POSITIVE INTEGER SOLUTIONS OF SOME DIOPHANTINE EQUATIONS INVOLVING LUCAS-BALANCING NUMBERS

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ABSTRACT. This paper is devoted to solving the equations $x^s - 8C_nxy + 16y^t = \pm 2^r$ for $(s,t) \in \{(2,2),(2,4),(4,2)\}$ in positive integers x and y. The solutions are obtained in terms of the balancing, Pell, and Lucas-Pell numbers.

1. Introduction

A pair of natural numbers (n,r), where n is a balancing number and r is the corresponding balancer, is a solution of the Diophantine equation $1+2+\cdots+(n-1) = (n+1)+\cdots+(n+r)$ [1]. If n is a balancing number, then $8n^2+1$ is a perfect square, and the positive square root of $8n^2+1$ is called a Lucas-balancing number. The nth balancing and the nth Lucas-balancing numbers are denoted by B_n and C_n respectively, and satisfy the binary recurrences $B_{n+1} = 6B_n - B_{n-1}$ with $B_0 = 0$, $B_1 = 1$, and $C_{n+1} = 6C_n - C_{n-1}$ with $C_0 = 1$ and $C_1 = 3$ [15, 18].

The Diophantine equations of the form $ax^2 + bxy + cy^2 = d$, for different values of a, b, c, and d have been studied by many authors [3, 4, 5]. Keskin, et al. [9] obtained the solutions of $x^2 - 5F_nxy - 5(-1)^ny^2 = \pm 5^r$ in positive integers x and y. Demirturk, et al. [7] proved that the equation $x^2 - L_nxy + (-1)^ny^2 = 5$ has positive integer solutions only when $n = 1,2,3,4$, whereas $x^2 - L_nxy + (-1)^ny^2 = -5$ has solutions in positive integers only for $n = 1,2,3$. Keskin, et al. [10] studied the Diophantine equation $x^2 - L_nxy + (-1)^ny^2 = \pm 5^r$. Karaatli, et al. [8] and Patel, et al. [17] studied some Diophantine equations involving balancing numbers. In the present work, we solve the equations $x^s - 8C_nxy + 16y^t = \pm 2^r$ in positive integers x and y when $(s,t) \in \{(2,2),(2,4),(4,2)\}$. These solutions are expressed in terms of Pell, Lucas-Pell, balancing and Lucas-balancing numbers.

The Pell sequence $\{P_n\}$ and Lucas-Pell sequence $\{Q_n\}$ are defined by means of the binary recurrences $P_n = 2P_{n-1} + P_{n-2}$, $P_0 = 0$, $P_1 = 1$, and $Q_n = 2Q_{n-1} + Q_{n-2}$, $Q_0 = 2$, $Q_1 = 2$. The Binet forms of these sequences are given by $P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $Q_n = \alpha^n + \beta^n$, respectively, where $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$. These two sequences share interesting relationships with balancing and Lucas-balancing sequences [16].

The generalized Fibonacci sequence $\{U_n\}$ is defined by the binary recurrence

$$U_n = AU_{n-1} - BU_{n-2}, \quad n \geq 2, \tag{1.1}$$

with initial terms $U_0 = 0$ and $U_1 = 1$. Similarly, the generalized Lucas sequence $\{V_n\}$ defined by

$$V_n = AV_{n-1} - BV_{n-2}, \quad n \geq 2, \tag{1.2}$$

with $V_0 = 2$ and $V_1 = A$. If $A = 6$ and $B = -1$, then the generalized Fibonacci and Lucas sequences coincide with the balancing sequence and the Lucas-balancing sequence, respectively.

2. Preliminaries

In this section, we recall some results about balancing, Lucas-balancing, Pell, and Lucas-Pell sequences. The contents of this section will help us develop the main results. We will refer back to these results whenever necessary, with or without further reference.

As usual, we denote the nth balancing, nth Lucas-balancing, nth Pell, and nth Lucas-Pell numbers by B_n, C_n, P_n, and Q_n, respectively, and $v_n = 2C_n$; $n = 0, 1, 2, \ldots$. Throughout this paper, unless otherwise mentioned, m, n, a, b, x, etc. denote integers.

Lemma 2.1. *If m and n are positive integers, then*

(a) $v_n^2 - 32B_n^2 = 4$,
(b) $B_{2n} = B_n v_n$,
(c) $Q_n^2 - 8P_n^2 = 4(-1)^n$,
(d) $B_m v_n + B_n v_m = 2B_{m+n}$,
(e) $\gcd(B_n, v_n) = 1$ *or* 2,
(f) $P_{2n} = P_n Q_n = 2B_n$,
(g) $\gcd(P_n, Q_n) = 1$ *or* 2,
(h) $P_m | P_n$ *if and only if* $m|n$,
(i) $Q_m | Q_n$ *if and only if* $\frac{n}{m}$ *is odd,*
(j) $Q_m | P_n$ *if and only if* $\frac{n}{m}$ *is even,*
(k) $5 \nmid Q_n$ *for any* n,
(l) $2 \nmid B_n \Leftrightarrow 2 \nmid n \Leftrightarrow 2 \nmid P_n$.

Some results of Lemma 2.1 are also true if either m or n equals 0.

The proofs of assertions (a), (b), (d), and (e) can be found in [18]. For proofs of the other assertions, see [11].

Lemma 2.2. *If n is an odd positive integer, then $B_n \equiv n \pmod{32}$.*

Proof. By virtue of Theorem 2.2 of [6], the identity

$$\sum_{r=1}^{n} B_r^2 = \frac{1}{32}\Big[B_{2n+1} - (2n+1)\Big] \tag{2.1}$$

holds for each positive integer n. The proof of the lemma follows directly from (2.1). □

Lemma 2.3. *If n is an even positive integer, then $B_n \equiv 3n \pmod{32}$.*

Proof. By virtue of Lemma 2.1(d), $B_{2n+2} + B_{2n-2} = 34B_{2n}$ and hence, the sequence $x_n = B_{2n}$, satisfies the recurrence relation

$$x_{n+1} = 34x_n - x_{n-1}, \quad x_0 = 0, \quad x_1 = 6. \tag{2.2}$$

Using the recurrence relation (2.2), it is easy to see that the sequence $y_n = x_n - 6n$ satisfies the nonhomogeneous recurrence

$$y_{n+1} = 34y_n - y_{n-1} + 192n, \quad y_0 = y_1 = 0. \tag{2.3}$$

With the help of mathematical induction, the conclusion of the lemma follows from (2.3). □

