ANOTHER PROOF FOR PARTIAL STRONG DIVISIBILITY PROPERTY OF LUCAS-TYPE POLYNOMIALS

RIGOBERTO FLÓREZ AND ROBINSON A. HIGUITA

Abstract. A second order polynomial sequence $\mathcal{L}_n(x)$ is of *Lucas-type* if its Binet formula has a structure similar to Lucas numbers. This sequence partially satisfies the strong divisibility property [1]. Thus, $\gcd(\mathcal{L}_n(x), \mathcal{L}_m(x))$ is 1, 2, or $\mathcal{L}_{\gcd(n,m)}(x)$. In this paper, we give a short, simple, and different proof of this property.

1. Introduction and Basic Definitions

A second order polynomial sequence is of *Lucas-type* (*Fibonacci-type*) if its Binet formula has a structure similar to Lucas (Fibonacci) numbers. Some known examples of Lucas-type polynomials are Lucas polynomials, Pell-Lucas polynomials, Fermat-Lucas polynomials, Chebyshev polynomials, and Vieta-Lucas polynomials.

A second order recursive sequence a_n satisfies the *strong divisibility property* if $\gcd(a_n, a_m) = a_{\gcd(n,m)}$. Flórez, et al. [1] proved that Lucas-type polynomials $\mathcal{L}_n(x)$ partially satisfy the strong divisibility property. Thus, $\gcd(\mathcal{L}_n(x), \mathcal{L}_m(x))$ is 1, 2, or $\mathcal{L}_{\gcd(n,m)}(x)$. In this paper, we give a shorter and simpler proof of this property (it is valid for both polynomials and Lucas numbers). It is based on the generalization of the numerical identity $L_nF_n = F_{2n}$ and the strong divisibility property of Fibonacci-type polynomials. Note that McDaniel [4] proved this property for Lucas numbers. If in (1.2) we set $p_0 = 2$ and $p_1(x) = x$, we have the Lucas polynomials. Lucas polynomials are a generalization of Lucas numbers. We can obtain these numbers by evaluating the Lucas polynomials at $x = 1$. Therefore, our proof is also a short proof of McDaniel result [4].

We now summarize some concepts given by the authors in earlier articles for generalized Fibonacci polynomials [1, 2]. If $d(x)$ and $g(x)$ are fixed non-zero polynomials in $\mathbb{Q}[x]$ and $n \geq 2$, then we define

$$\mathcal{F}_0(x) = 0,\ \mathcal{F}_1(x) = 1,\ \text{and}\ \mathcal{F}_n(x) = d(x)\mathcal{F}_{n-1}(x) + g(x)\mathcal{F}_{n-2}(x). \tag{1.1}$$

A second order polynomial recurrence relation is of *Fibonacci-type* if it satisfies the relation given in (1.1), and of *Lucas-type* if

$$\mathcal{L}_0(x) = p_0,\ \mathcal{L}_1(x) = p_1(x),\ \text{and}\ \mathcal{L}_n(x) = d(x)\mathcal{L}_{n-1}(x) + g(x)\mathcal{L}_{n-2}(x), \tag{1.2}$$

where $|p_0| = 1$ or 2 and $p_1(x)$, $d(x) = \alpha p_1(x)$, and $g(x)$ are fixed non-zero polynomials in $\mathbb{Q}[x]$ with α an integer of the form $2/p_0$.

If $n \geq 0$ and $d^2(x) + 4g(x) > 0$, then the Binet formulas for the recurrence relations in (1.1) and (1.2) are $\mathcal{F}_n(x) = (a^n(x) - b^n(x))\,/\,(a(x) - b(x))$ and $\mathcal{L}_n(x) = (a^n(x) + b^n(x))\,/\alpha$. (For details on the construction of the two Binet formulas, see [1].)

A sequence of Lucas-type (Fibonacci-type) is *equivalent* or *conjugate* to a sequence of Fibonacci-type (Lucas-type), if their recursive sequences are determined by the same polynomials $d(x)$ and $g(x)$. Note that two equivalent polynomials have the same $a(x)$ and $b(x)$ in their Binet representations. In this paper, we suppose that $\mathcal{F}_t(x)$ and $\mathcal{L}_t(x)$ are equivalent if they are used in the same statement.

Most of the following conditions were required in the papers that we are citing. Therefore, we require here that $\gcd(d(x), g(x)) = 1$ for both types of sequences and $\gcd(p_0, p_1(x)) = 1$, $\gcd(p_0, d(x)) = 1$, and $\gcd(p_0, g(x)) = 1$ for Lucas type polynomials.

