to check that $\Lambda_4 \neq 0$. If it were, then it would follow that

$$\frac{L_{m_1}^{k_2}}{L_{m_2}^{k_1}} = 2^{k_1-k_2}\alpha^{k_2n_1-k_1n_2}. \tag{4.21}$$

We consider the following Lemma.

Lemma 4.4. *Equation* (4.21) *has only many small positive integer solutions* (k_i, n_i, m_i) *for* $i = \{1, 2\}$ *with* $k_1 < k_2$ *and* $m_1 \leq m_2 \leq 6$. *Furthermore, none of these solutions lead to a valid solution to the original Diophantine equation* (2.1).

Proof. Suppose that (4.21) holds and assume that $\gcd(k_1, k_2) = 1$. Since $\alpha^{k_2n_1-k_1n_2} \in \mathbb{Q}$, it follows that $k_2n_1 = k_1n_2$. Thus, if one of the n_1 or n_2 is zero, so is the other. Since $n_i \geq m_i$ for $i \in \{1, 2\}$, it follows that $n_1 = n_2 = 0$ and $m_1 = m_2 = 0$, so $x_{k_1} = x_{k_2}$. Therefore $k_1 = k_2$, a contradiction. Thus, n_1 and n_2 are both positive integers. Next, $L_{m_1}^{k_2}/L_{m_2}^{k_1} = 2^{k_1-k_2} < 1$. Thus, $L_{m_1}^{k_2} < L_{m_2}^{k_1} < L_{m_2}^{k_2}$, so $L_{m_1} < L_{m_2}$. This implies that $(m_1, m_2) = (1, 0)$ or $m_1 < m_2$. The case $(m_1, m_2) = (1, 0)$ gives $1/2^{k_1} = 2^{k_1-k_2}$. Thus, $k_2 = 2k_1$ and since $\gcd(k_1, k_2) = 1$, we get $k_1 = 1$ and $k_2 = 2$, so $n_2 = 2n_1$. But then $x_2 = x_{k_2} = L_{n_2}L_{m_2} = L_{2n_1}L_0 = 2L_{2n_1}$ is even, a contradiction because $x_2 = 2x_1 \pm 1$ (by Example 3.6 (i)) is odd. Thus, $m_1 < m_2$. If $m_2 > 6$, the Carmichael Primitive Divisor Theorem for Lucas numbers shows that L_{m_2} is divisible by a prime $p > 7$, which does not divide L_{m_1}. This is impossible because it contradicts the assumption that (4.21) holds. Thus, $m_2 \leq 6$. Furthermore, because $L_{m_1}^{k_2}/L_{m_2}^{k_1} = 1/2^{k_2-k_1}$, it follows that $L_{m_1}^{k_1} \mid L_{m_1}^{k_2} \mid L_{m_2}^{k_1}$, so $L_{m_1} \mid L_{m_2}$. So, there are three cases that we must analyze:

Case 1. $m_1 = 0$ and $m_2 \in \{3, 6\}$. If $(m_1, m_2) = (0, 3)$, then $2^{k_2}/4^{k_1} = 1/2^{2k_1-k_2} = 1/2^{k_2-k_1}$. This gives $2k_2 = 3k_1$ and because k_1 and k_2 are coprime, it follows that $k_1 = 2$ and $k_2 = 3$. Then $x_2 = x_{k_1} = L_{n_1}L_{m_1} = L_{n_1}L_0 = 2L_{n_1}$ is even, a contradiction because $x_2 = 2x_1 \pm 1$ is odd. If $(m_1, m_2) = (0, 6)$, then $2^{k_2}/18^{k_1} = 1/2^{k_2-k_1}$, which is impossible because examining the exponent of 3 we would get $k_1 = 0$, a contradiction.

Case 2. $m_1 = 2$ and L_{m_2} is a power of 2. The case $m_2 = 0$ has been treated so the only other case left is $m_2 = 3$. In this case, $1/4^{k_1} = 1/2^{k_2-k_1}$, giving $k_2 = 3k_1$. Thus, since $\gcd(k_1, k_2) = 1$, $k_1 = 1$, and $k_2 = 3$. Since $k_2n_1 = k_1n_2$, we get $n_2 = 3n_1$. Thus, $x_1 = L_{n_1}L_1 = L_{n_1}$ and $x_3 = L_{3n_1}L_3 = 4L_{3n_1}$. Now, $x_3 = x_1(4x_1^2 \pm 3)$ (by Example 3.6 (ii)) and the second factor is odd, so the power of 2 dividing $4L_{3n_1}$ divides $x_1 = L_{n_1}$. But, $4L_{3n_1}$ is a multiple of 8 because L_{3n_1} is even. Thus, $8 \mid L_{n_1}$, which is false.

