THE FIBONACCI QUILT GAME

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ABSTRACT. Zeckendorf [13] proved that every positive integer can be expressed as the sum of nonconsecutive Fibonacci numbers. This theorem inspired a beautiful game, the Zeckendorf Game [2]. Two players begin with n 1's and take turns applying rules inspired by the Fibonacci recurrence, $F_{n+1} = F_n + F_{n-1}$, until a decomposition without consecutive terms is reached; whoever makes the last move wins. We look at a game resulting from a generalization of the Fibonacci numbers, the Fibonacci Quilt sequence [3]. This sequence arises from the two-dimensional geometric property of tiling the plane through the Fibonacci spiral. Beginning with 1 in the center, we place integers in the squares of the spiral such that each square contains the smallest positive integer that does not have a decomposition as the sum of previous terms that do not share a wall. This sequence eventually follows two recurrence relations, allowing us to construct a variation on the Zeckendorf Game, the Fibonacci Quilt Game. Whereas some properties of the Fibonacci sequence are inherited by this sequence, the nature of its recurrence leads to others, such as Zeckendorf's theorem, no longer holding. Thus, it is of interest to investigate the generalization of the game in this setting to see which behaviors persist. We prove, similar to the original game, that this game also always terminates in a legal decomposition. We give a lower bound on game lengths, showing that, depending on strategies, the length of the game can vary and either player could win. Finally, we give a conjecture on the length of a random game.

1. INTRODUCTION

1.1. **History.** The Fibonacci numbers are one of the most famous sequences of all time; appearing throughout mathematics and nature [9]. Zeckendorf [13] proved that every positive integer has a unique representation as a sum of nonconsecutive Fibonacci numbers, which are defined by $F_1 = 1, F_2 = 2$, and $F_{n+1} = F_n + F_{n-1}$; conversely, an equivalent definition of the Fibonacci numbers is the unique sequence of integers such that every positive integer can be uniquely written as a sum of nonconsecutive terms. Here, we set the initial conditions $F_1 = 1$ and $F_2 = 2$ rather then $F_1 = F_2 = 1$ to preserve uniqueness. This is the first of many interplays between notions of legal decomposition and definitions of a sequence, expanded to a large class of linear recurrences (see [6, 7, 11, 12] for examples).

1.1.1. *The Zeckendorf Game.* We can use these notions of legal decomposition to create interesting games. The first, the Zeckendorf Game, was defined based on the recurrence relation of the Fibonacci sequence $\{F_n\}$. We briefly summarize the results from [2, 1].

We first introduce some notation. Let $\{F_1^n\}$ denote n copies of F_1, and in general $\{F_i^n\}$ denote n copies of F_i; because we never raise Fibonacci numbers to a power, there should be no confusion as to what is meant. For example, $\{F_1^3 \wedge F_4^2 \wedge F_5^1\}$ would be three copies of $F_1 = 1$, two copies of $F_4 = 5$, and one copy of $F_5 = 8$. For simplicity, we omit exponents of 1, so $\{F_i\} = \{F_i^1\}$.

Definition 1.1 (The Two-Player Zeckendorf Game)**.** *At the beginning of the game, there is an unordered list of n 1's. Let $F_1 = 1$, $F_2 = 2$, and $F_{i+1} = F_i + F_{i-1}$; therefore, the initial list is $\{F_1^n\}$. On each turn, a player can do one of the following moves.*

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(1) *If the list contains two consecutive Fibonacci numbers, F_{i-1} and F_i, then a player can remove these and replace with F_{i+1}. We denote this move $\{F_{i-1} \wedge F_i \to F_{i+1}\}$.*
(2) *If the list has two (or more) of the same Fibonacci number, F_i, then*
  (a) *if $i = 1$, a player can change two F_1's to F_2, denoted by $\{F_1^2 \to F_2\}$,*
  (b) *if $i = 2$, a player can change two F_2's to F_1 and F_3, denoted by $\{F_2^2 \to F_1 \wedge F_3\}$, and*
  (c) *if $i \geq 3$, a player can change two F_i's to F_{i-2} and F_{i+1}, denoted by $\{F_i^2 \to F_{i-2} \wedge F_{i+1}\}$.*

The players take turns moving. The game ends when no more moves are possible, and the last player to move wins.