Lemma 2.4. ([8], Corollary 3.4). *If n is a nonnegative integer, then*

$$C_n \equiv \begin{cases} 1 \pmod{16}, & \text{if } 2|n; \\ 3 \pmod{16}, & \text{if } 2 \nmid n. \end{cases} \tag{2.4}$$

Lemma 2.5. ([8], Theorem 3.9). *For $n \geq 0$, there is no positive integer x such that $v_n = x^2$.*

Lemma 2.6. ([8], Theorem 3.12). *For $m \geq 0$ and $n \geq 1$, there is no positive integer x such that $B_n = 2v_m x^2$.*

Lemma 2.7. ([19], Theorem 3.3). *If $m \geq 0$ and $n \geq 1$, then C_n divides B_m if and only if $\frac{m}{n}$ is an even integer.*

Lemma 2.8. ([15], Theorem 2.8, Theorem 2.13). *If $m \geq 1$ and $n \geq 1$, then B_n divides B_m if and only if $n|m$. Consequently, $\gcd(B_m, B_n) = B_{\gcd(m,n)}$.*

Lemma 2.9. ([19], Theorem 3.2). *If $m \geq 1$ and $n \geq 1$, then C_n divides C_m if and only if $\frac{m}{n}$ is an odd integer.*

Lemma 2.10. ([17], Theorem 3.2). *If $m \geq 1$ and $n \geq 1$, then $B_n = B_m x^2$ has no solution except $x = 1$.*

Lemma 2.11. ([8], Theorem 3.10). *If $n \geq 0$ and $x > 0$ are integers such that $v_n = 2x^2$, then $(n, x) = (0, 1)$.*

Lemma 2.12. ([14], Theorem 3). *If for $0 \leq m < n$, $Q_m Q_n = x^2$, then $n = 3m$, $3 \nmid m$, and m is odd.*

Lemma 2.13. *If $n \geq 1$, then the equation $P_n = x^2$ has the positive integer solutions $(n, x) = (1, 1)$ or $(7, 13)$.*

Proof. For the proof of Lemma 2.13, see [12]. □

Lemma 2.14. ([2], Lemma 2.5). *If n, y, and m are positive integers with $m \geq 2$, then the equation $Q_n = 2y^m$ has the only integer solution $(n, y) = (1, 1)$.*

Lemma 2.15. ([13], p.1). *Let $m = 2^a m'$ and $n = 2^b n'$, where m' and n' are odd and $a, b \geq 0$ with $d = \gcd(m, n)$. Then,*

$$\gcd(U_m, U_n) = U_d.$$

$$\gcd(V_m, V_n) = \begin{cases} V_d, & \text{if } a = b; \\ 1 \text{ or } 2, & \text{if } a \neq b. \end{cases}$$

$$\gcd(U_m, V_n) = \begin{cases} V_d, & \text{if } a > b; \\ 1 \text{ or } 2, & \text{if } a \leq b. \end{cases}$$

Lemma 2.16. ([13], p.1). *If $d = \gcd(m, n)$, then*

$$\gcd(P_m, Q_n) = \begin{cases} Q_d, & \text{if } m/d \text{ is even}; \\ 1 \text{ or } 2, & \text{otherwise}. \end{cases}$$

Theorem 2.17. ([8], Theorem 4.1). *If k is a nonnegative integer, then all positive integer solutions of the equation $u^2 - 2v^2 = 2^k$ are given by*

$$(u, v) = \begin{cases} (2^{\frac{k-2}{2}} v_m, 2^{\frac{k+2}{2}} B_m), & \text{if } k \text{ is even}; \\ (2^{\frac{k+1}{2}} P_{2m+1}, 2^{\frac{k-3}{2}} Q_{2m+1}), & \text{if } k \text{ is odd}. \end{cases}$$

Furthermore, all positive integer solutions of the equation $u^2 - 2v^2 = -2^k$ *are given by*

$$(u,v) = \begin{cases} (2^{\frac{k-2}{2}} Q_{2m+1}, 2^{\frac{k}{2}} P_{2m+1}), & \text{if } k \text{ is even;} \\ (2^{\frac{k+3}{2}} B_m, 2^{\frac{k-3}{2}} v_m), & \text{if } k \text{ is odd.} \end{cases}$$

with $m \geq 0$.

Lemma 2.18. *If* $n \geq 1$ *and* $k \geq 0$*, then* $B_n = 2^k$ *only if* $n = 1$ *and* $k = 0$.

Proof. If $n = 1$ and $k = 0$, then $B_n = 2^k$. Assume to the contrary that for some positive integer n, $B_n = 2^k$ for some $k > 0$. Hence, B_n is even and by Lemma 2.8, n is also even. Consequently, $6 = B_2 | B_n$ which implies that $3 | B_n = 2^k$, which is a contradiction. □

Lemma 2.19. *If* m *and* n *are positive integers and* $n|m$*, then* $B_n | C_m$ *if and only if* $n = 1$.

Proof. If $n = 1$, then $B_m | C_n$. Conversely if $n|m$, then $B_n | B_m$. Since $gcd(B_m, C_m) = 1$ and $B_n | B_m$, it follows that $gcd(B_n, C_m) = 1$ and hence, $B_n | C_m$ only if $n = 1$. □

Lemma 2.20. *If* n *and* t *are positive integers, then* $2^t | B_n$ *if and only if* $2^t | n$.

Proof. If $2^t | n$, then by Lemma 2.8, $B_{2^t} | B_n$. But, by Lemma 2.1(b), $B_{2^t} = 2^t B_1 C_1 C_2 \cdots C_{2^{t-1}}$ and hence, $2^t | B_{2^t} | B_n$. Conversely, assume that $2^t \nmid n$. If n is odd, then B_n is also odd and $2^t \nmid B_n$. If n is even, then $n = 2^s l$, where s and l are natural numbers and $s < t$ and l is odd. By Lemma 2.1(b), $B_n = B_{2^s l} = 2^s B_l C_l C_{2l} \cdots C_{2^{s-1} l}$. Since $B_l C_l C_{2l} \cdots C_{2^{s-1} l}$ is odd and $s < t$, it follows that $2^t \nmid B_n$. □

Lemma 2.21. *If* m*,* n*, and* t *are positive integers, then* $2^t B_m | B_n$ *if and only if* $2^t m | n$.

Proof. If $2^t m | n$, then by Lemma 2.8, $B_{2^t m} | B_n$. But, by virtue of Lemma 2.1(b), $B_{2^t m} = 2^t B_m C_m C_{2m} \cdots C_{2^{t-1} m}$ and hence, $2^t B_m | B_{2^t m} | B_n$. Conversely, let $2^t B_m | B_n$, $m = 2^s u$, $s \geq 0$, and u be odd. Since $2^s | B_m$, by Lemma 2.20, $2^s | m$ and $u | m$ implies $B_u | B_m$. Hence, $2^{s+t} B_u | B_n$. Again by Lemma 2.20, $2^{s+t} | B_n$ implies $2^{s+t} | n$ and by Lemma 2.8, $B_u | B_n$ implies $u | n$ and therefore, $2^{s+t} u = 2^t m | n$. □

Lemma 2.22. *Let* m*,* n*, and* t *be positive integers. Then,* $B_m | 2^t B_n$ *if and only if* $m | n$.