Case 3. $m_1 = 2$ and $m_2 = 6$. We get $3^{k_2}/(2.3^2)^{k_1} = 1/2^{k_2-k_1}$. Examining at the exponent of 3, we get $k_2 = 2k_1$ and examining the exponent of 2 we also get $k_2 = 2k_1$, so $k_1 = 1$ and $k_2 = 2$. Also, $n_2 = 2n_1$. Thus, $x_1 = L_{n_1}L_{m_1} = 3L_{n_1}$ and $x_2 = L_{n_2}L_{m_2} = 18L_{2n_1}$ is even, contradicting $x_2 = 2x_1^2 \pm 1$ is odd. □

So, by Lemma 4.4, we have $\Lambda_4 \neq 0$. Thus, we can now apply Theorem 3.1 with the data

$$t = 3, \quad \gamma_1 = 2L_{m_1}, \quad \gamma_2 = 2L_{m_2}, \quad \gamma_3 = \alpha, \quad b_1 = k_2,$$
$$b_2 = -k_1, \quad b_3 = -(k_2n_1 - k_1n_2).$$

We have $\mathbb{K} = \mathbb{Q}(\sqrt{5})$, which has degree $D = 2$. Also, using (4.5), we can take $B = 4n_4^2$. We can also take $A_1 = 2(2 + m_1 \log\alpha) \le 4m_1 \log\alpha$, $A_2 = 2(2 + m_2 \log\alpha) \le 4m_2 \log\alpha$ and $A_3 = \log\alpha$. Theorem 3.1 states that

$$\begin{aligned} \log|\Lambda_4| &> -1.4 \times 30^6 \times 3^{4.5} \times 2^2(1+\log 2)(1+\log(4n_4^2))(4m_1\log\alpha)(4m_2\log\alpha)\log\alpha, \\ &> -3.44 \times 10^{12} m_1 m_2 (1+\log(2n_4)). \end{aligned}$$

By comparing this with the inequality (4.20), we get

$$2n_3 \log\alpha - \log(48k_2) < 3.44 \times 10^{12} m_1 m_2 (1+\log(2n_4)).$$

Since $k_2 < 4n_4$ and $n_4 > 10$, we get that $\log(48k_2) < 2(1+\log(2n_4))$. Thus,

$$n_3 < 3.58 \times 10^{12} m_1 m_2 (1+\log(2n_4)). \tag{4.22}$$

All this was done under the assumption that $\alpha^{2n_3} > 48k_2$. But if that inequality fails, then

$$n_3 < c_1 \log(48k_2) < 12(1+\log(2n_4)),$$

which is much better than (4.22). Thus, (4.22) holds in all cases. We have just proved the following lemma.

Lemma 4.5. *Assuming* $n_3 > 100$, *we have*

$$n_3 < 3.58 \times 10^{12} m_1 m_2 (1+\log(2n_4)).$$

We now start finding effective bounds for our variables.

Case 1. $m_4 \le 100$.

Then $m_1 < 100$ and $m_2 < 100$. By Lemma 4.5, we get that

$$n_3 < 3.58 \times 10^{16}(1+\log(2n_4)).$$

By Lemma 4.1, we get

$$\log\delta < 4n_3\log\alpha < 6.89 \times 10^{16}(1+\log(2n_4)).$$

By inequality (4.4), we have that

$$\begin{aligned} n_4 &\le n_4 + m_4 - 1 \\ &< k_2 c_1 \log\delta \\ &< 1.72 \times 10^{27} c_1 (1+\log(2n_4))^2 (\log\delta)^3 \quad \text{(by (4.5) and Lemma 4.3)} \\ &< \frac{1}{\log\alpha}(1.72 \times 10^{27}(1+\log(2n_4))^2)(6.89 \times 10^{16}(1+\log(2n_4)))^3 \\ &< 1.17 \times 10^{78} \log(1+\log(2n_4))^5. \end{aligned}$$

With the help of *Mathematica*, we get that $n_4 < 4.6 \times 10^{89}$. Thus, using (4.5), we get

$$\max\{k_2, n_4\} < 4.6 \times 10^{89}.$$

We have just proved the following lemma.

Lemma 4.6. *If* $m_4 = \max\{m_1, m_2\} \le 100$, *then*

$$\max\{k_2, n_4\} < 4.6 \times 10^{89}.$$

Case 2. $m_4 > 100$.

Note that $m_3 \le 100$ or $m_3 > 100$ in which case, by Lemma 4.2 and inequality (4.5), we have $m_3 \le 160(1+\log(4n_4))^2$, provided that $m_4 > 10000$, which we now assume.

We let $i \in \{1,2\}$ be such that $m_i = m_3$ and j be such that $\{i,j\} = \{1,2\}$. We assume that $n_3 > 100$. We work with (4.17) for i and (4.10) for j and note that conditions $n_i > 100$ and $m_j = m_4 > 100$ are satisfied. That is,

$$\begin{aligned} |k_i \log\delta + \log(2L_{m_i}) - n_i \log\alpha| &< \frac{12}{\alpha^{2n_i}}, \\ |k_j \log\delta - \log 2 - (n_j + m_j)\log\alpha| &< \frac{12}{\alpha^{2m_j}}. \end{aligned}$$