2. Lucas-Type Polynomials Partial Divisibility Property

In this section, we prove that the Lucas-type polynomials partially satisfy the strong divisibility property. However, we need some results from [1, 3].

For brevity, throughout the rest of the paper, we present polynomials without the "x". For example, instead of $\mathcal{F}_n(x)$ and $\mathcal{L}_n(x)$ we use $\mathcal{F}_n$ and $\mathcal{L}_n$.

Lemma 2.1 ([1])**.** *Let p, q, r, and s be polynomials in $\mathbb{Q}[x]$. If $\gcd(p, r) = 1$ and $\gcd(q, s) = 1$, then $\gcd(pq, rs) = \gcd(p, s)\gcd(q, r)$.*

Lemma 2.2. *If $m > 0$, then*

(1) $\alpha\mathcal{L}_m\mathcal{F}_m = \mathcal{F}_{2m}$.
(2) $\gcd(g, \mathcal{L}_m) = \gcd(g, \mathcal{L}_1) = 1$.
(3) If q is odd and $q \mid m$, then $\mathcal{L}_{m/q}$ divides $\mathcal{L}_m$.
(4) There is a polynomial T_m such that $\mathcal{L}_{2^m q} = \mathcal{L}_q T_m + \mathcal{L}_0 g^{q2^{m-1}}$.
(5) If $n|m$, then

$$\gcd(\mathcal{L}_n, \mathcal{L}_m) = \begin{cases} \mathcal{L}_n, & \text{if } m/n \text{ is odd;} \\ \gcd(\mathcal{L}_n, \mathcal{L}_0), & \text{otherwise.} \end{cases}$$

Proof. The proof of Part (1) is in [3] and the proofs of Parts (2), (3), and (4) are in [1]. We prove Part (5). If m/n is odd, then the proof follows from Part (3). Suppose that $m = 2^k nl$, where $k > 0$ and l is odd. From Part (3) we have $\mathcal{L}_{nl} = \mathcal{L}_n Q$ for some polynomial $Q \in \mathbb{Q}[x]$. This and Part (4) imply that there is a polynomial T such that $\mathcal{L}_m = \mathcal{L}_{2^k nl} = \mathcal{L}_{nl}T + \mathcal{L}_0 g^{nl2^{k-1}} = \mathcal{L}_n TQ + \mathcal{L}_0 g^{nl2^{k-1}}$. This implies that $\gcd(\mathcal{L}_n, \mathcal{L}_m) = \gcd(\mathcal{L}_n, \mathcal{L}_0 g^{nl2^{k-1}}) = \gcd(\mathcal{L}_n, \mathcal{L}_0)$. □

Theorem 2.3. *If $n, m > 0$, $\delta = \gcd(n, m)$, and $\nu_2(n)$ is the 2-adic valuation of n, then*

$$\gcd(\mathcal{L}_n, \mathcal{L}_m) = \begin{cases} \mathcal{L}_\delta, & \text{if } \nu_2(n) = \nu_2(m); \\ \gcd(\mathcal{L}_\delta, \mathcal{L}_0), & \text{otherwise.} \end{cases}$$

Proof. Let D be $\gcd(\mathcal{F}_m\mathcal{F}_{2n}, \mathcal{F}_{2m}\mathcal{F}_n)$. Since $\delta = \gcd(n, m)$, it is easy to see that

$$D = \mathcal{F}_{2\delta}\mathcal{F}_\delta \gcd\left(\left(\mathcal{F}_m/\mathcal{F}_\delta\right)\left(\mathcal{F}_{2n}/\mathcal{F}_{2\delta}\right), \left(\mathcal{F}_n/\mathcal{F}_\delta\right)\left(\mathcal{F}_{2m}/\mathcal{F}_{2\delta}\right)\right).$$

This and Lemma 2.1 imply that

$$D = \mathcal{F}_{2\delta}\mathcal{F}_\delta \gcd\left(\left(\mathcal{F}_m/\mathcal{F}_\delta\right), \left(\mathcal{F}_{2m}/\mathcal{F}_{2\delta}\right)\right)\gcd\left(\left(\mathcal{F}_{2n}/\mathcal{F}_{2\delta}\right), \left(\mathcal{F}_n/\mathcal{F}_\delta\right)\right).$$