Proof. If $m | n$, then by Lemma 2.8, $B_m | 2^t B_n$. Conversely, assume that $B_m | 2^t B_n$. If $n < m$, then by the primitive divisor theorem (see [21]), there exists an odd prime divisor of B_m that does not divide B_n and consequently, $B_m \nmid 2^t B_n$. If $m = n$, then $B_m | 2^t B_n$. Now, let $n > m$ and assume, to the contrary, that $B_m | 2^t B_n$, but $m \nmid n$. Let $d = gcd(m, n)$. By Lemma 2.8, $B_d = \gcd(B_m, B_n)$. Consequently, $B_m | 2^t B_n$ is equivalent to $\frac{B_m}{B_d} | 2^t \frac{B_n}{B_d}$. Since $\gcd(\frac{B_m}{B_d}, \frac{B_n}{B_d}) = 1$, it follows that $\frac{B_m}{B_d} | 2^t$. Once again by the primitive divisor theorem, there exists an odd prime factor of B_m that does not divide B_d. Hence, this prime factor divides 2^t, which is a contradiction. □

3. Main Results

In this section, we solve the Diophantine equations $x^s - 8C_n xy + 16y^t = \pm 2^r$ in positive integers x and y with limited choices for s and t. Throughout this section, solution(s) means solution(s) in positive integers.

To begin, we prove a theorem that addresses the solutions of $x^2 - 8C_n xy + 16y^2 = 2^k$.

Theorem 3.1. *Let k be a nonnegative integer and n be a positive integer. If $k \geq 4$ is even, then all solutions of the equation*

$$x^2 - 8C_nxy + 16y^2 = 2^k \tag{3.1}$$

are given by $(x, y) = (2^{\frac{k}{2}} \frac{B_{m\pm n}}{B_n}, 2^{\frac{k-4}{2}} \frac{B_m}{B_n})$ when $n|m$. If $k = 0$, then the solutions are given by $(x, y) = (\frac{B_{m\pm n}}{B_n}, \frac{B_m}{4B_n})$ when $4n|m$. Furthermore, if $k = 2$, then the solutions are given by $(x, y) = (\frac{2B_{m\pm n}}{B_n}, \frac{B_m}{2B_n})$ when $2n|m$.

If $k \geq 7$ is odd, then (3.1) has a solution only for $n = 1$, and this solution is given by

$$(x, y) = \begin{cases} (3 \cdot 2^{\frac{k-5}{2}} Q_{2m+1} + 2^{\frac{k+1}{2}} P_{2m+1},\ 2^{\frac{k-9}{2}} Q_{2m+1}), & m \geq 0; \\ (3 \cdot 2^{\frac{k-5}{2}} Q_{2m+1} - 2^{\frac{k+1}{2}} P_{2m+1},\ 2^{\frac{k-9}{2}} Q_{2m+1}), & m > 0. \end{cases}$$

However, if $k = 1, 3, 5$, then (3.1) has no solution.

Proof. The substitutions $u = |2x - 8C_ny|$, $v = 16B_ny$ with the identity $8B_n^2 + 1 = C_n^2$ converts (3.1) to $u^2 - 2v^2 = 2^{k+2}$. If $k \geq 0$ is even, then using Theorem 2.17, we get

$$u = |2x - 8C_ny| = 2^{\frac{k}{2}} Q_{2m},$$
$$v = 16B_ny = 2^{\frac{k+4}{2}} B_m, \tag{3.2}$$

with $m \geq 0$. First, assume that $u = 2x - 8C_ny$. If $k \geq 4$, then by virtue of Lemma 2.22, $16B_ny = 2^{\frac{k+4}{2}} B_m$ is solvable if and only if $n|m$ and in this case, the possible values of y are given by $y = 2^{\frac{k-4}{2}} \frac{B_m}{B_n}$. Substituting y into $u = 2x - 8C_ny$, we get

$$2x - 8C_n 2^{\frac{k-4}{2}} \frac{B_m}{B_n} = 2^{\frac{k}{2}} Q_{2m}.$$

Since $Q_{2m} = 2C_m$, the above equation can be written as

$$2x = 2^{\frac{k+2}{2}} \left(\frac{B_mC_n + C_mB_n}{B_n} \right).$$

Now, using Lemma 2.1(d), we find $x = 2^{\frac{k}{2}} (\frac{B_{m+n}}{B_n})$, when $n|m$. A similar calculation shows that if $m \geq 1$ and $u = -2x + 8C_ny$, then $x = 2^{\frac{k}{2}} (\frac{B_{m-n}}{B_n})$ when $n|m$. Furthermore, if $k = 0$, then (3.2) is solvable if $4B_n|B_m$, and by virtue of Lemma 2.21, this is possible only if $4n|m$. In this case, the solutions of (3.1) are given by $x = \frac{B_{m\pm n}}{B_n}$ and $y = \frac{B_m}{4B_n}$. Similarly, if $k = 2$, then (3.2) is solvable only if $2n|m$ and the solutions are given by $x = \frac{2B_{m\pm n}}{B_n}$ and $y = \frac{B_m}{2B_n}$.

Next, assume that k is odd. By virtue of Theorem 2.17, $u = |2x - 8C_ny| = 2^{\frac{k+3}{2}} P_{2m+1}$ and $v = 16B_ny = 2^{\frac{k-1}{2}} Q_{2m+1}$. Since $2||Q_{2m+1}$, the equation $16B_ny = 2^{\frac{k-1}{2}} Q_{2m+1}$ has no solution if $k = 1, 3, 5$. Now, let $k \geq 7$. If $u = 2x - 8C_ny$, then

$$B_ny = 2^{\frac{k-7}{2}} \frac{Q_{2m+1}}{2}. \tag{3.3}$$

Since by Lemma 2.1(f), $B_n = P_nq_n$, we can write (3.3) as

$$P_nq_ny = 2^{\frac{k-7}{2}} q_{2m+1}, \tag{3.4}$$

where $q_i = Q_i/2$ for $i \geq 0$. Since by Lemma 2.15, $\gcd(q_n, q_{2m+1}) = 1$ and (3.4) implies that $q_n|q_{2m+1}$, we must have $q_n = 1$ and consequently, $n = 0$ or $n = 1$. If $n = 0$, then (3.4) can be written as $q_{2m+1} = 0$, which is not true. If $n = 1$, then (3.4) reduces to $y = 2^{\frac{k-7}{2}} q_{2m+1}$. Hence, for $k \geq 7$ is odd, (3.3) is solvable only if $n = 1$ and the solutions of (3.1) are given by

$$x = \frac{u + 8C_n y}{2} = 3 \cdot 2^{\frac{k-5}{2}} Q_{2m+1} + 2^{\frac{k+1}{2}} P_{2m+1},\ y = 2^{\frac{k-9}{2}} Q_{2m+1}, \quad m \geq 0.$$