Baird-Smith, Epstein, Flint, and Miller [2, 1] proved that this game always terminates in a finite number of moves in the Zeckendorf decomposition of n, and then bounded the game length. One of the key ingredients in their proof is that there is no decomposition involving sums of Fibonacci numbers with fewer summands than the Zeckendorf decomposition. This is proved using a monovariant related to the number and indices of each term and has been generalized to many other sequences [5].

Theorem 1.2. *The shortest game reaches the Zeckendorf decomposition in $n - Z(n)$ moves, where $Z(n)$ is the number of terms in the Zeckendorf decomposition of n. The longest game is bounded by $i \times n$, where i is the index of the largest Fibonacci number less than or equal to n.*

Because there is a large range between the lower and upper bounds, they also conjectured on the length of a random game.

Conjecture 1.3. *As n goes to infinity, the number of moves in a random game, when all legal moves are equally likely, converges to a Gaussian.*

Finally, they found that for $n > 2$, Player 2 has the winning strategy; interestingly, however, the proof is nonconstructive. Although it is known that Player 2 can win, it is not known how they should play.

In this paper, we generalize their results by replacing the Fibonacci numbers with the Fibonacci Quilt. We define this sequence in the next section, and explain why this is an interesting extension.

1.1.2. *The Fibonacci Quilt Sequence.* Previous work extended Zeckendorf's theorem to a wide class of recurrence relations (see [6, 7]), and has extensively studied the behavior of these decompositions. Lekkerkerker [10] proved the mean number of terms needed in a decomposition grows linearly with the largest index in the decomposition, and Koloğlu, Kopp, Miller, and Wang [8, 11, 12] expanded this to show the distribution of the number of terms in a decomposition of n between two consecutive terms of the sequence is Gaussian. This work, however, is done on a class of recurrences called PLRS (for Positive Linear Recurrence Sequences). Briefly, these are fixed depth constant coefficient linear recurrences where the coefficients are nonnegative integers, the first coefficient in the recurrence is positive, and the initial conditions are chosen appropriately; if the first coefficient is not positive, then different behavior can happen, in particular unique decomposition is often lost.

When looking to expand this work further, Catral, Ford, Harris, Miller, and Nelson [3] wanted to explore new patterns. The Fibonacci Quilt sequence arises from a two-dimensional construction and is eventually dictated by a recurrence relation with first coefficient zero; thus, the previous work is not applicable here and although some properties are the same, we will see others are different.

Recall the alternative definition of the Fibonacci numbers stated above; they are the unique sequence of integers such that every positive integer can be uniquely written as a sum of nonconsecutive terms. The Fibonacci Quilt sequence is similarly defined on the Fibonacci spiral, where each term added is the smallest positive integer that cannot be expressed as the sum of nonadjacent previous terms.

The spiral is known in quilting communities as the Log Cabin pattern, giving this sequence its name. To construct the sequence begin with 1 in the q_1 position, then spiral out adding the smallest

