A NEW COMBINATORIAL INTERPRETATION OF THE FIBONACCI NUMBERS SQUARED. PART II.

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Abstract. We give further combinatorial proofs of identities related to the Fibonacci numbers squared by considering the tiling of an n-board (a $1 \times n$ array of square cells of unit width) with half-squares ($\frac{1}{2} \times 1$ tiles) and $(\frac{1}{2}, \frac{1}{2})$-fence tiles. A (w, g)-fence tile is composed of two $w \times 1$ rectangular subtiles separated by a gap of width g. In addition, we construct a Pascal-like triangle whose (n, k)th entry is the number of tilings of an n-board that contain k fences. Elementary combinatorial proofs are given for some properties of the triangle and we show that reversing the rows gives the $(1/(1-x^2), x/(1-x)^2)$ Riordan array. Finally, we show that tiling an n-board with $(\frac{1}{4}, \frac{1}{4})$- and $(\frac{1}{4}, \frac{3}{4})$-fences also generates the Fibonacci numbers squared.

1. Introduction

In [6], we showed that the number of ways to tile an n-board using half-squares ($\frac{1}{2} \times 1$ tiles, denoted by h) and $(\frac{1}{2}, \frac{1}{2})$-fences (denoted by f) is F_{n+1}^2, where F_n is the nth Fibonacci number ($F_0 = 1$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$). A (w, g)-fence tile is composed of two pieces (referred to as posts) of size $w \times 1$ separated by a gap of size $g \times 1$. We used these tilings to formulate combinatorial proofs of identities relating the Fibonacci numbers squared to themselves and to other number sequences. Here, after proving the main theorem of [6] another way, we give combinatorial proofs of further identities, again drawing in part on methods described in [4]. We then construct a Pascal-like triangle whose rows sum to give the Fibonacci numbers squared and show that it is closely related to a Riordan array. Finally, we present an alternative combinatorial interpretation of the Fibonacci numbers squared, this time by tiling with $(\frac{1}{4}, \frac{1}{4})$- and $(\frac{1}{4}, \frac{3}{4})$-fences.

We begin by proving Theorem 3.2 of [6] in a more combinatorial manner. The proofs of this and Identity 3.2 require the following lemma.

Lemma 1.1. *There is a bijection between the tilings of an n-board using half-squares and $(\frac{1}{2}, \frac{1}{2})$-fences and the tilings of an ordered pair of n-boards using squares and dominoes.*

Proof. For each h (f) occupying the left side of a cell, place a square (domino) on the first of the pair of n-boards. For each h (f) occupying the right side of a cell, place a square (domino) on the second n-board to be tiled with squares and dominoes. Note that each fence occupies two consecutive left sides or right sides of a cell and each fence corresponds to one domino. The process is clearly reversible and so the mapping is a bijection. □

Theorem 1.2. *Let A_n be the number of ways to tile an n-board using half-squares and fences. Then $A_n = F_{n+1}^2$.*

Proof. There are F_{n+1} ways to tile an n-board using squares and dominoes [4]. From Lemma 1.1, A_n is the same as the number of ways to tile an ordered pair of n-boards using squares and dominoes, which is F_{n+1}^2. □

2. Identities

It is generally advantageous to describe tilings that involve fences in terms of metatiles, which are groupings of tiles that cover an integral number of cells and cannot be split to make smaller metatiles [5]. When tiling an n-board with h and f, the possible metatiles are ff (two interlocking fences we call a bifence), and the families $h(ff)^j h$, $fh(ff)^j h$, $h(ff)^j fh$, and $fh(ff)^j fh$, where $j \geq 0$, which have lengths of $2j+1$, $2j+2$, $2j+2$, and $2j+3$, respectively [6]. Notice that because all metatiles contain 0 or 2 half-squares, all tilings contain an even number of half-squares. The fh at the start of the second family, at the end of the third family, and at both ends of the fourth family is referred to as a filled fence (a fence with its gap filled by a half-square). If a half-square is inside (outside) a fence, it is said to be captured (free). On tiling a board, a tile of either type may always follow a tile of either type before the final h is placed. After the final h is placed, the remainder of the board must be filled with bifences, if anything. In other words, in the symbolic representation of the tiling as a string of f and h, these symbols may be placed in any order before the final h. After this, there cannot be an odd number of f because this would imply an unfilled fence at the end.