By a similar procedure as before, we eliminate the term involving $\log\delta$. We multiply the first inequality by k_j, the second inequality by k_i, subtract the resulting inequalities, and apply the triangle inequality to get

$$\begin{aligned} |k_j \log(2L_{m_i}) - k_i \log 2 - (k_j n_i - k_i(n_j + m_j))\log\alpha| &< \frac{12k_j}{\alpha^{2m_i}} + \frac{12k_i}{\alpha^{2l_j}} \\ &< \frac{24k_2}{\alpha^{2\min\{n_i,m_j\}}}. \end{aligned} \tag{4.23}$$

Assume that $\alpha^{2\min\{n_i,m_j\}} > 48k_2$. We put

$$\Gamma_5 = k_j \log(2L_{m_i}) - k_i \log 2 - (k_j n_i - k_i(n_j + m_j))\log\alpha.$$

We can write $\Lambda_5 = (2L_{m_i})^{k_j} 2^{-k_i} \alpha^{(k_j n_i - k_i(n_j+m_j))} - 1$. Under the above assumption and using (4.23), we get that

$$|\Lambda_5| = |e^{\Gamma_5} - 1| < 2|\Gamma_5| < \frac{48k_2}{\alpha^{2\min\{n_i,m_j\}}}. \tag{4.24}$$

We are now set to apply Theorem 3.1 on Λ_5. First, we need to check that $\Lambda_5 \neq 0$. If it were, then we would get that

$$L_{m_i}^{k_j} = 2^{k_i - k_j} \alpha^{(k_j n_i - k_i(n_j + m_j))}. \tag{4.25}$$

We consider the following lemma.

Lemma 4.7. *The equation* (4.25) *has only many small positive integer solutions* $(k_i, k_j, n_i, n_j, m_i, m_j)$ *for* $i, j = \{1,2\}$ *with* $k_1 < k_2$ *and* $m_1 \le m_2 \le 6$. *Furthermore, none of these solutions lead to a valid solution to the original Diophantine equation* (2.1).

Proof. Suppose that (4.25) holds, and assume that $\gcd(k_1,k_2) = 1$. Since $\alpha^{(k_j n_i - k_i(n_j+m_j))} \in \mathbb{Q}$, $k_j n_i = k_i(n_j + m_j)$. Next, $L_{m_i}^{k_j} = 2^{k_i - k_j}$. Thus, $k_i \ge k_j$, so $i = 2$, $j = 1$, $k_2 > k_1$, and $m_2 \neq 1$. Because $L_{m_2} > 1$ is a power of 2, it follows that $m_2 \in \{0,3\}$. Suppose $m_2 = 0$. Then, $L_{m_2}^{k_1} = 2^{k_1} = 2^{k_2 - k_1}$, so $k_2 = 2k_1$. Hence, $k_1 = 1$ and $k_2 = 2$. Furthermore, $n_2 = 2(n_1 + m_1)$. Thus, $x_2 = x_{k_2} = L_{n_2} L_{m_2} = 2L_{2(n_1+m_1)}$ is even, which is false because $x_2 = 2x_1^2 \pm 1$ is odd. Next, suppose $m_2 = 3$. Then, $4^{k_1} = 2^{k_2 - k_1}$. Thus, $k_2 = 3k_1$, so $k_1 = 1$ and $k_2 = 3$. Next, $n_2 = 3(n_1 + m_1)$. Hence, $x_1 = x_{k_1} = L_{n_1} L_{m_1}$ and $x_3 = x_{k_2} = L_{n_2} L_{m_2} = 4L_{3(n_1+m_1)}$. By the previous argument in the proof of Lemma 4.4, 8 divides $x_3 = x_1(4x_1^2 \pm 1)$, so $8 \mid x_1$. Since $x_1 = L_{n_1} L_{m_1}$ and $8 \nmid L_n$ for any n, it follows that L_{n_1} and L_{m_1} are both even. Thus, $3 \mid n_1$ and $3 \mid m_1$. Furthermore, one of L_{n_1} or L_{m_1} is a multiple of 4, so one of n_1 or m_1 is odd. Suppose both are odd. Then $4 \mid L_{n_1}$, $4 \mid L_{m_1}$, so $16 \mid x_1 \mid x_3 \mid 4L_{3(n_1+m_1)}$. This implies that $4 \mid L_{3(n_1+m_1)}$, which is false because $3(n_1 + m_1)$ is an even multiple of 3, and $2 \| L_{6m}$ for any m. Now, suppose that one of n_1 or m_1 is an even multiple of 3, and the other is odd. Then $\operatorname{ord}_2(x_1) = 3$, where $\operatorname{ord}_2(x)$ is the exponent at which 2 appears in the factorization of x. Hence,

$$3 = \operatorname{ord}_2(x_3) = \operatorname{ord}_2(4L_{3(n_1+m_1)}) = 2 + \operatorname{ord}_2(L_{3(n_1+m_1)}),$$

giving $\mathrm{ord}_2(L_{3(n_1+m_1)}) = 1$. This is again false because $3(n_1+m_1)$ is an odd multiple 3, so it is a number of the form $3+6m$, and for such numbers we have $4\|L_{3+6m}$. Hence, in all instances we have obtained a contradiction. □