(Recall that if $\mathcal{F}_t$ and $\mathcal{L}_t$ are in the same statement, they are equivalent.) From Lemma 2.2 Part (1), we have that D is equal to

$$\mathcal{F}_{2\delta}\mathcal{F}_\delta \gcd\left(\frac{\mathcal{F}_m}{\mathcal{F}_\delta}, \frac{\mathcal{F}_m\mathcal{L}_m}{\mathcal{F}_\delta\mathcal{L}_\delta}\right)\gcd\left(\frac{\mathcal{F}_n}{\mathcal{F}_\delta}, \frac{\mathcal{F}_n\mathcal{L}_n}{\mathcal{F}_\delta\mathcal{L}_\delta}\right) = \frac{\mathcal{F}_n\mathcal{F}_m\mathcal{F}_{2\delta}\mathcal{F}_\delta}{(\mathcal{F}_\delta\mathcal{L}_\delta)^2}\gcd\left(\mathcal{L}_\delta, \mathcal{L}_m\right)\gcd\left(\mathcal{L}_\delta, \mathcal{L}_n\right).$$

Therefore,

$$\gcd(\mathcal{F}_m\mathcal{F}_{2n}, \mathcal{F}_{2m}\mathcal{F}_n) = \left(\alpha\mathcal{F}_n\mathcal{F}_m/\mathcal{L}_\delta\right)\gcd\left(\mathcal{L}_\delta, \mathcal{L}_m\right)\gcd\left(\mathcal{L}_\delta, \mathcal{L}_n\right). \tag{2.1}$$

We now consider two cases.

Case 1. $\nu_2(n) = \nu_2(m)$. Thus, $\nu_2(m) = \nu_2(n) = \nu_2(\delta)$. This, (2.1), and Lemma 2.2 Part (3) imply that $\gcd(\mathcal{F}_m\mathcal{F}_{2n}, \mathcal{F}_{2m}\mathcal{F}_n) = (\alpha\mathcal{F}_n\mathcal{F}_m/\mathcal{L}_\delta)\,\mathcal{L}_\delta^2 = \alpha\mathcal{F}_n\mathcal{F}_m\mathcal{L}_\delta$. From this and Lemma 2.2 Part (1), we have

$$D = \gcd(\mathcal{F}_m\mathcal{F}_{2n}, \mathcal{F}_{2m}\mathcal{F}_n) = \gcd(\mathcal{F}_m\alpha\mathcal{L}_n\mathcal{F}_n, \alpha\mathcal{L}_m\mathcal{F}_m\mathcal{F}_n) = \alpha\mathcal{F}_n\mathcal{F}_m\gcd(\mathcal{L}_n, \mathcal{L}_m).$$

Therefore, $\gcd(\mathcal{L}_n, \mathcal{L}_m) = \mathcal{L}_\delta$.

Case 2. $\nu_2(n) \neq \nu_2(m)$. Without loss of generality, suppose that $\nu_2(n) < \nu_2(m)$. So, $\nu_2(\delta) = \nu_2(n)$. This implies that m/δ is even, therefore, by Lemma 2.2 Part (5), we have $\gcd(\mathcal{L}_\delta, \mathcal{L}_m) = \gcd(\mathcal{L}_\delta, \mathcal{L}_0)$. Lemma 2.2 Part (3) and $\nu_2(\delta) = \nu_2(n)$ imply that there is a $Q \in \mathbb{Q}[x]$ such that $\mathcal{L}_n = \mathcal{L}_\delta Q$. Therefore, $\gcd(\mathcal{L}_\delta, \mathcal{L}_n) = \mathcal{L}_\delta$. This, (2.1), and $\gcd(\mathcal{L}_\delta, \mathcal{L}_m) = \gcd(\mathcal{L}_\delta, \mathcal{L}_0)$ imply that $\gcd(\mathcal{L}_n, \mathcal{L}_m) = \gcd(\mathcal{L}_\delta, \mathcal{L}_0)$. □

References

[1] R. Flórez, R. Higuita, and A. Mukherjee, *Characterization of the strong divisibility property for generalized Fibonacci polynomials*, Integers, **18** (2018), Paper No. A14.

[2] R. Flórez, R. Higuita, and A. Mukherjee, *The star of David and other patterns in Hosoya polynomial triangles*, Journal of Integer Sequences, **21** (2018), Article 18.4.6.

[3] R. Flórez, N. McAnally, and A. Mukherjee, *Identities for the generalized Fibonacci polynomial*, Integers, **18B** (2018), Paper No. A2.

[4] W. McDaniel, *The g.c.d. in Lucas sequences and Lehmer number sequences*, The Fibonacci Quarterly, **29.1** (1991), 24–29.

MSC2010: Primary 11B39; Secondary 11B83.

Department of Mathematical Sciences, The Citadel, Charleston, SC, USA
E-mail address: rflorez1@citadel.edu

Instituto de Matemáticas, Universidad de Antioquia, Medellín
E-mail address: robinson.higuita@udea.edu.co