Identity 2.1. *For $n \geq 0$,*

$$F_{n+1}^2 = \begin{cases} \sum_{k=1}^{n} kF_{n+1-k}^2, & n \text{ odd}, \\ 1 + \sum_{k=1}^{n} kF_{n+1-k}^2, & n \text{ even}. \end{cases}$$

Proof. How many tilings of an n-board contain at least two half-squares? *Answer 1*: A_n when n is odd, and $A_n - 1$ when n is even because the all-bifence tiling only occurs for even n. *Answer 2*: Following the method introduced in [2], we condition on the location of the second h. Because the symbolic representation of all non-bifence metatiles end in h, if the kth cell in the n-board contains the second h, the symbolic representation of the tiling of the first k cells must end in h. This leaves one h that may be placed anywhere among the $k-1$ fences and so, there are k ways to tile these first k cells. There are A_{n-k} ways to tile the rest of the board. Summing over all possible k gives $\sum_{k=1}^{n} kA_{n-k}$. After equating this to Answer 1, the identity follows from Theorem 1.2. □

To generalize Identity 2.1 we first need the following definition and lemma. Let $B_n^{(q)}$ be the number of tilings of an n-board that contain exactly $2q$ half-squares. Thus for $n \geq 1$, $B_n^{(0)} = 1$ if n is even (the all-bifence tiling) and is 0 if n is odd. Evidently, $B_n^{(q)} = 0$ if $n < q$ and for convenience, we set $B_0^{(0)} = 1$.

Lemma 2.2. *For $n \geq q > 0$,*

$$B_n^{(q)} = B_{n-2}^{(q)} + \binom{n+q-1}{2q-1}. \tag{2.1}$$

Proof. The symbolic representation of a tiling must end in either h or ff. If it ends in h, we are free to place the remaining $2q-1$ half-squares and $n-q$ fences in any order; this gives $\binom{n+q-1}{2q-1}$ possibilities. If it ends in ff, there are $B_{n-2}^{(q)}$ ways to tile the remaining cells. □

As will be shown at the end of Section 3, $B_n^{(q)}$ is, for $n \geq q \geq 0$, the (n,q)th entry of a Riordan array and the generating function for it is known.

Identity 2.3. *For $p > 0$,*

$$F_{n+1}^2 = \sum_{q=0}^{p-1} B_n^{(q)} + \sum_{k=p}^{n} \binom{k+p-1}{2p-1} F_{n+1-k}^2.$$

Proof. How many tilings of an n-board contain at least $2p$ half-squares? *Answer 1*: The total number of tilings minus the tilings that contain less than $2p$ half-squares, i.e.,

$$A_n - \sum_{q=0}^{p-1} B_n^{(q)}.$$

Answer 2: We condition on the location of the $2p$th half-square. If it occurs in the kth cell, the symbolic representation of the tiling up to that cell must end in h. There are $\binom{k+p-1}{2p-1}$ ways to order the remaining $2p-1$ half-squares and $k-p$ fences and A_{n-k} ways to tile the rest of the board. Summing over all possible k and equating the result to Answer 1 gives

$$A_n - \sum_{q=0}^{p-1} B_n^{(q)} = \sum_{k=p}^{n} \binom{k+p-1}{2p-1} A_{n-k},$$