Similarly, if $u = -(2x - 8C_n y)$, a similar argument shows that

$$x = 3 \cdot 2^{\frac{k-5}{2}} Q_{2m+1} - 2^{\frac{k+1}{2}} P_{2m+1},\ y = 2^{\frac{k-9}{2}} Q_{2m+1}, \quad m \geq 0.$$

This ends the proof.

□

The next theorem is similar to Theorem 3.1, except that on the right side of (3.1), 2^k has been replaced by -2^k.

Theorem 3.2. *Let n be a positive integer and k a nonnegative integer.*
(a) If k is even, then the equation

$$x^2 - 8C_n xy + 16y^2 = -2^k \tag{3.5}$$

is solvable only if $k \geq 6$ and $n = 1$ and the solutions are given by

$$(x, y) = \left(2^{\frac{k-2}{2}}(3P_{2m+1} \pm Q_{2m+1}), 2^{\frac{k-6}{2}} P_{2m+1}\right); \quad m = 0, 1, 2, \ldots$$

(b) If k and n are odd, then (3.5) is solvable only if $k \geq 7$ and $n = 1$ and the solutions are given by

$$(x, y) = \left(3 \cdot 2^{\frac{k-5}{2}} Q_{2m} \pm 2^{\frac{k+3}{2}} B_m, 2^{\frac{k-9}{2}} Q_{2m}\right), \quad m = 0, 1, 2, \ldots \tag{3.6}$$

Furthermore, if k is odd and n is even, then (3.5) is solvable only if $k \geq 9$ and $n = 2$ and the solutions are

$$x = \frac{17}{3} \cdot 2^{\frac{k-7}{2}} Q_{2m} \pm 2^{\frac{k+3}{2}} B_m,\ y = \frac{1}{3} \cdot 2^{\frac{k-11}{2}} Q_{2m}; \quad m = 1, 3, 5, \ldots$$

Proof. With suitable algebraic manipulations, we can rewrite (3.5) as

$$(2x - 8C_n y)^2 - 2(16B_n y)^2 = -2^{k+2}.$$

(a) If $k \geq 0$ is even, then by virtue of Theorem 2.17, we have

$$u = |2x - 8C_n y| = 2^{\frac{k}{2}} Q_{2m+1},$$
$$v = 16B_n y = 2^{\frac{k+2}{2}} P_{2m+1}; \quad m \geq 0. \tag{3.7}$$

First, let $u = 2x - 8C_n y$. If $k = 0, 2, 4$, then (3.7) reduces to $2^{\frac{6-k}{2}} B_n y = P_{2m+1}$, which has no solution because the left side is even, and the right side is odd. If $k \geq 6$, then (3.7) is equivalent to $P_{2n} y = 2^{\frac{k-4}{2}} P_{2m+1}$. Since $\gcd(P_{2n}, P_{2m+1}) = 1$, from Lemma 2.15, it follows that P_{2n} divides $2^{\frac{k-4}{2}}$. But, by Lemmas 2.18 and 2.1(f), this is possible only when $n = 1$. Hence, in this case, the solutions are given by

$$x = 2^{\frac{k-2}{2}}(Q_{2m+1} + 3P_{2m+1}),\ y = 2^{\frac{k-6}{2}} P_{2m+1}; \quad m \geq 0.$$

If k is even and $u = -(2x - 8C_n y)$, then a similar argument shows that (3.5) is solvable only if $k \geq 6$ and $n = 1$ and the solutions are given by

$$x = 2^{\frac{k-2}{2}}(-Q_{2m+1} + 3P_{2m+1}),\ y = 2^{\frac{k-6}{2}} P_{2m+1}; \quad m \geq 0.$$

(b) Now, let k be odd. By Theorem 2.17,

$$u = |2x - 8C_n y| = 2^{\frac{k+5}{2}} B_m,$$
$$v = 16B_n y = 2^{\frac{k-1}{2}} v_m = 2^{\frac{k+1}{2}} C_m. \tag{3.8}$$

If $u = 2x - 8C_n y$, then by virtue of (3.8)

$$B_n y = 2^{\frac{k-7}{2}} C_m. \tag{3.9}$$

If $k = 1, 3, 5$, then (3.8) reduces to $2^{\frac{7-k}{2}} B_n y = C_m$, which has no solution because the left side is even, and the right side is odd. Now, if $k \geq 7$ and n are odd, then by Lemma 2.15, $\gcd(B_n, C_m) = 1$ and (3.9) is equivalent to $B_n | 2^{\frac{k-7}{2}}$. But, in view of Lemma 2.18, $B_n | 2^{\frac{k-7}{2}}$ is possible only if $n = 1$. Hence, in this case, the solutions of (3.5) are given by

$$x = 3 \cdot 2^{\frac{k-3}{2}} C_m + 2^{\frac{k+3}{2}} B_m, \ y = 2^{\frac{k-7}{2}} C_m; \quad m \geq 0.$$

These solutions are also expressible in the form

$$x = 3 \cdot 2^{\frac{k-5}{2}} Q_{2m} + 2^{\frac{k+3}{2}} B_m, \ y = 2^{\frac{k-9}{2}} Q_{2m}; \quad m \geq 0.$$

A similar calculation shows that if $u = -2x + 8C_n y$, then (3.5) is solvable only if $k \geq 7$ and $n = 1$, and the solutions are given by

$$x = 3 \cdot 2^{\frac{k-5}{2}} Q_{2m} - 2^{\frac{k+3}{2}} B_m, \ y = 2^{\frac{k-9}{2}} Q_{2m}; \quad m \geq 0.$$

Next, assume that k is odd and n is even, say $n = 2^a n_1$, n_1 is odd. Any use of Lemma 2.1(b) converts (3.9) to

$$2^a B_{n_1} C_{n_1} C_{2n_1} \cdots C_{2^{a-1} n_1} y = 2^{\frac{k-7}{2}} C_m. \tag{3.10}$$