Thus, by Lemma 4.7, we have that $\Lambda_5 \neq 0$. So, we can apply Theorem 3.1 with the data

$$t = 3, \quad \gamma_1 = 2L_{m_i}, \quad \gamma_2 = 2, \quad \gamma_3 = \alpha, \quad b_1 = k_j,$$
$$b_2 = -k_i, \quad b_3 = -(k_j n_i - k_i(n_j + m_j)).$$

From the previous calculations, we know that $\mathbb{K} = \mathbb{Q}(\sqrt{2})$, which has degree $D = 2$ and $A_1 = 4m_i \log\alpha$, $A_2 = 2\log 2$, and $A_3 = \log\alpha$. We also take $B = 4n_4^2$. By Theorem 3.1, we get that

$$\begin{aligned} \log|\Lambda_5| &> -1.4 \times 30^6 \times 3^{4.5} \times 2^2(1+\log 2)(1+\log(4n_4^2))(4m_i\log\alpha)(2\log 2)\log\alpha, \\ &> -5.18 \times 10^{12} m_i(1+\log(2n_4)). \end{aligned}$$

Comparing the above inequality with (4.24), we get

$$2\min\{n_i, m_j\}\log\alpha - \log(48k_2) < 5.12 \times 10^{12} m_i(1+\log(2n_4)).$$

Since $m_4 > 100$, we get, using (4.5) ($k_2 < 4n_4$), that

$$\min\{n_i, n_j\} < 5.38 \times 10^{12}(160(1+\log(4n_4))^2)(1+\log(2n_4)) + \frac{c_1}{2}\log(192n_4),$$

which implies that

$$\min\{n_i, m_j\} < 1.72 \times 10^{15}(1+\log(2n_4))^3. \tag{4.26}$$

All this was under the assumptions that $n_4 > 10000$, and that $\alpha^{2\min\{n_i,m_j\}} > 48k_2$. But, still under the condition that $n_4 > 10000$, if $\alpha^{2\min\{n_i,m_j\}} < 48k_2$, then we get an inequality for $\min\{n_i, n_j\}$ that is much better than (4.26). So, (4.26) holds provided that $n_4 > 10000$. Suppose $\min\{n_i, m_j\} = m_j$. Then, we get

$$m_3 < 160(1+\log(4n_4))^2, \quad m_4 < 1.72 \times 10^{15}(1+\log(2n_4))^3.$$

By Lemma 4.5, since $m_3 > 100$, we get

$$\begin{aligned} n_3 &< (3.58 \times 10^{12})(160(1+\log(4n_4))^2)(1+\log(2n_4)) \\ &\quad \times 1.72 \times 10^{15}(1+\log(2n_4))^3 \\ &< 1.98 \times 10^{30}(1+\log(2n_4))^6. \end{aligned}$$

With Lemma 4.1, we get

$$\log\delta < 3.80 \times 10^{30}(1+\log(2n_4))^6,$$

which with Lemma 4.3 gives

$$n_4 < 4.30 \times 10^{26}(1+\log(2n_4))^2(3.80 \times 10^{30}(1+\log(2n_4))^6)^2,$$

which implies that

$$n_4 < 6.21 \times 10^{87}(1+\log(2n_4))^{14}. \tag{4.27}$$

With the help of *Mathematica*, we obtain $n_4 < 1.30 \times 10^{122}$. This was proved under the assumption that $n_4 > 10000$, but the situation $n_4 \leq 10000$ already provides a better bound than $n_4 < 1.30 \times 10^{122}$. Hence,

$$\max\{k_2, n_1, n_2\} < 1.30 \times 10^{122}. \tag{4.28}$$

This was when $m_j = \min\{n_i, m_j\}$. Now, assume $n_i = \min\{n_i, m_j\}$. Then, we get

$$n_i < 1.72 \times 10^{15}(1 + \log(2n_4))^3.$$

By Lemma 4.1, we get

$$\log \delta < 3.31 \times 10^{15}(1 + \log(2n_4))^3.$$

Now by Lemma 4.3, with Lemma 4.1 to bound l_4, give

$$\begin{aligned} n_4 &< 4.30 \times 10^{26}(1 + \log(2n_4)))^2(3.31 \times 10^{15}(1 + \log(2n_4))^3)^2 \\ &< 4.72 \times 10^{57}(1 + \log(2n_4))^{10}. \end{aligned}$$

This gives $n_4 < 2.44 \times 10^{80}$, which is a better bound than 1.30×10^{122}. We state what we have proved.

Lemma 4.8. *If $m_4 = \max\{m_1, m_2\} > 100$ and $n_3 = \min\{n_1, n_2\} > 100$, then*

$$\max\{k_2, n_1, n_2\} < 1.30 \times 10^{122}.$$

The remaining case is when $m_4 > 100$ and $n_3 \leq 100$. But then, by Lemma 4.1, we get $\log \delta < 192$. Now Lemma 4.1, with Lemma 4.3, give

$$n_4 < 1.56 \times 10^{31}(1 + \log(2n_4))^2,$$

which implies that $n_4 < 10^{36}$ and further $\max\{k_1, n_1, n_2\} < 10^{40}$. We state what we have proved.