and the identity follows from Theorem 1.2. □

Identity 2.4. *For $n > 0$,*

$$F_{n+1}^2 = 2n - 1 + \sum_{m=2}^{n-1} (2m^2 - 2m - 1) F_{n-m}^2.$$

Proof. How many tilings of an n-board use at least two fences? *Answer 1*: $A_n - 1 - 2(n-1)$ because the all-h tiling uses no fences and there are $2(n-1)$ ways to place a single filled fence on an n-board otherwise filled with half-squares. *Answer 2*: We condition on the position of the start of the first metatile that contains the second fence. If the first k cells are filled with half-squares and cell $k+1$ is the start of a metatile of length $j = 2, \ldots, n-k$ containing at least two fences, on summing over all possible j and noting that there are two such metatiles for $j > 2$ but only one for $j = 2$, we have $\sum_{j=2}^{n-k} (2 - \delta_{j,2}) A_{n-k-j}$ ways to tile from cell $k+1$ onwards. If the first k cells contain exactly one fence, there are $2(k-1)$ ways to place this fence. The metatile of length j starting at cell $k+1$ needs to contain one or more fences. Hence, there are $\sum_{j=2}^{n-k} (2 + \delta_{j,2}) A_{n-k-j}$ ways to tile the remaining cells. Combining these possibilities, summing over all possible k, and then equating to Answer 1 gives

$$\begin{aligned}
A_n - 2n + 1 &= \sum_{k=0}^{n-2} \sum_{j=2}^{n-k} (2 - \delta_{j,2}) A_{n-k-j} + 2 \sum_{k=1}^{n-2} (k-1) \left(\sum_{j=2}^{n-k} (2 + \delta_{j,2}) A_{n-k-j} \right) \\
&= \sum_{j=2}^{n} (2 - \delta_{j,2}) A_{n-j} + \sum_{k=1}^{n-2} A_{n-k-2} + 2 \sum_{k=1}^{n-3} \sum_{j=3}^{n-k} A_{n-k-j} \\
&\quad + 2 \sum_{k=2}^{n-2} (k-1) A_{n-k-2} + 4 \sum_{k=2}^{n-2} (k-1) \sum_{j=2}^{n-k} A_{n-k-j} \\
&= A_{n-2} + 2 \sum_{m=2}^{n-1} A_{n-m-1} + \sum_{m=2}^{n-1} A_{n-m-1} + 2 \sum_{m=3}^{n-1} (m-2) A_{n-m-1} \\
&\quad + 2(A_{n-4} + \cdots + A_0 + A_{n-5} + \cdots + A_0 + \cdots + A_0) \\
&\quad + 4(A_{n-4} + \cdots + A_0 + 2(A_{n-5} + \cdots + A_0) + \cdots + (n-3)A_0) \\
&= A_{n-2} + \sum_{m=2}^{n-1} (2m-1) A_{n-m-1} + 2 \sum_{r=0}^{n-4} (n-3-r) A_r + 4 \sum_{s=1}^{n-3} (1 + \cdots + s) A_{n-s-3}
\end{aligned}$$

$$= A_{n-2} + \sum_{m=2}^{n-1}(2m-1)A_{n-m-1} + 2\sum_{m=3}^{n-1}(m-2)A_{n-m-1} + 4\sum_{m=3}^{n-1}\tfrac{1}{2}(m-2)(m-1)A_{n-m-1}$$
$$= A_{n-2} + \sum_{m=2}^{n-1}(2m^2-2m-1)A_{n-m-1},$$

where we used $r = n - m - 1$ and $s = m - 2$. The identity follows from Theorem 1.2. □

Although the following two identities are certainly already known, we include them as the method of proof appears to be new.

Identity 2.5. *For $n > 3$, $F_{n+1}^2 = F_n^2 + 4F_{n-1}^2 + F_{n-2}^2 - F_{n-3}^2$.*

Proof. For $n > 3$, how many tilings of an n-board are there? *Answer 1*: A_n. *Answer 2*: We condition on the end tiles. If the first and last tiles are free half-squares, there are A_{n-1} ways to tile the remaining cells in between. If the tiling starts and ends with a filled fence, there are A_{n-3} ways to tile the remaining cells. If the first tile is an h and the last is a filled fence or vice versa or the tiling starts or ends with a bifence, in each of these four cases there are A_{n-2} ways to tile the remaining cells. We have counted tilings that start and end in a bifence (of which there are A_{n-4}) twice and so we must subtract these to leave, on equating both answers, $A_n = A_{n-1} + 4A_{n-2} + A_{n-3} - A_{n-4}$, and the identity follows from Theorem 1.2. □