Since n_1 is odd, $\gcd(B_{n_1}, C_m) = 1$ by Lemma 2.15 and hence $n_1 = 1$ and $n = 2^a$. Furthermore, $n_1 = 1$ reduces (3.10) to

$$2^a C_1 C_2 \cdots C_{2^{a-1}} y = 2^{\frac{k-7}{2}} C_m, \tag{3.11}$$

which implies $C_1 C_2 \cdots C_{2^{a-1}} | C_m$ and consequently, $C_i | C_m$ for $i = 1, 2, \ldots, 2^{a-1}$. Using Lemma 2.9, we conclude that $a = 1$ and hence $n = 2$, and (3.11) reduces to

$$2C_1 y = 2^{\frac{k-7}{2}} C_m. \tag{3.12}$$

If $k \leq 7$, then (3.12) reduces to $3 \cdot 2^{\frac{9-k}{2}} y = C_m$, which has no solution because the left and right sides are opposite parity. Thus, if k is odd and n is even, (3.5) is solvable only if $k \geq 9$ and $n = 2$ and the solutions in this case are given by

$$x = \frac{17}{3} \cdot 2^{\frac{k-7}{2}} Q_{2m} + 2^{\frac{k+3}{2}} B_m, \ y = \frac{1}{3} \cdot 2^{\frac{k-11}{2}} Q_{2m}; \quad m \geq 1.$$

A similar calculation shows that if k is odd, n is even, and $u = -2x + 8C_n y$, then (3.5) is solvable only if $k \geq 9$ and $n = 2$ and the solutions are given by

$$x = \frac{17}{3} \cdot 2^{\frac{k-7}{2}} Q_{2m} - 2^{\frac{k+3}{2}} B_m, \ y = \frac{1}{3} \cdot 2^{\frac{k-11}{2}} Q_{2m}; \quad m \geq 1.$$

□

The following theorem is similar to Theorem 3.1 except that, in the left side of (3.1), the exponent of y has been doubled.

Theorem 3.3. *If k and n are natural numbers, then the Diophantine equation*

$$x^2 - 8C_nxy^2 + 16y^4 = 2^k \tag{3.13}$$

has no solution if $k \equiv 1, 2 \pmod 4$. If $k \equiv 0 \pmod 4$, then (3.13) is solvable only if $k \geq 4$ and the solutions are given by $x = 2^{\frac{k}{2}}v_n,\ y = 2^{\frac{k-4}{4}},\ n \geq 0$. Furthermore, if $k \equiv 3 \pmod 4$, then (3.13) is solvable only if $k \geq 7$ and has just one solution given by $(x, y) = \left(7 \cdot 2^{\frac{k-3}{2}}, 2^{\frac{k-7}{4}}\right)$.

Proof. If $k \geq 4$ is even, then by virtue of Theorem 3.1, the solutions of (3.13) in (x, y^2) are given by

$$(x, y^2) = \left(2^{\frac{k}{2}}\frac{B_{m\pm n}}{B_n}, 2^{\frac{k-4}{2}}\frac{B_m}{B_n}\right)$$

where $n|m$. Thus,

$$2^{\frac{k-4}{2}}B_m = B_ny^2, \quad n|m. \tag{3.14}$$

If $k = 0$, then by virtue of Theorem 3.1,

$$(x, y^2) = \left(\frac{B_{m\pm n}}{B_n}, \frac{B_m}{4B_n}\right), \quad 4n|m \tag{3.15}$$

and if $k = 2$, then by virtue of Theorem 3.1,

$$(x, y^2) = \left(\frac{2B_{m\pm n}}{B_n}, \frac{B_m}{2B_n}\right), \quad 2n|m. \tag{3.16}$$

Moreover, if k is odd, then in view of Theorem 3.1, (3.13) is solvable in (x, y^2) only if $k \geq 7$ and the solutions are given by

$$(x, y^2) = \left(3 \cdot 2^{\frac{k-5}{2}}Q_{2m+1} \pm 2^{\frac{k+1}{2}}P_{2m+1}, 2^{\frac{k-9}{2}}Q_{2m+1}\right)$$

and hence,

$$y^2 = 2^{\frac{k-9}{2}}Q_{2m+1}. \tag{3.17}$$

Now, we distinguish the following seven cases:

Case 1: $k \geq 4$, $k \equiv 0 \pmod 4$, and y is even. In this case, $\frac{k-4}{2} = 2t$ for some nonnegative integer t, and (3.14) takes the form

$$2^{2t}B_m = B_ny^2. \tag{3.18}$$

Because y is even, it can be written as $y = 2^sl$, $s \geq 1$, and $l \geq 1$ is odd. Now, we can write (3.18) as $2^{2t}B_m = B_n2^{2s}l^2$, which, consequently, takes the form $B_m = B_n(2^{s-t}l)^2$. By Lemma 2.10, this is possible only if $2^{s-t}l = 1$, or, equivalently, $s - t = 0$ and $l = 1$. Thus, $B_n = B_m$ and hence, $n = m$ and in this case, $y = 2^t = 2^{\frac{k-4}{4}}$. Using Lemma 2.1(b), we can write the solution as

$$(x, y) = \left(2^{\frac{k}{2}}v_n,\ 2^{\frac{k-4}{4}}\right);\ n \geq 0.$$

Case 2: $k \geq 4$, $k \equiv 0 \pmod 4$, and y is odd. If $k = 4$, (3.14) reduces to $B_m = B_ny^2$. By virtue of Lemma 2.10, this is possible only if $y = 1$. Hence, $B_m = B_n$ and consequently, $m = n$. Now using Lemma 2.1(b), the solutions are given by

$$(x, y) = (4v_n,\ 1);\ n \geq 0.$$

If $k > 4$, then $\frac{k-4}{4} > 1$, and (3.14) can be written as $y^2 = 2^{\frac{k-4}{2}}\frac{B_m}{B_n}$, which is not solvable because the left and right sides are of opposite parity.