Lemma 4.9. *If $m_4 > 100$ and $n_3 \leq 100$, then*

$$\max\{k_1, n_1, n_2\} < 10^{40}.$$

5. The Final Computations

5.1. **The First Reduction.** In this subsection, we reduce the bounds for k_1, m_1, n_1 and k_2, m_2, n_2 to cases that can be computationally treated. For this, we return to the inequalities for Γ_2, Γ_4, and Γ_5.

We return to (4.15), and we set $s = k_2 - k_1$ and $r = k_2(n_1 + m_1) - k_1(n_2 + m_2)$ and divide both sides by $s \log \alpha$ to get

$$\left| \frac{\log 2}{\log \alpha} - \frac{r}{s} \right| < \frac{24k_2}{\alpha^{2m_3} s \log \alpha}. \tag{5.1}$$

We assume that l_3 is so large that the right side of the inequality in (5.1) is smaller than $1/(2s^2)$. This certainly holds if

$$\alpha^{2m_3} > 48k_2^2/\log \alpha. \tag{5.2}$$

Since $k_2 < 1.3 \times 10^{122}$, it follows that the last inequality (5.2) holds provided that $m_3 \geq 589$, which we now assume. In this case, r/s is a convergent of the continued fraction of $\tau = \frac{\log 2}{\log \alpha}$ and $s < 1.30 \times 10^{122}$. We are now set to apply Lemma 3.3.

We write $\tau = [a_0; a_1, a_2, a_3, \ldots] = [1, 2, 3, 1, 2, 3, 2, 4, 2, 1, 2, 11, 2, 1, 11, 1, 1, 134, 2, 2, \ldots]$ for the continued fraction of τ and p_k/q_k for the kth convergent. We get that $r/s = p_j/q_j$ for some $j \leq 237$. Furthermore, putting $a(M) = \max\{a_j : j = 0, 1, \ldots, 237\}$, we get $a(M) = 880$. By Lemma 3.3, we get

$$\frac{1}{882s^2} = \frac{1}{(a(M)+2)s^2} \leq \left|\tau - \frac{r}{s}\right| < \frac{24k_2}{\alpha^{2m_3} s \log \alpha},$$

giving

$$\alpha^{2m_3} < \frac{882 \times 24k_2^2}{\log \alpha} < \frac{882 \times 24 \times (1.30 \times 10^{122})^2}{\log \alpha},$$

leading to $m_3 \le 1190$. We state what we have just proved.

Lemma 5.1. *We have* $m_3 = \min\{m_1, m_2\} \le 1190$.

If $m_1 = m_3$, then we have $i = 1$ and $j = 2$; otherwise $m_2 = m_3$ implying that we have $i = 2$ and $j = 1$. In both cases, the next step is the application of Lemma 3.5 (LLL algorithm) for (4.23), where $n_i < 1.30 \times 10^{112}$ and $|k_j n_i - k_i(n_j + m_j)| < 10^{116}$. For each $m_j \in [1, 1190]$ and

$$\Gamma_5 = k_j \log(2L_{m_i}) - k_i \log 2 - (k_j n_i - k_i(n_j + m_j)) \log \alpha, \tag{5.3}$$

we apply the LLL-algorithm on Γ_3 with the data

$$t = 3, \quad \tau_1 = \log(2L_{m_i}), \quad \tau_2 = \log 2, \quad \tau_3 = \log \alpha,$$
$$x_1 = k_j, \quad x_2 = -k_i, \quad x_3 = k_j n_i - k_i(n_j + m_j).$$

Furthermore, we set $X = 10^{116}$ as an upper bound to $|x_i|$ for $i = 1, 2, 3$, and $C = (5X)^5$. A computer search in *Mathematica* allows us to conclude, with the inequality (4.23), that

$$2 \times 10^{-480} < \min_{1 \le \min\{n_i, m_j\} \le 1190} |\Gamma_5| < \frac{24k_2}{\alpha^{2\min\{n_i, m_j\}}}. \tag{5.4}$$

Thus, $\min\{n_i, m_j\} \le 1419$. We first assume that $i = 1$ and $j = 2$. Thus, $n_1 \le 1419$ or $m_j = \min\{n_i, m_j\} \le 1419$.

Next, we suppose that $m_j = \min\{n_i, m_j\} \le 1419$. Because $m_1 = m_3 \le 1190$, we have

$$m_3 = \min\{m_1, m_2\} \le 1190 \quad \text{and} \quad m_4 = \max\{m_1, m_2\} \le 1419.$$

Now, returning to the inequality (4.19), which involves

$$\Gamma_4 = k_2 \log(2L_{m_1}) - k_1 \log(2L_{m_2}) - (k_2 n_1 - k_1 n_2) \log \alpha, \tag{5.5}$$

we again use the LLL algorithm to estimate the lower bound for $|\Gamma_4|$ and thus, find a bound for n_1 that is better than the one given in Lemma 4.8. We distinguish the cases $m_3 < m_4$ and $m_3 = m_4$.