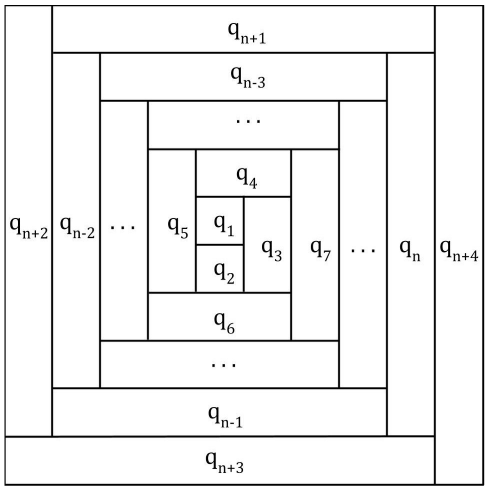

FIGURE 1. Log Cabin Quilt Pattern

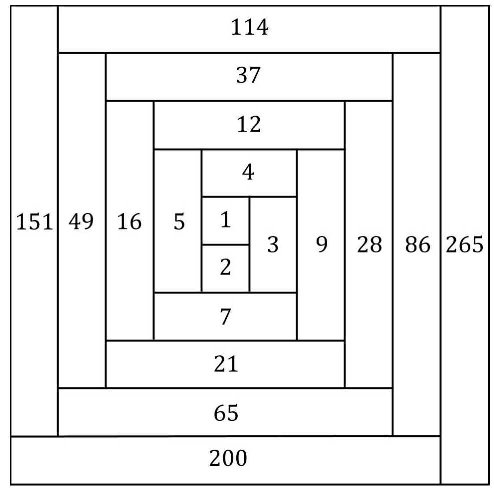

FIGURE 2. The Fibonacci Quilt Sequence

positive integer that cannot be expressed as the sum of nonadjacent previous terms; two terms are adjacent if they share part of a wall. To display more terms, we adjust the size of the spiral and make all horizontal distances 1 unit, and have the vertical distances the appropriate size from the spiral. For example, the first positive integer we do not add is 6, because it can be expressed as $2+4$. To formalize our definition of this sequence, we must first formalize what it means to be expressed as the sum of nonadjacent terms.

Definition 1.4 (FQ-legal decomposition)**.** [3] *Let an increasing sequence of positive integers $\{q_i\}_{i=1}^{\infty}$ be given. We declare a decomposition of an integer*

$$m \;=\; q_{\ell_1} + q_{\ell_2} + \cdots + q_{\ell_t} \tag{1.1}$$

(where $q_{\ell_i} > q_{\ell_{i+1}}$) to be an FQ-legal decomposition if, for all i and j, we have $|\ell_i - \ell_j| \neq 0, 1, 3, 4$ and $\{1,3\} \not\subset \{\ell_1, \ell_2, \ldots, \ell_t\}$.

To better understand this definition, see Figure 1. Looking at terms less than or equal to itself, q_{n+4} is adjacent to itself, q_n, q_{n+1}, and q_{n+3}, thus, if any of these were present in the decomposition of some m, then q_{n+4} could not be present without violating this definition. Further, q_1 and q_3 cannot both be present because they are adjacent in the center of the quilt, although no other q_n and q_{n+2} are. Unlike the Fibonacci numbers, not all integers have a unique FQ-legal decomposition; for example, $8 = 1 + 7 = 3 + 5$ are both FQ-legal decompositions of 8.

With this, we can now formalize the definition of the Fibonacci Quilt Sequence.

Definition 1.5 (Fibonacci Quilt Sequence)**.** [3] *An increasing sequence of positive integers $\{q_i\}_{i=1}^{\infty}$ is called the Fibonacci Quilt sequence if every q_i ($i \geq 1$) is the smallest positive integer that does not have an FQ-legal decomposition using the elements $\{q_1, \ldots, q_{i-1}\}$.*

Although this definition is mathematically precise, in practice it is still computation and time intensive to determine q_n, even knowing $q_1, q_2, \ldots, q_{n-1}$. Fortunately, after a short time, the behavior of the sequence can be explained by recurrence relations.

Theorem 1.6 (Recurrence Relations)**.** [3] *Let q_n denote the nth term in the Fibonacci Quilt. Then,*

$$\begin{aligned} q_{n+1} &= q_n + q_{n-4} \text{ for } n \geq 6, \\ q_{n+1} &= q_{n-1} + q_{n-2} \text{ for } n \geq 5, \\ \sum_{i=1}^{n} q_i &= q_{n+5} - 6. \end{aligned} \tag{1.2}$$

Note that the recurrence relation of *minimal* length is the second one above, and because the leading coefficient there (the q_n term) is zero, we do not have a PLRS.[1]

From these recurrence relations we can build our game, which we describe in the next section. Similar to the Zeckendorf Game, the rules follow from the recurrence relations that describe the sequence. However, new features arise from the nonuniqueness of decompositions, and the different behavior of the quilt at the center coming from its two-dimensional definition.

1.2. **Main Results.** Although the Fibonacci Quilt Game is adapted from the Zeckendorf Game, it requires many more moves. This is firstly because in a Zeckendorf Decomposition, there are only two criteria required for legality: no duplicate terms, and no consecutive terms. The Fibonacci Quilt Game requires five criteria, which are direct results of Definition 1.4: no duplicate terms, no consecutive terms, no terms of distance 3 apart, no terms of distance 4 apart, and 1 and 3 cannot both be present. Each of these requirements creates a new rule.