Case 3: $k \geq 4$, $k \equiv 2 \pmod 4$, and y is even. Because $k \equiv 2 \pmod 4$, $k = 4t + 2$ for some $t \geq 1$ and (3.14) can be written as

$$2^{2t-1} B_m = B_n y^2. \tag{3.19}$$

If n is odd, then $2^{2t-1}|y^2$ and from (3.19), we get $B_m = 2B_n w^2$, where $w = y/2^t$. We claim that $B_m = 2B_n w^2$ has no solution. Because of the presence of 2 in the right side of the last equation, B_m is even and hence m is even, that is $m = 2m_1$. Using Lemma 2.1(b), we can write (3.19) as $\frac{B_{m_1} C_{m_1}}{B_n} = w^2$. Because $gcd(B_{m_1}, C_{m_1}) = 1$, it follows that $gcd(\frac{B_{m_1}}{B_n}, C_{m_1}) = 1$ and hence, $\frac{B_{m_1}}{B_n} = w_1^2$ and $C_{m_1} = w_2^2$ for some positive integers w_1 and w_2 such that $w = w_1 w_2$. In view of Lemma 2.10, $\frac{B_{m_1}}{B_n} = w_1^2$ holds only when $w_1 = 1$. Furthermore, by Lemma 2.11, $C_{m_1} = w_2^2$ has no solution other than $m_1 = 0$. Because we are interested in positive integer solutions in x and y, in this case, (3.13) is not solvable. If n is even, then from (3.19), we get $B_m = 2B_n u^2$, where $u = \frac{y}{2^t}$. Using a similar argument as above, it can be shown that $B_m = 2B_n u^2$ is also not solvable. This implies that (3.13) has no solution.

Case 4: $k \geq 4$, $k \equiv 2 \pmod 4$, and y is odd. If n is even, then it follows from (3.19) that $2^{2t-1}|B_n$, which implies $B_n = 2^{2t-1+k_3} m_3$ for some $k_3 \geq 0$ and for some odd positive integer m_3. Because $n|m$, by Lemma 2.8, $B_n|B_m$ and hence, $B_m = (2^{2t-1+k_3} m_3)(2^{k_4} m_4)$ for some $k_4 \geq 0$ and for some odd positive integer m_4. Thus, $2^{2t-1+k_3+k_4}|B_m$ and (3.19) can be written as $2^{2t-1+k_4}(\frac{B_m}{2^{2t-1+k_3+k_4}}) = (\frac{B_n}{2^{2t-1+k_3}})y^2$. Because the terms on the right side are odd, we must have $2t - 1 + k_4 = 0$, which is possible only when $t = 0$ and $k_4 = 1$. But, this implies that $B_m = 2B_n y^2$, and we have already checked that this equation has no solution in positive integer y. If n is odd, then the right side of (3.19) is odd, whereas the left side is even. Hence, in this case, (3.13) has no solution.

Case 5: $k = 0, 2$. If $k = 0$, from (3.15) we have $y^2 = \frac{B_m}{4B_n}$, which reduces to

$$B_m = B_n u_3^2, \tag{3.20}$$

where $u_3 = 2y$. In view of Lemma 2.10, (3.20) is solvable only if $u_3 = 1$, which is not possible since $u_3 = 2y$. If $k = 2$, then from (3.16), we have $y^2 = \frac{B_m}{2B_n}$, which is equivalent to

$$B_m = 2B_n y^2. \tag{3.21}$$

Using a similar argument as in Case 3, it is easy to see that (3.21) has no solution. Hence, for $k = 0$ and $k = 2$, (3.13) has no solution.

Case 6: $k \geq 7$ and $k \equiv 1 \pmod 4$. In this case, $k = 4t + 1$ for some $t \geq 2$ and (3.17) can be written as $2^{2t-4} Q_{2m+1} = y^2$. This implies that Q_{2m+1} is a perfect square, which is not possible since $2||Q_{2m+1}$. So, if $k \equiv 1 \pmod 4$, (3.13) has no solution.

Case 7: $k \geq 7$ and $k \equiv 3 \pmod 4$. If $k = 7$, then (3.17) takes the form $2y^2 = Q_{2m+1}$. By ([8], Theorem 4.4), the last equation is solvable only for $m = 0$, which implies that $y = 1$ and $x = 28$. If $k > 7$, then we can write k as $k = 4t + 3$ with $t > 1$, and (3.17) can be written as $2^{2t-3} Q_{2m+1} = y^2$, which reduces to

$$2Q_{2m+1} = u^2, \tag{3.22}$$

where $u = (\frac{4y}{2^t})$. Thus, u is even, that is $u = 2s_2$, where $s_2 \geq 1$. Therefore, (3.22) reduces to $Q_{2m+1} = 2s_2^2$. In view of ([8], Theorem 4.4), this is possible only if $m = 0$. Thus,

$$(x, y^2) = \left(3 \cdot 2^{\frac{k-3}{2}} \pm 2^{\frac{k+1}{2}}, 2^{\frac{k-7}{2}}\right)$$

and hence, the solutions of (3.13) are given by

$$(x, y) = \left(3 \cdot 2^{\frac{k-3}{2}} \pm 2^{\frac{k+1}{2}}, 2^{\frac{k-7}{4}}\right).$$

□

Next, we consider a Diophantine equation that generalizes (3.5) in the sense that y has been replaced by y^2.

Theorem 3.4. *The Diophantine equation*

$$x^2 - 8C_nxy^2 + 16y^4 = -2^k \tag{3.23}$$

has no solution if $k \equiv 0, 1 \pmod 4$. *If* $k \equiv 2 \pmod 4$, (3.23) *is solvable only if* $k \geq 6$ *and the solutions are given by* $(x, y) = (5 \cdot 2^{\frac{k-2}{2}}, 2^{\frac{k-6}{4}})$ *and* $(x, y) = (985 \cdot 2^{\frac{k-2}{2}}, 13 \cdot 2^{\frac{k-6}{4}})$. *Furthermore, if* $k \equiv 3 \pmod 4$, *there exist just one solution only if* $k \geq 7$ *and is given by*

$$(x, y) = \left(3 \cdot 2^{\frac{k-3}{2}}, 2^{\frac{k-7}{4}}\right).$$

Proof. We distinguish three cases.

Case 1: k is even.
If k is even, then in view of Theorem 3.2(a), (3.23) is solvable for (x, y^2) only if $k \geq 6$ and the solutions are given by $x = 2^{\frac{k-2}{2}}(Q_{2m+1} + 3P_{2m+1})$ or $x = 2^{\frac{k-2}{2}}P_{2m-1}$ and

$$y^2 = 2^{\frac{k-6}{2}}P_{2m+1}. \tag{3.24}$$

If $k \equiv 0 \pmod 4$, then $k = 4t$ for some positive integer $t \geq 2$, y is even, and (3.24) can be written as $2u_1^2 = P_{2m+1}$, where $u_1 = \frac{y}{2^{t-1}}$. But, the last equation has no solution since P_{2m+1} is odd. If $k \equiv 2 \pmod 4$, then (3.24) can be written in the form

$$2^{2t-2}P_{2m+1} = y^2 \tag{3.25}$$

for some positive integer t. Substituting $w_3 = \frac{y}{2^{t-1}}$, (3.25) takes the form $w_3^2 = P_{2m+1}$. In view of Lemma 2.13, there are two possibilities, either $2m+1 = 1$ or $2m+1 = 7$ and hence, $m = 0$ or $m = 3$. If $m = 0$, then $x = 5.2^{\frac{k-2}{2}}$ or $x = 2^{\frac{k-2}{2}}$ and $y = 2^{\frac{k-6}{4}}$ and if $m = 3$, then $x = 985 \cdot 2^{\frac{k-2}{2}}$ or $x = 29 \cdot 2^{\frac{k-2}{2}}$ and $y = 13 \cdot 2^{\frac{k-6}{4}}$.