5.1.1. *The Case* $m_3 < m_4$. We take $m_1 = m_3 \in [1, 1190]$ and $m_2 = m_4 \in [m_3 + 1, 1419]$ and apply Lemma 3.5 with the data:

$$t = 3, \quad \tau_1 = 2L_{m_1}, \quad \tau_2 = 2L_{m_2}, \quad \tau_3 = \log \alpha,$$
$$x_1 = k_2, \quad x_2 = -k_1, \quad x_2 = k_1 n_2 - k_2 n_1.$$

We also put $X = 10^{116}$ and $C = (20X)^9$. After a computer search in *Mathematica*, with the inequality (4.19), we confirm that

$$2 \times 10^{-1120} \le \min_{\substack{1 \le m_3 \le 1190 \\ m_3 + 1 \le m_4 \le 1419}} |\Gamma_4| < 24k_2 \alpha^{-2n_3}.$$

This leads to inequality

$$\alpha^{2n_3} < 12 \times 10^{1120} k_2.$$

Substituting for the bound k_2 given in Lemma 4.8, we get that $n_1 = n_3 \le 2950$.

5.1.2. *The Case $m_3 = m_4$.* In this case, $m_1 = m_2 \leq 1419$, and we have

$$\Gamma_4 = (k_2 - k_1)\log(2L_{m_1}) - (k_2 n_1 - k_1 n_2)\log\alpha \neq 0.$$

This is similar to the case we addressed in the previous steps and yields the bound on n_1, which is less than 2950. In both cases, we have $n_1 \leq 2950$. From

$$\log\delta \leq k_1 \log\delta \leq 4n_1 \log\alpha < 5678,$$

and by considering the inequality given in Lemma 4.3, we conclude that

$$n_2 < 1.4 \times 10^{34}(1 + \log(2n_2))^2,$$

which, with the help of *Mathematica*, yields $n_2 < 1.12 \times 10^{38}$. We summarize the first cycle of our reductions:

$$\max\{k_1, m_1\} \leq n_1 < 2950 \quad \text{and} \quad \max\{k_2, m_2\} \leq n_2 < 1.12 \times 10^{38}. \tag{5.6}$$

From (5.6), we note that the upper bound on n_2 represents a good reduction of the bound given in Lemma 4.8. Hence, we expect that if we restart our reduction cycle with the new bound on n_2, then we get better bounds on n_1 and n_2. Thus, we return to inequality (5.1) and take $M = 1.12 \times 10^{38}$. A computer search in *Mathematica* reveals that

$$q_{82} > M > n_2 > k_2 - k_1 \quad \text{and} \quad a(M) = \max\{a_i : 0 \leq i \leq 82\} = a_{12} = 134.$$

It follows that $m_3 \leq 100$. We now return to (5.3) and we put $X = 1.12 \times 10^{40}$ and $C = (20X)^5$ and then apply the LLL algorithm in Lemma 3.5 to $m_3 \in [1, 100]$. After a computer search in *Mathematica*, we get

$$1.04 \times 10^{-139} < \min_{1 \leq m_3 \leq 100} |\Gamma_4| < 24k_2\alpha^{-2\min\{n_i, m_j\}}.$$

Then, $\min\{n_i, m_j\} \leq 410$. By continuing under the assumption that $m_j = \min\{n_i, m_j\} \leq 426$, we return to (5.5) and put $X = 1.12 \times 10^{40}$, $C = (20X)^5$, and $M = 1.12 \times 10^{38}$ for the case $m_3 < m_4$ and the case $m_3 = m_4$. After a computer search, we confirm that

$$4.39 \times 10^{-168} < \min_{\substack{1 \leq m_3 \leq 100 \\ m_3+1 \leq m_4 \leq 426}} |\Gamma_4| < 24k_2\alpha^{-2n_3}. \tag{5.7}$$

This gives $n_1 \leq 494$, which holds in both cases. Hence, by a similar procedure given in the first cycle, we get that $n_2 < 3 \times 10^{36}$.

We state what we have proved.

Lemma 5.2. *Let $(k_i, n_i, m_i\}$ be a solution to the Diophantine equation $x_{k_i} = L_{n_i}L_{m_i}$, with $0 \leq m_i \leq n_i$ for $i = 1, 2$ and $1 \leq k_1 \leq k_2$. Then,*

$$\max\{k_1, m_1\} \leq n_1 \leq 494 \quad \textit{and} \quad \max\{k_2, m_2\} \leq n_2 < 3 \times 10^{36}.$$

5.2. **The Final Reduction.** Returning to (4.9) and (4.17) and using (x_1, y_1) as the smallest positive solution to the Pell equation (1.3), we obtain

$$\begin{aligned} x_k &= \frac{1}{2}(\delta^k + \eta^k) = \frac{1}{2}\left(\left(x_1 + y_1\sqrt{d}\right)^k + \left(x_1 - y_1\sqrt{d}\right)^k\right) \\ &= \frac{1}{2}\left(\left(x_1 + \sqrt{x_1^2 \mp 1}\right)^k + \left(x_1 - \sqrt{x_1^2 \mp 1}\right)^k\right) = P_k^{\pm}(x_1). \end{aligned}$$

Thus, we return to the Diophantine equation $x_{k_1} = L_{n_1}L_{m_1}$ and consider the equations

$$P_{k_1}^{+}(x_1) = L_{n_1}L_{m_1} \quad \text{and} \quad P_{k_1}^{-}(x_1) = L_{n_1}L_{m_1}, \tag{5.8}$$

with $k_1 \in [1, 500]$, $m_1 \in [0, 500]$, and $n_1 \in [m_1 + 1, 500]$.

