A NONLINEAR RECURRENCE AND ITS RELATIONS TO CHEBYSHEV POLYNOMIALS

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Abstract. We solve a second order recurrence; the solutions are second order linear sequences. The solutions are related to evaluated partial sums of Chebyshev polynomials.

1. Nonlinear Recurrence

For constants p and q with initial values $u_0 = a$ and $u_1 = b$, consider the nonlinear recurrence relation

$$u_{n+1}(u_n + u_{n-1}) = pu_n^2 - u_n u_{n-1} - q. \tag{1}$$

Lemma 1.1.

$$\frac{u_{n-1}^2 + u_n^2 - (p-1)u_n u_{n-1} - q}{u_{n-1} + u_n} = \frac{a^2 + b^2 - (p-1)ab - q}{b + a}.$$

Proof. For $n = 1$, the equation of Lemma 1.1 follows immediately from the definition.

Assume this has been shown for n, and we check this for $n + 1$. Substitute the recurrence relation

$$u_{n+1} = \frac{pu_n^2 - u_n u_{n-1} - q}{(u_n + u_{n-1})} \text{ into } \frac{u_n^2 + u_{n+1}^2 - (p-1)u_{n+1}u_n - q}{u_n + u_{n+1}}$$

and simplify to get

$$\frac{u_{n-1}^2 + u_n^2 - (p-1)u_n u_{n-1} - q}{u_{n-1} + u_n}.$$

The result follows. □

Theorem 1.2. *The solutions to this nonlinear recurrence satisfy the second order linear recurrence*

$$u_{n+1} = (p-1)u_n - u_{n-1} + \frac{a^2 + b^2 - (p-1)ab - q}{a + b}. \tag{2}$$

Proof. Using

$$u_{n+1} - (p-1)u_n + u_{n-1} = \frac{pu_n^2 - u_n u_{n-1} - q}{(u_n + u_{n-1})} - (p-1)u_n + u_{n-1} = \frac{u_{n-1}^2 + u_n^2 - (p-1)u_n u_{n-1} - q}{u_{n-1} + u_n}$$

and Lemma 1.1 gives the desired result. □

1.1. Periodic Sequences.

$p = 0$: The sequence is periodic of period 3.
$p = 1$: The sequence is periodic of period 4.
$p = 2$: The sequence is periodic of period 6.

2. Chebyshev Related Sequences

The Chebyshev U-polynomials, also called of the second kind [3], are $U_0 = 1$, $U_1 = 2p$, and $U_2 = 4p^2 - 1$, where $U_{n+1} = 2p \cdot U_n - U_{n-1}$; the nth partial sum $S_n(p)$ is defined as $\sum_{k=0}^{n} U_k$.

Theorem 2.1. *For $n \geq 0$,*

$$u_{n+2}(u_0 + u_1) = -A_n(p)q + A_{n-1}(p)u_0^2 + A_{n+1}(p)u_1^2 + B_n(p)u_0u_1.$$

$A_n(2p+1)$ is the nth partial sum $S_n(p)$ of Chebyshev polynomials and $B_n + 2A_n = 1$.

Proof. It is easy to see that $A_{-1} = 0$, $A_0 = 1$, $A_1 = p$, $A_2 = p^2 - p$, $A_3 = p^3 - 2p^2 + 1$ and $B_0 = -1$, $B_1 = 1 - 2p$, $B_2 = -2p^2 + 2p + 1$, $B_3 = -2p^3 + 4p^2 - 1$. We show that $A_{n+1} = (p-1)A_n - A_{n-1} + 1$ and that $B_n + 2A_n = 1$ by induction.

$$\begin{aligned}
u_{n+4}(u_0 + u_1) &= (p-1)u_{n+3}(u_0 + u_1) - u_{n+2}(u_0 + u_1) + u_0^2 + u_1^2 - (p-1)u_0u_1 - q \\
&= (p-1)(-A_{n+1}q + A_nu_0^2 + A_{n+2}u_1^2 + B_{n+1}u_0u_1) \\
&+ A_nq - A_{n-1}u_0^2 - A_{n+1}u_1^2 - B_nu_0u_1 + u_0^2 + u_1^2 - (p-1)u_0u_1 - q \\
&= -q((p-1)A_{n+1} - A_n + 1) + u_0^2((p-1)A_n - A_{n-1} + 1) \\
&+ u_1^2((p-1)A_{n+2} - A_{n+1} + 1) + u_0u_1((p-1)B_{n+1} - B_n - (p-1)) \\
&= -qA_{n+2} + u_0^2A_{n+1} + u_1^2A_{n+3} + u_0u_1B_{n+2}
\end{aligned}$$

because

$$\begin{aligned}
B_{n+2} &= (p-1)B_{n+1} - (p-1) - B_n \\
&= (p-1)(B_{n+1} - 1) - B_n \\
&= -2(p-1)A_{n+1} + 2A_n - 1 = -2A_{n+2} + 1
\end{aligned}$$

The recurrence

$$A_{n+2} = (p-1)A_{n+1} - A_n + 1$$

yields

$$A_{n+2}(2p+1) = 2pA_{n+1}(2p+1) - A_n(2p+1) + 1.$$

Using generating functions, we show that $A_{n+2}(2p+1)$ has the same recurrence relation and initial values as the partial sums of Chebyshev polynomials.

The generating series $\alpha(x) = \sum_{n\geq 0} A_nx^n$ is determined from

$$\sum_{n\geq 0} A_{n+2}x^{n+2} = (p-1)x\sum_{n\geq 0} A_{n+1}x^{n+1} - x^2\sum_{n\geq 0} A_nx^n + x^2\sum_{n\geq 0} x^n,$$

which gives

$$\alpha(x) - 1 - px = (p-1)x(\alpha(x) - 1) - \alpha(x)x^2 + \frac{x^2}{1-x},$$

so

$$\alpha(x) = \frac{1}{(1-x)(1-(p-1)x+x^2)}.$$

Replacing p by $2p+1$ gives the generating series for the partial sums of the Chebyshev polynomials. □

We can apply similar methods to prove analogues for the recurrence appearing in [1]. In that case, alternating partial sums of Chebyshev polynomials are the analogue; the details are left for the interested reader.

2.1. **Related Examples from OEIS,** [2].

$p = 12$: A097826, $p = 7$: A053142, $p = 6$: A089817, $p = 5$: A061278, $p = 4$: A027941.
$p = 3$: A145910, $q = 0$, $u_0 = -1$, $u_1 = 2$.
$p = -2$: Express sequence as

$$u_n = \pm \frac{a_n q + b_n u_0^2 + c_n u_1^2 + d_n u_0 u_1}{u_0 + u_1}$$

for $n \geq 2$, then $a_n = 1, 2, 6, 15, 40, \ldots$, $b_n = 0, 1, 2, 6, 15, 40, \ldots$, $c_n = 2, 6, 15, 40, 104, \ldots$ are part of A001654 and $d_n = 1, 5, 11, 31, 79, 209, \ldots$ is A236428.
$p = -3$: A109437, A217233.
$p = -4$: A099025.

References

[1] R. C. Alperin, *A family of nonlinear recurrences and their linear solutions*, The Fibonacci Quarterly, **57.4** (2019), 318-321.
[2] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, `https://oeis.org`.
[3] T. J. Rivlin, *Chebychev Polynomials*, Wiley, 1974.

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