Identity 2.6. *For $n > 2$,*

$$F_{n+1}^2 = F_{n-2}^2 + 4\sum_{j=1}^{n-1} F_j^2.$$

Proof. How many tilings of an n-board contain at least one fence? *Answer 1*: $A_n - 1$ because the only tiling not containing a fence is the all-h tiling. *Answer 2*: For $n \geq 4$, such a board can be tiled in the following ways: $h[[n-1]]h$, $h[n-2]fh$, $fh[n-2]h$, $f^2[n-2]$, $[n-2]f^2$, or $fh[n-3]fh$, where $[m]$ and $[[m]]$ represent an arbitrary m-board and an m-board containing at least one fence, respectively. Hence, if C_m is the number of tilings of an m-board that contain at least one fence, then

$$C_n = C_{n-1} + 4A_{n-2} + A_{n-3} - A_{n-4}, \tag{2.2}$$

where the $-A_{n-4}$ term is to compensate for $f^2[n-2]$ with $[n-2]f^2$ counting the $f^2[n-4]f^2$ tiling twice. This duplication does not occur for $n < 4$ and so, $C_3 = C_2 + 4A_1 + A_0$ and $C_2 = 3 = 4A_0 - 1$ (because the tilings of a two-board containing a fence are hfh, fhh, and f^2). Using (2.2) to replace the first term on the right side of (2.2) recursively, and equating the result to Answer 1 gives $A_n - 1 = A_{n-3} - 1 + 4\sum_{j=0}^{n-2} A_j$, and the identity follows from Theorem 1.2. □

Identity 2.7. *For $n \geq 0$,*

$$\sum_{k=1}^{n}(-1)^k F_{k+1}^2 = (-1)^n \left\{ F_n^2 + 2\sum_{j=0}^{\lfloor n/2-1 \rfloor} F_{n-2j-1}^2 \right\}.$$

Proof. Following the description-involution-exception (DIE) method of [3], the 'description' is all possible positive-length tilings of boards not greater than length n. The sign-reversing involution I for any metatile x satisfies $x = I(I(x))$ and the lengths of x and $I(x)$ are of opposite parity. We make $I(h^2) = f^2$, $I(h(f^2)^j fh) = h(f^2)^{j+1}h$, and $I(fh(f^2)^j h) = fh(f^2)^j fh$ with $j \geq 0$. All of these cases give a mapping to a metatile of length one more. The remaining cases (which follow from $x = I(I(x))$) give a mapping to a metatile of length one less. I is applied to the final metatile of a tiling of a k-board to generate a tiling of a $(k-1)$- or $(k+1)$-board. The exceptions are when an n-board ends in h^2, $h(f^2)^j fh$, or $fh(f^2)^j h$ because the involution would then generate a tiling of an $(n+1)$-board. The number of tilings of an n-board ending in these metatiles are A_{n-1}, A_{n-2j-2}, and A_{n-2j-2}, respectively. Hence

$$\sum_{k=1}^{n}(-1)^k A_k = (-1)^n \left\{ A_{n-1} + 2\sum_{j=0}^{\lfloor n/2-1 \rfloor} A_{n-2j-2} \right\},$$

and the identity follows from Theorem 1.2. □

The general identity that follows can be used to relate the squares of the Fibonacci numbers to other sequences. As described in the following lemma, the sequences arise when we consider tilings that lack one or more types of metatile and specific examples of the general identity were given as Identities 4.5, 4.7, and 4.9 in [6] (and appear again here as the 2nd, 8th, and 9th identities, respectively, in Table 1). We first define $S_n^{(P,m,l_0,d)}$ to be the number of ways to tile an n-board using half-squares and $(\frac{1}{2}, \frac{1}{2})$-fences such that only metatiles drawn from the disjoint sets P and Q are used. Set P contains finitely many metatiles. If $m = 0$, set Q is empty. Otherwise, it contains infinitely many metatiles: Q contains m metatiles of each length $l_0 + jd$ for $j = 0, 1, 2, \ldots$, where $l_0 \geq 3$. When tiling with h and f, there are at most two metatiles of each length $l_0 + jd$ and so, m cannot exceed 2. Using this notation, we already have that $S_n^{(P,2,3,1)} = A_n$, where $P = \{h^2, f^2, hfh, fhh\}$ is the set of all possible metatiles of length less than 3, and the remaining parameters describe the lengths and numbers of all metatiles of length greater than 2, namely, two of each length $3, 4, \ldots$.