Case 2: k and n are odd.
In this case, by Theorem 3.2 (b), (3.23) is solvable for (x, y^2) only if $k \geq 7$ and the solutions are

$$x = 3 \cdot 2^{\frac{k-5}{2}}Q_{2m} \pm 2^{\frac{k+3}{2}}B_m$$

and

$$y^2 = 2^{\frac{k-9}{2}}Q_{2m}. \tag{3.26}$$

If $k \equiv 1 \pmod 4$, then $k = 4t+1$ for some integer $t \geq 2$, and (3.26) reduces to $y^2 = 2^{2t-4}Q_{2m}$. Writing $h = \frac{y}{2^{t-2}}$, the last equation can be written as

$$h^2 = Q_{2m} = 2C_m. \tag{3.27}$$

Thus, h is even that is, $h = 2t_1$ and (3.27) takes the form $2t_1^2 = C_m$, which is not solvable because C_m is odd. Let $k \equiv 3 \pmod 4$. If $k = 7$, then (3.26) reduces to

$$2y^2 = Q_{2m}, \tag{3.28}$$

and if $k > 7$, then (3.26) reduces to

$$2w_4^2 = Q_{2m}, \tag{3.29}$$

where $w_4 = \frac{y}{2^{t-1}}$. By virtue of ([8], Theorem 3.10), (3.28) and (3.29) are solvable only for $m = 0$, and the solution is given by

$$(x, y) = \left(3 \cdot 2^{\frac{k-3}{2}}, 2^{\frac{k-7}{4}}\right).$$

Case 3: k is odd and n is even.
In this case, by virtue of Theorem 3.2, (3.23) is solvable for (x, y^2) only if $k \geq 9$ and the solutions are

$$x = \frac{17}{3} \cdot 2^{\frac{k-7}{2}} Q_{2m} \pm 2^{\frac{k+3}{2}} B_m,$$

and

$$3y^2 = 2^{\frac{k-11}{2}} Q_{2m}; \quad m = 1, 3, 5, \ldots. \tag{3.30}$$

If $k \equiv 1 \pmod 4$, then $k = 4t + 1$ for some integer $t \geq 2$ and (3.30) reduces to $Q_2^2 y^2 = 2^{2t-4} Q_2 Q_{2m}$, which is equivalent to

$$Q_2 Q_{2m} = w_5^2, \tag{3.31}$$

where $w_5 = \frac{Q_2 y}{2^{t-2}}$. But, by virtue of Lemma 2.12, (3.31) is not solvable. If $k \equiv 3 \pmod 4$, then $k = 4t + 3$ for some integer $t \geq 2$ and (3.30) can be written as

$$3y^2 = 2^{2t-3} C_m; \quad m = 1, 3, 5, \ldots. \tag{3.32}$$

Substituting $w_6 = \frac{y}{2^{t-2}}$ in (3.32), we get

$$2C_m = 3w_6^2. \tag{3.33}$$

But, (3.33) has no solution because C_m is always odd. □

In the following theorem, we study a Diophantine equation that resembles the one appearing in Theorem 3.1. The only difference is that the exponent of x has been doubled.

Theorem 3.5. *The Diophantine equation*

$$x^4 - 8C_n x^2 y + 16y^2 = 2^k \tag{3.34}$$

has no solution if $k \equiv 1, 2 \pmod 4$. *If* $k \equiv 0 \pmod 4$, *then the solutions of* (3.34) *exist only if* $k \geq 4$ *and are given by*

$$(x, y) = \left(2^{\frac{k}{4}}, 2^{\frac{k-4}{2}} v_n\right).$$

Furthermore, if $k \equiv 3 \pmod 4$, (3.34) *is solvable and has just one solution only if* $k \geq 7$. *This solution is given by*

$$(x, y) = (2^{\frac{k-3}{4}}, 7.2^{\frac{k-7}{2}}).$$

Proof. If $k \geq 4$ is even, then in view of Theorem 3.1, the solutions of (3.34) for (x^2, y) are given by

$$(x^2, y) = \left(2^{\frac{k}{2}} \frac{B_{m\pm n}}{B_n}, 2^{\frac{k-4}{2}} \frac{B_m}{B_n}\right), \quad n | m.$$

Thus,

$$B_n x^2 = 2^{\frac{k}{2}} B_{m\pm n}. \tag{3.35}$$

If $k = 0$, then

$$(x^2, y) = \left(\frac{B_{m\pm n}}{B_n}, \frac{B_m}{4B_n}\right), \quad 4n|m \tag{3.36}$$

and if $k = 2$, then

$$(x^2, y) = \left(\frac{2B_{m\pm n}}{B_n}, \frac{B_m}{2B_n}\right), \quad 2n|m. \tag{3.37}$$

If k is odd, then by Theorem 3.1, (3.34) is solvable for (x^2, y) only if $k \geq 7$ and these solutions are given by

$$x^2 = 2^{\frac{k-5}{2}}(3Q_{2m+1} \pm 8P_{2m+1}) \tag{3.38}$$

and

$$y = 2^{\frac{k-9}{2}} Q_{2m+1}.$$

Now, we distinguish the following eight cases.

Case 1: $k \geq 4$, $k \equiv 0 \pmod 4$, and x is even.
In this case, $k = 4t$ for some $t \geq 1$ and (3.35) reduces to $2^{2t}B_{m\pm n} = B_n x^2$. Because x is even, $x = 2^{s_2} l_1$ for some $s_2 \geq 1$ and $l_1 \geq 1$ and (3.35) can be written as $2^{2t}B_{m\pm n} = B_n 2^{2s_2} l_1^2$. The last equation further reduces to $B_{m\pm n} = B_n(2^{s_2-t}l_1)^2$. By Lemma 2.10, this is possible if $2^{s_2-t}l_1 = 1$, which implies that $s_2 - t = 0$ and $l_1 = 1$. Thus, in this case, $x = 2^t = 2^{\frac{k}{4}}$ and $y = 2^{\frac{k-4}{2}} v_n$.