Besides the trivial case $k_1 = 1$, with the help of a computer search in *Mathematica* on the above equations in (5.8), we list the only nontrivial solutions in Table 1. We also note that

$$7 + 5\sqrt{2} = (1 + \sqrt{2})^3,$$

so these solutions come from the same Pell equation with $d = 2$.

$Q^+_{k_1}(x_1)$				
k_1	x_1	y_1	d	δ
2	2	1	3	$2 + \sqrt{3}$
2	5	2	6	$5 + 2\sqrt{6}$
2	10	3	11	$10 + 3\sqrt{11}$
2	4	1	15	$4 + \sqrt{15}$
2	6	1	35	$6 + \sqrt{35}$

$Q^-_{k_1}(x_1)$				
k_1	x_1	y_1	d	δ
2	1	1	2	$1 + \sqrt{2}$
2	2	1	5	$2 + \sqrt{5}$
2	7	5	2	$7 + 5\sqrt{2}$
2	4	1	17	$4 + \sqrt{17}$
2	26	1	677	$26 + \sqrt{677}$
2	179	1	32042	$179 + \sqrt{32042}$

TABLE 1. Solutions to $P^{\pm}_{k_1}(x_1) = L_{n_1} L_{m_1}$

From Table 1, we set each $\delta = \delta_t$ for $t = 1, 2, \ldots 10$. We then work on the linear forms in logarithms Γ_1 and Γ_2, to reduce the bound on n_2 given in Lemma 5.2. From the inequality (4.10), for $(k, n, m) = (k_2, n_2, m_2)$, we write

$$\left| k_2 \frac{\log \delta_t}{\log \alpha} - (n_2 + m_2) + \frac{\log 2}{\log(\alpha^{-1})} \right| < \left(\frac{12}{\log \alpha} \right) \alpha^{-2m_2}, \tag{5.9}$$

for $t = 1, 2, \ldots 10$.

We put

$$\tau_t = \frac{\log \delta_t}{\log \alpha}, \qquad \mu_t = \frac{\log 2}{\log(\alpha^{-1})}, \qquad \text{and} \quad (A_t, B_t) = \left(\frac{12}{\log \alpha}, \alpha \right).$$

We note that τ_t is transcendental by the Gelfond-Schneider's Theorem and thus, τ_t is irrational. We can rewrite the above inequality, (5.9), as

$$0 < |k_2 \tau_t - (n_2 + m_2) + \mu_t| < A_t B_t^{-2m_2}, \quad \text{for} \quad t = 1, 2, \ldots, 10. \tag{5.10}$$

We take $M = 3 \times 10^{36}$, the upper bound on n_2 according to Lemma 5.2, and apply Lemma 3.4 to the inequality (5.10). As before, for each τ_t with $t = 1, 2, \ldots, 10$, we compute its continued fraction $[a_0^{(t)}, a_1^{(t)}, a_2^{(t)}, \ldots]$ and its convergents $p_0^{(t)}/q_0^{(t)}, p_1^{(t)}/q_1^{(t)}, p_2^{(t)}/q_2^{(t)}, \ldots$. For each case, by means of a computer search in *Mathematica*, we find an integer s_t such that

$$q_{s_t}^{(t)} > 18 \times 10^{36} = 6M \qquad \text{and} \qquad \varepsilon_t = ||\mu_t q^{(t)}|| - M||\tau_t q^{(t)}| > 0.$$

We finally compute all the values of $b_t = \lfloor \log(A_t q_{s_t}^{(t)}/\epsilon_t)/\log B_t \rfloor/2$. The values of b_t correspond to the upper bounds on m_2, for each $t = 1, 2, \ldots, 10$, according to Lemma 3.4.

Note that we have a problem at $\delta_7 = 2 + \sqrt{5}$. This is because

$$2 + \sqrt{5} = 2 \left(\frac{1 + \sqrt{5}}{2} \right)^2 = 2\alpha^2.$$

In this case, we have $\Gamma_1 = (k_2 - 1) \log 2 - (n_2 + m_2 - 2k_2) \log \alpha$. Thus,

$$\left| \frac{\log 2}{\log \alpha} - \frac{n_2 + m_2 - 2k_2}{k_2 - 1} \right| < \frac{12}{(k_2 - 1)\alpha^{2m_2} \log \alpha}.$$

By a similar procedure given in Subsection 5.1 with $M = 3\times 10^{36}$, we get that $q_{77} > M$ and $a(M) = \max\{a_i : 0 \le i \le 77\} = 134$. From this, we can conclude that $m_2 \le 96$.