Lemma 2.8. *Writing $S_n^{(P,m,l_0,d)}$ as S_n for brevity,*

$$S_n = \begin{cases} \delta_{0,n} + \sum_{i=1}^{|P|} S_{n-l_i}, & m = 0, \\ \delta_{0,n} - \delta_{d,n} + S_{n-d} + mS_{n-l_0} + \sum_{i=1}^{|P|}(S_{n-l_i} - S_{n-d-l_i}), & m > 0, \end{cases} \tag{2.3}$$

where l_i is the length of the ith metatile in P and $S_{n<0} = 0$.

Proof. Conditioning on the last metatile gives

$$S_n = \delta_{0,n} + \sum_{i=1}^{|P|} S_{n-l_i} + m\sum_{j=0}^{\infty} S_{n-l_0-jd} \tag{2.4}$$

with $S_{n<0} = 0$. If $m > 0$, subtracting (2.4) with n replaced by $n-d$ from (2.4) gives (2.3). □

Identity 2.9. *For $n \geq 0$,*

$$F_{n+1}^2 = S_n + \sum_i \sum_{k=l_i}^{n} F_{k-l_i+1}^2 S_{n-k},$$

where l_i is the length of the ith metatile in the set $\overline{P \cup Q}$.

Proof. How many tilings of an n-board contain at least one metatile that is not in the set $P \cup Q$? *Answer 1*: $A_n - S_n$. *Answer 2*: We condition on the location of the last metatile not in the set $P \cup Q$. If this metatile has length l, then the number of tilings when it lies on cells $k - l + 1$ to k (for $k = l, \ldots, n$) is $A_{k-l}S_{n-k}$. Summing over the possible k and possible metatiles and equating the two answers gives

$$A_n - S_n = \sum_i \sum_{k=l_i}^{n} A_{k-l_i} S_{n-k}.$$

The identity follows from Theorem 1.2. □

Choosing $P = \{h^2, f^2\}$ and $m = 0$ gives $S_n = F_{n+1}$. From Identity 2.9, one then obtains the following identity.

Identity 2.10. *For $n > 0$,*

$$F_{n+1}^2 = F_{n+1} + 2\sum_{l=2}^{n}\sum_{k=l}^{n} F_{k-l+1}^2 F_{n+1-k}.$$

Further identities relating the Fibonacci numbers squared to other number sequences are summarized in Table 1.

TABLE 1. Particular cases of Identity 2.9 where $S_n = \delta_{0,n} - (\delta_{m,1} + \delta_{m,2})\delta_{d,n} + c_1 S_{n-1} + c_2 S_{n-2} + c_3 S_{n-3}$. The OEIS sequence numbers are for each S_n.