Each of these rules also requires many base case rules, which is due to the construction of the Fibonacci Quilt sequence. The quilt behaves differently in the center causing the recurrence relations in (1.2) to begin later. The base rules are largely intuitive, e.g., $1 \wedge 2 \rightarrow 3$ not 4, as it would be in the general rules. The general rules arise from how the recurrence relation combines terms. The most interesting is Rule (2a) below, which states a certain move can only be done if no other moves are available; without this addition, the game need not terminate. It is similar in spirit to the Greedy-6 decomposition from [3] (which leads to unique decompositions). We will see later that we can associate an almost monovariant to the game; it breaks down for Rule (2a), but our requirements imply that this rule is used at most once, and thus, our quantity is effectively as good as a true monovariant.

The notation used for the Fibonacci Quilt Game is similar to that of the Zeckendorf Game. Let $\{1^n\}$ or $\{q_1^n\}$ be n copies of 1, and in general $\{q_i^n\}$ be n copies of q_i. For example, $\{q_1^3 \wedge q_3^2 \wedge q_4^1\}$ would be three copies of 1, two copies of 3, and one copy of 4.

Definition 1.7 (The Two-Player Fibonacci Quilt Game)**.** *At the beginning of the game, there is an unordered list of n 1's. Let $q_1 = 1$, $q_2 = 2$, $q_3 = 3$, $q_4 = 4$, and, for $i \geq 5$, $q_i = q_{i-3} + q_{i-2}$; therefore, the initial list is $\{q_1^n\}$. Players alternate turns, and on each turn can make one of the following moves.*

(1) *If the list contains two consecutive Fibonacci Quilt terms, q_i and q_{i+1}, then*
   (a) *if $i = 1$, a player can change q_1 and q_2 to q_3, denoted $\{q_1 \wedge q_2 \rightarrow q_3\}$, and*
   (b) *if $i \geq 2$, a player can change q_i and q_{i+1} to q_{i+3}, denoted $\{q_i \wedge q_{i+1} \rightarrow q_{i+3}\}$.*

(2) *If the list contains two Fibonacci Quilt terms of distance 4 apart, q_i and q_{i+4}, then*
   (a) *if $i = 1$, and no other moves are possible, a player can change q_1 and q_5 to q_2 and q_4, denoted $\{q_1 \wedge q_5 \rightarrow q_2 \wedge q_4\}$, and*
   (b) *if $i \geq 2$, a player can change q_i and q_{i+4} to q_{i+5}, denoted $\{q_i \wedge q_{i+4} \rightarrow q_{i+5}\}$.*

(3) *If the list contains two (or more) of the same Fibonacci Quilt term, q_i, then*
   (a) *if $i = 1$, a player can change q_1 and q_1 to q_2, denoted $\{q_1^2 \rightarrow q_2\}$,*
   (b) *if $i = 2$, a player can change q_2 and q_2 to q_4, denoted $\{q_2^2 \rightarrow q_4\}$,*

[1] The first recurrence relation is a PLRS, but the initial conditions for the Fibonacci Quilt come from the second relation, and thus, although this could generate a PLRS, it does not generate a PLRS for our situation because of the different initial conditions.

(c) *if* $i = 3$*, a player can change* q_3 *and* q_3 *to* q_2 *and* q_4*, denoted* $\{q_3^2 \to q_2 \wedge q_4\}$*,*
(d) *if* $i = 4$*, a player can choose to change* q_4 *and* q_4 *to* q_1 *and* q_6 ***or*** q_3 *and* q_5*, denoted* $\{q_4^2 \to q_1 \wedge q_6\}$ *and* $\{q_4^2 \to q_3 \wedge q_5\}$*, respectively,*
(e) *if* $i = 5$*, a player can change* q_5 *and* q_5 *to* q_1 *and* q_7*, denoted* $\{q_5^2 \to q_1 \wedge q_7\}$*,*
(f) *if* $i = 6$*, a player can choose to change* q_6 *and* q_6 *to* q_2 *and* q_8 ***or*** q_3 *and* q_7*, denoted* $\{q_6^2 \to q_2 \wedge q_8\}$ *and* $\{q_6^2 \to q_3 \wedge q_7\}$*, respectively, and*
(g) *if* $i \geq 7$*, a player can change* q_i *and* q_i *to* q_{i-5} *and* q_{i+2}*, denoted* $\{q_i^2 \to q_{i-5} \wedge q_{i+2}\}$*.*