The results of the computation for each t are recorded in Table 2.

t	δ_t	s_t	q_{s_t}	$\varepsilon_t >$	b_t
1	$2+\sqrt{3}$	68	2.07577×10^{37}	0.319062	94
2	$5+2\sqrt{6}$	91	8.19593×10^{37}	0.087591	97
3	$10+3\sqrt{11}$	67	2.25831×10^{38}	0.316767	96
4	$4+\sqrt{15}$	70	2.78896×10^{37}	0.329388	94
5	$6+\sqrt{35}$	74	1.75745×10^{38}	0.409752	96
6	$1+\sqrt{2}$	76	2.02409×10^{37}	0.263855	94
7	$2+\sqrt{5}$	–	–	–	96
8	$4+\sqrt{17}$	78	4.76137×10^{37}	0.131771	96
9	$26+\sqrt{677}$	65	3.17521×10^{37}	0.356148	94
10	$179+\sqrt{32042}$	77	3.45317×10^{37}	0.384127	94

Table 2. First reduction computation results

By replacing $(k,n,m) = (k_2,n_2,m_2)$ in the inequality (4.17), we can write

$$\left| k_2\frac{\log\delta_t}{\log\alpha} - n_2 + \frac{\log(2L_{m_2})}{\log(\alpha^{-1})}\right| < \left(\frac{12}{\log\alpha}\right)\alpha^{-2n_2}, \tag{5.11}$$

for $t = 1,2,\ldots,10$.

We now put

$$\tau_t = \frac{\log\delta_t}{\log\alpha},\quad \mu_{t,m_2} = \frac{\log(2L_{m_2})}{\log(\alpha^{-1})},\quad \text{and}\quad (A_t,B_t) = \left(\frac{12}{\log\alpha},\alpha\right).$$

With the above notations, we can rewrite (5.11) as

$$0 < |k_2\tau_t - n_2 + \mu_{t,m_2}| < A_tB_t^{-2n_2},\qquad \text{for}\quad t = 1,2,\ldots 10. \tag{5.12}$$

We again apply Lemma 3.4 to the inequality (5.12), with

$$t = 1,2,\ldots,10,\quad m_2 = 1,2,\ldots,b_t,\quad \text{with}\quad M = 3\times 10^{36}.$$

We take

$$\varepsilon_{t,m_2} = ||\mu_t q^{(t,m_2)}|| - M||\tau_t q^{(t,m_2)}|| > 0,$$

and

$$b_t = b_{t,m_2} = \lfloor \log(A_t q_{s_t}^{(t,m_2)}/\epsilon_{t,m_2})/\log B_t\rfloor/2.$$

The case $\delta_7 = 2+\sqrt{5}$ is again treated individually by a similar procedure as in the previous step. With the help of *Mathematica*, we record the results of the computation in Table 3.

t	1	2	3	4	5	6	7	8	9	10
$\varepsilon_{t,m_2} >$	0.0145	0.0002	0.0006	0.0034	0.0106	0.0005	–	0.0009	0.0019	0.0010
b_{t,m_2}	97	103	102	99	99	100	102	100	99	100

Table 3. Final reduction computation results

$$\text{Therefore,}\quad \max\{b_{t,m_2} : t = 1, 2, \ldots, 10 \quad \text{and} \quad m_2 = 1, 2, \ldots b_t\} \leq 103.$$

Thus, by Lemma 3.4, we have that $n_2 \leq 103$ for all $t = 1, 2, \ldots, 10$. From $\delta^k \leq \alpha^{n+m+6}$, we conclude that $k_1 < k_2 \leq 198$. Collecting everything together, our problem is reduced to search for the solutions for (2.1) in the following ranges

$$1 \leq k_1 < k_2 \leq 200, \quad 0 \leq m_1 \leq n_1 \leq 200, \quad \text{and} \quad 0 \leq m_2 \leq n_2 \leq 200.$$

After a computer search on the equation (2.1) with the above ranges, we obtained the following solutions, which are the only solutions for the exceptional d cases we stated in Theorem 2.1:

For the $+1$ case:

$$\begin{aligned} (d = 3) &\qquad x_1 = 2 = L_1 L_0, \quad x_2 = 7 = L_4 L_1; \\ (d = 15) &\qquad x_1 = 4 = L_3 L_1 = L_0 L_0, \quad x_5 = 15124 = L_{11} L_9; \\ (d = 35) &\qquad x_1 = 6 = L_2 L_0, \quad x_3 = 846 = L_8 L_6. \end{aligned}$$

For the -1 case:

$$\begin{aligned} (d = 2) &\qquad x_1 = 1 = L_3 L_3, \quad x_2 = 3 = L_2 L_1, \quad x_3 = 7 = L_4 L_1, \quad x_9 = 1393 = L_{11} L_4; \\ (d = 5) &\qquad x_1 = 2 = L_1 L_0, \quad x_2 = 9 = L_2 L_2; \\ (d = 17) &\qquad x_1 = 4 = L_3 L_1 = L_0 L_0, \quad x_2 = 33 = L_5 L_2. \end{aligned}$$

This completes the proof of Theorem 2.1. □

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