P	m, l_0, d	c_1, c_2, c_3	OEIS sequence	identity: $F_{n+1}^2 - S_n =$
$\{f^2, hfh, fhh\}$	$2,3,1$	$1,3,-1$	A052973	$\sum_{k=1}^{n} F_k^2 S_{n-k}$
$\{h^2, hfh, fhh\}$	$2,3,1$	$2,1,0$	A001333	$\sum_{k=2}^{n} F_{k-1}^2 S_{n-k}$
$\{h^2\}$	$2,3,1$	$2,-1,2$	A007909	$3\sum_{k=2}^{n} F_{k-1}^2 S_{n-k}$
$\{f^2\}$	$2,3,1$	$1,1,1$	A001590	$S_{n-1} + \sum_{k=2}^{n}(F_k^2 + 2F_{k-1}^2)S_{n-k}$
$\{h^2, f^2\}$	$2,3,1$	$2,0,1$	A052980	$2\sum_{k=2}^{n} F_{k-1}^2 S_{n-k}$
$\{hfh, fhh\}$	$2,3,1$	$1,2,0$	A078008	$S_{n-1} + \sum_{k=2}^{n}(F_k^2 + F_{k-1}^2)S_{n-k}$
$\{h^2, f^2, hfh\}$	$1,3,1$	$2,1,-1$	A077998	$\sum_{l=2}^{n}\sum_{k=l}^{n} F_{k-l+1}^2 S_{n-k}$
$\{h^2, fhh, hfh, fhfh\}$	0,-,-	$1,2,1$	A002478	$\sum_{l=2}^{n}\sum_{k=l}^{n}(2 - \delta_{l,2} - \delta_{l,3})F_{k-l+1}^2 S_{n-k}$
$\{h^2\}$	$2,3,2$	$1,1,1$	A000213	$\sum_{j=1}^{\lfloor n/2\rfloor}\sum_{k=2j}^{n}(2 + \delta_{j,1})F_{k-2j+1}^2 S_{n-k}$

3. Pascal-Like Triangle

$n \setminus k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	0								
2	1	2	1							
3	1	4	4	0						
4	1	6	11	6	1					
5	1	8	22	24	9	0				
6	1	10	37	62	46	12	1			
7	1	12	56	128	148	80	16	0		
8	1	14	79	230	367	314	130	20	1	
9	1	16	106	376	771	920	610	200	25	0

FIGURE 1. A Pascal-like triangle with entries $\left[{n \atop k}\right]$ (A123521 in [9]).

As in [5], we form a Pascal-like triangle by tabulating $\left[{n \atop k}\right]$, the number of tilings of an n-board that use k fences (Figure 1). The choice $\left[{0 \atop 0}\right] = 1$ is justified in the proof of Identity 3.7.

Identity 3.1. *For* $n \geq 0$,

$$F_{n+1}^2 = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}. \tag{3.1}$$

Proof. The right side of (3.1) is the sum of row n, which gives all possible ways to tile an n-board. The result then follows from Theorem 1.2. □

Identity 3.2. *For* $n \geq k \geq 0$,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j=k-m}^{m} \binom{n-j}{j} \binom{n-(k-j)}{k-j},$$

where $m = \min(\lfloor n/2 \rfloor, k)$.

Proof. From Lemma 1.1, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ is also the number of square-domino tilings of an ordered pair of n-boards that use a total of k dominoes. The number of ways to tile an n-board with j dominoes (and $n-2j$ squares) is $\binom{n-j}{j}$. If one of the n-boards has j dominoes, the other will have $k-j$ dominoes. Hence, there are $\binom{n-j}{j}\binom{n-(k-j)}{k-j}$ ways to tile the n-boards if the first board has j dominoes. Evidently, j cannot exceed k or $\lfloor n/2 \rfloor$ and so $m \geq j \geq k-m$. We then sum over all possible values of j. □

Identity 3.3. *For* $n \geq 0$, $\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right] = 1$.

Proof. This corresponds to the all half-square tiling of an n-board. □

Identity 3.4. *For* $n \geq 0$, $\left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right]$ *is 1 if* n *is even and is 0 otherwise.*

Proof. A bifence is of length 2 (and is composed of two fences) and is the only metatile without half-squares. Thus, the fence-only tiling can only occur when n is even. □

Identity 3.5. *For* $n \geq 1$, $\left[\begin{smallmatrix} n \\ 1 \end{smallmatrix}\right] = 2(n-1)$.

Proof. Only the hfh and fhh metatiles contain one fence. Both metatiles are length 2. There are $n-1$ ways to place a length-2 metatile on an n-board (with the remaining cells occupied by h^2 metatiles). □

Identity 3.6. *For* $n \geq q \geq 0$, $\left[\begin{smallmatrix} n \\ n-q \end{smallmatrix}\right] = B_n^{(q)}$.