Case 2: $k \geq 4$, $k \equiv 0 \pmod 4$, and x is odd.
In this case, $k = 4t$ for some $t \geq 1$ and (3.35) reduces to

$$2^{2t}B_{m\pm n} = B_n x^2. \tag{3.39}$$

Because x is odd, it follows from (3.39) that $2^{2t}|B_n$, which implies $B_n = 2^{2t+k_6}m_6$ for some odd positive integer m_6 and $k_6 \geq 0$. Since $n|m$, it follows that $n|(m \pm n)$ and by Lemma 2.8, $B_n|B_{m\pm n}$. Hence, $B_{m\pm n}$ can be written as $(2^{2t+k_6}m_6)(2^{k_7}m_7)$ for some $k_7 \geq 0$. This implies that $2^{2t+k_6+k_7}|B_{m\pm n}$ and (3.39) can be written as

$$2^{2t+k_7} \cdot \frac{B_{m\pm n}}{2^{2t+k_6+k_7}} = \frac{B_n}{2^{2t+k_6}} \cdot x^2. \tag{3.40}$$

Because the terms on the right side of (3.40) are odd, we must have $2t + k_7 = 0$ and hence $k_7 = t = 0$, which is impossible since $k \geq 4$.

Case 3: $k \geq 4$, $k \equiv 2 \pmod 4$, and x is even.
In this case, $k = 4t + 2$ for some $t \geq 1$ and (3.35) takes the form

$$2^{2t+1}B_{m\pm n} = B_n x^2. \tag{3.41}$$

Because x is even, $x = 2^{k_9}m_9$ for some odd positive integer m_9. On substituting in (3.35), we get $2B_{m\pm n} = B_n(2^{k_9-t}m_9)^2$, which is of the form $2B_{m\pm n} = B_n u^2$. While proving Theorem 3.3, we have seen that the latter equation has no solution in positive integer u.

Case 4: $k \geq 4$, $k \equiv 2 \pmod 4$, and x is odd.
In this case, (3.35) takes the form

$$2^{2t+1}B_{m\pm n} = B_n x^2. \tag{3.42}$$

Because x is odd, it follows from (3.42) that $2^{2t+1}|B_n$, which implies that $B_n = 2^{2t+1+k_{10}}m_{10}$ for some $k_{10} \geq 0$ and for some odd positive integer m_{10}. Since $n|m$, by Lemma 2.8, $B_n|B_{m\pm n}$

and hence, $B_{m\pm n}$ can be written as $(2^{2t+1+k_{10}}m_{10})(2^{k_{11}}m_{11})$ for some $k_{11} \geq 0$ and for some odd positive integer m_{11}. This implies $2^{2t+1+k_{10}+k_{11}}|B_{m\pm n}$ and (3.35) can be written as $2^{2t+1+k_{11}}(\frac{B_{m\pm n}}{2^{2t+1+k_{10}+k_{11}}}) = (\frac{B_n}{2^{2t+1+k_{10}}})x^2$. Because the terms on the right side are odd, we must have $2t+1+k_{11} = 0$, which is not possible. Hence, in this case (3.34) is not solvable.

Case 5: $k = 0, 2$.
If $k = 0$, then by (3.36), $x^2 = \frac{B_{m\pm n}}{B_n}$, which is equivalent to

$$B_{m\pm n} = B_n x^2 \tag{3.43}$$

and by Lemma 2.10, this is possible only if $x = 1$. This implies that $m \pm n = n$, which leads to either $m = 2n$ or $m = 0$. If $m = 0$, then from (3.36), $y = 0$, which is not a positive integer. If $m = 2n$, then (3.36) reduces to $2y = C_n$, which is clearly not solvable. Similarly, if $k = 2$, then by (3.37), $x^2 = \frac{2B_{m\pm n}}{B_n}$ which is equivalent to

$$B_{m\pm n} = 2B_n z^2, \tag{3.44}$$

where $x = 2z$ and it has been already proved that equations of the form of (3.44) are not solvable.

Case 6: $k \geq 7$, $k \equiv 1 \pmod 4$, and x is even.
In this case, we can write $k = 4t+1$, where $t \geq 2$ and (3.38) reduces to

$$x^2 = 2^{2(t-1)}(3Q_{2m+1} \pm 8P_{2m+1}). \tag{3.45}$$

Because x is even, $x = 2g_1$ for some $g_1 \geq 1$ and (3.45) reduces to $2^{2(2-t)}g_1^2 = 3Q_{2m+1} \pm 8P_{2m+1}$, which is of the form $s_3^2 = 3Q_{2m+1} \pm 8P_{2m+1}$, where $s_3 = 2^{2-t}g_1$. Furthermore, using the Binet forms, it is easy to see that $3Q_{2m+1} + 8P_{2m+1} = Q_{2m+3}$ and $3Q_{2m+1} - 8P_{2m+1} = Q_{2m-1}$. Thus, we get $Q_{2m+3} = s_3^2$ and $Q_{2m-1} = s_3^2$, which are impossible since $Q_n \equiv 2, 6 \pmod 8$ for all n.

Case 7: $k \geq 7$, $k \equiv 3 \pmod 4$, and x is even.
Letting $k = 4t+3$, where $t \geq 1$, (3.38) takes the form

$$x^2 = 2^{2t-1}(3Q_{2m+1} \pm 8P_{2m+1}). \tag{3.46}$$

Since $3Q_{2m+1} + 8P_{2m+1} = Q_{2m+3}$ and $3Q_{2m+1} - 8P_{2m+1} = Q_{2m-1}$, we get from (3.46) that

$$Q_{2m+3} = 2s_4^2 \tag{3.47}$$

or

$$Q_{2m-1} = 2s_4^2, \tag{3.48}$$

where $s_4 = \frac{x}{2^t}$. (3.47) is impossible by Lemma 2.14 since $m \geq 0$. (3.48) holds only when $m = 1$ by Lemma 2.14. Thus, in this case, the only solution of (3.34) is given by

$$(x, y) = (2^{\frac{k-3}{4}}, 7.2^{\frac{k-7}{2}}).$$

Case 8: Both k and x are odd.
If k is odd, then from (3.38), it follows that x is even. Therefore, in this case, no solution exists.

□

Conclusion

In this work, we solved the Diophantine equations $x^2 - 8C_nxy + 16y^2 = \pm 2^r$, $x^2 - 8C_nxy + 16y^4 = \pm 2^r$, and $x^4 - 8C_nxy + 16y^2 = 2^r$ in positive integers x and y. One notices that these equations admit further generalizations. It will be an interesting idea to investigate the conditions under which $x^k - 8C_nxy + 16y^l = \pm 2^r$ can be solved in positive integers x and y or, one can explore its solvability for other particular values of k and l.

4. Acknowledgment

We thank the anonymous referee for the valuable comments and constructive suggestions that improved the quality and presentation of this paper.

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MSC2010: 11B37, 11B39, 11D09

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