Proof. The result follows from the definition of $B_n^{(q)}$ because $\left[\begin{smallmatrix} n \\ n-q \end{smallmatrix}\right]$ is also the number of tilings containing $2q$ half-squares. □

Identity 3.7.

$$\begin{bmatrix} n \\ k \end{bmatrix} = \delta_{n,0}\delta_{k,0} - \delta_{n,1}\delta_{k,1} + \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-2 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}, \tag{3.2}$$

where $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = 0$ *if* $n < k$ *or* $n < 0$.

Proof. We condition on the last metatile. If that metatile is of length l and contains j fences, there are $\left[\begin{smallmatrix} n-l \\ k-j \end{smallmatrix}\right]$ ways to tile the remaining cells using $k-j$ fences. Considering all possible metatiles gives

$$\begin{bmatrix} n \\ k \end{bmatrix} = \delta_{n,0}\delta_{k,0} + \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} + 2\sum_{j=1}^{\infty} \begin{bmatrix} n-j-1 \\ k-j \end{bmatrix}. \tag{3.3}$$

If $n = l$ and $k = j$, there is exactly one way to tile the whole board (i.e., by using that single metatile) and so, we let $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] = 1$. Replacing n by $n-1$ and k by $k-1$ in (3.3) and subtracting the result from (3.3) gives (3.2). □

A $(p(x), q(x))$ Riordan array is a lower triangular matrix whose (n,k)th entry is the coefficient of x^n in the series for $p(x)\{q(x)\}^k$ [8]. Our triangle has the following simple relation to the $(1/(1-x^2), x/(1-x)^2)$ Riordan array (whose entries are the absolute value of the triangle A158454 in [9]).

Theorem 3.8. *If $R(n,k)$ is the (n,k)th entry of the $(1/(1-x^2), x/(1-x)^2)$ Riordan array, then*

$$\begin{bmatrix} n \\ k \end{bmatrix} = R(n, n-k). \tag{3.4}$$

Proof. Let $p = 1/(1-x^2)$ and $q = x/(1-x)^2$. Then $R(n-l, k-j)$ is the coefficient of x^n in the expansion of $x^l pq^{k-j}$. Multiplying the identity $q = xq + x + x^2q + x^2 - x^3q$ by pq^{k-1} and taking the coefficient of x^n gives $R(n,k) = R(n-1,k) + R(n-1,k-1) + R(n-2,k) + R(n-2,k-1) - R(n-3,k)$ for $n > 2$, $k > 0$. Taking $R(n<0,k) = R(n<k,k) = 0$ and including terms to arrive at a relation that is also compatible with the values of $R(k,n)$ for $0 \le n \le 2$ and $k = 0$ gives

$$R(n,k) = \delta_{n,0}\delta_{k,0} - \delta_{n,1}\delta_{k,0} + R(n-1,k) + R(n-1,k-1) + R(n-2,k) + R(n-2,k-1) - R(n-3,k), \tag{3.5}$$

which is then valid for all n and k. Substituting (3.4) into (3.2), replacing k by $n-k$, and noting that $\delta_{n,0}\delta_{n-k,0} - \delta_{n,1}\delta_{n-k,1}$ can be rewritten as $\delta_{n,0}\delta_{k,0} - \delta_{n,1}\delta_{k,0}$ gives (3.5). □

As an immediate consequence of Theorem 3.8, we see that $F_{n+1}^2 = \sum_{k=0}^{n} R(n,k)$ for $n \ge 0$ (which was also noted in [1]), and from Identity 3.6, $R(n,k) = B_n^{(k)}$. In other words, a combinatorial interpretation of $R(n,k)$ is the number of half-square, $(\frac{1}{2}, \frac{1}{2})$-fence tilings of an n-board that use $2k$ half-squares. Then from Lemma 2.2, we have, for $n \ge k \ge 0$,

$$R(n,k) = R(n-2,k) + \binom{n+k-1}{2k-1}. \tag{3.6}$$

4. Tiling with $(\frac{1}{4}, \frac{1}{4})$- and $(\frac{1}{4}, \frac{3}{4})$-fences

There is a bijection between the tiling of an n-board with half-squares and $(\frac{1}{2}, \frac{1}{2})$-fences and an n-board tiling using $(\frac{1}{4}, \frac{1}{4})$- and $(\frac{1}{4}, \frac{3}{4})$-fences. Hence, the latter tiling gives an alternative combinatorial interpretation of the Fibonacci numbers squared. To see the bijection, we construct directed pseudographs (or 'digraphs') for both types of tiling (Figure 2). These allow one to determine all the possible metatiles systematically [7]. Each arc corresponds to the addition of a tile or tiles to the yet-to-be-completed metatile at the next available gap. The 0-node represents the starting configuration (an empty board) and the end configuration (a completed metatile, which means that an integral number of cells have been filled with no gaps). The names of the other nodes indicate the configuration of the remaining cells on the board, starting from the first gap in the tiling so far. The digit 0 (1) represents an empty (filled) sub-cell of width w (when tiling with (w,g)-fences). A bar over the first 0 indicates that the left side of the gap does not coincide with a cell boundary. Each walk, starting and ending at the 0-node without visiting it in between, corresponds to a distinct metatile. Hence, if there is a cycle that does not include the 0-node (e.g., the one connecting nodes $\bar{0}$ and 01 in Figure 2(a)), there will be an infinite number of metatiles because a walk can follow the cycle an arbitrary number of times before returning to the 0-node. The crucial property of a metatile is its length. This is obtained by summing the lengths corresponding to each arc on the walk. For the half-square and $(\frac{1}{2}, \frac{1}{2})$-fence tiling, h and f contribute lengths of $\frac{1}{2}$ and 1, respectively. For the $(\frac{1}{4}, \frac{1}{4})$- and $(\frac{1}{4}, \frac{3}{4})$-fence tiling, both types of fence contribute lengths of $\frac{1}{2}$.

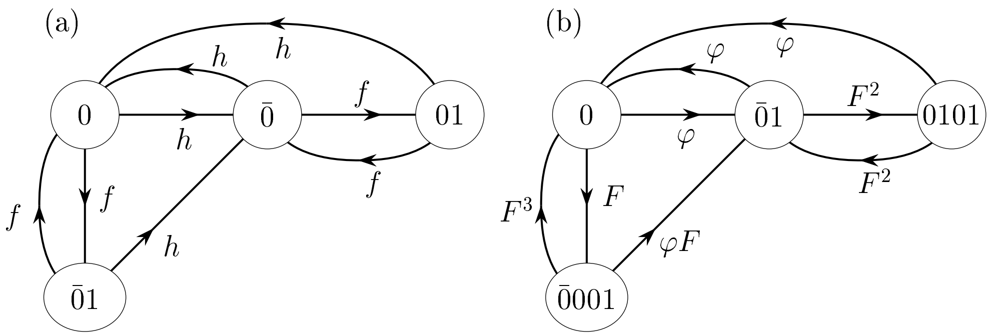

FIGURE 2. Digraphs for generating all possible metatiles when tiling with: (a) half-squares (h) and $(\frac{1}{2}, \frac{1}{2})$-fences ($f$); (b) $(\frac{1}{4}, \frac{1}{4})$-fences ($\varphi$) and $(\frac{1}{4}, \frac{3}{4})$-fences ($F$).

There is a bijection between the two types of tiling because, other than the labels on the arcs and nodes, the two digraphs are identical and any walk on one graph and the corresponding walk on the other generate metatiles of the same length. More explicitly, the possible metatiles for the φ-F tiling are F^4 (represented by the F arc from the 0 node to the $\bar{0}001$ node and the F^3 arc back), and for all $j \geq 0$, where j is the number of times the cycle composed of the two F^2 arcs is traversed, $\varphi(F^4)^j\varphi$, $F\varphi F(F^4)^j\varphi$, $\varphi(F^4)^jF^2\varphi$, and $F\varphi F(F^4)^jF^2f$, and their respective lengths are 2, $2j+1$, $2j+2$, $2j+2$, and $2j+3$.

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