(4) *If the list contains two Fibonacci Quilt terms of distance* 3 *apart,* q_i *and* q_{i+3}*, then*
(a) *if* $i = 1,\ 2$*, a player can change* q_i *and* q_{i+3} *to* q_{i+4}*, denoted* $\{q_i \wedge q_{i+3} \to q_{i+4}\}$*,*
(b) *if* $i = 3$*, a player can change* q_3 *and* q_6 *to* q_1 *and* q_7*, denoted* $\{q_3 \wedge q_6 \to q_1 \wedge q_7\}$*,*
(c) *if* $i = 4,\ 5$*, a player can change* q_i *and* q_{i+3} *to* q_1 *and* q_{i+4}*, denoted* $\{q_i \wedge q_{i+3} \to q_1 \wedge q_{i+4}\}$*,*
(d) *if* $i = 6$*, a player can change* q_6 *and* q_9 *to* q_2 *and* q_{10}*, denoted* $\{q_6 \wedge q_9 \to q_2 \wedge q_{10}\}$*, and*
(e) *if* $i \geq 7$*, a player can change* q_i *and* q_{i+3} *to* q_{i-5} *and* q_{i+4}*, denoted* $\{q_i \wedge q_{i+3} \to q_{i-5} \wedge q_{i+4}\}$*.*

(5) *If the list contains* q_1 *and* q_3*, a player can change* q_1 *and* q_3 *to* q_4*, denoted* $\{q_1 \wedge q_3 \to q_4\}$*.*

The game ends when there are no possible moves, and whomever made the last move wins.

The moves for this game may seem random, but they are a direct result of the recurrence relations stated in Theorem 1.6, and Definition 1.4 (FQ-legal decomposition). Each rule, when applied, takes two terms that could not be in a legal decomposition together and changes them to a legal term or pair of terms. For example, Rule 1 takes terms that are distance 1 apart, or q_i and q_j such that $j - i = 1$, and changes them to a single term.

There are many cases for each rule because the Fibonacci Quilt sequence does not follow the recurrence relations of (1.2) at the very beginning, and thus, the same rules cannot be applied there. Each base rule is created to change terms to a legal term or pair of terms while preserving that the sum of the list is n.

Two important things to note are Rule (2a) and Rules (3d) and (3f). Rule (2a) can only be applied when no other moves are possible, that is the list contains no other illegal pairs besides (q_1, q_5). For Rules (3d) and (3f), the player has two options because the Fibonacci Quilt Sequence lacks uniqueness, so $n = 8$ can be decomposed into $1 + 7$ or $3 + 5$, both of which are legal. We will show later that for $i \geq 7$, $2q_i$ can only be decomposed into two terms legally by Rule (3g).

With this construction, we first show that it is well-defined, and then study the length of a game.

Theorem 1.8. *Every game terminates in a finite number of moves at an FQ-legal decomposition.*

Knowing that the game terminates, we can also ask how quickly it can end. We give a result for the shortest game because we are able to associate a monovariant to the game. By looking at the smallest change possible for the summands that can be in play (i.e., we can never have a summand larger $q_m > n$), one could isolate an upper bound as well.

Theorem 1.9. *The shortest game on* n *arrives at an FQ-legal decomposition in* $n - L(n)$ *moves, where* $L(n)$ *is the maximum number of terms in an FQ-legal decomposition of* n*.*

We can also look at the length of a completely random game.

Conjecture 1.10. *As* n *goes to infinity, the number of moves in a random game decomposing* n *into its Zeckendorf expansion, when all legal moves are equally likely, converges to a Gaussian.*

The next section will provide proofs for each of these theorems, starting with key lemmas and building up, as well as evidence to support our conjecture. Finally, we will pose some questions we still hope to answer, as well as possible future work.

2. The Fibonacci Quilt Game

2.1. **The Game Is Playable.** The game as stated in Definition 1.7 has two rules where the player can choose between two possible decompositions. Specifically, if there are two q_4, the player may choose to make the move $\{q_4^2 \to q_1 \wedge q_6\}$ or the move $\{q_4^2 \to q_3 \wedge q_5\}$, and if there are two q_6, the player may choose to make the move $\{q_6^2 \to q_2 \wedge q_8\}$ or the move $\{q_6^2 \to q_3 \wedge q_7\}$. To ensure that the given definition of the game encompasses all possible moves, we first verify that for $n \geq 7$, $\{q_i^2 \to q_{i-5} \wedge q_{i+2}\}$ is the only possible move.

Proposition 2.1. *Given q_i^2 for $i \geq 7$, the only legal way to decompose q_i^2 into two terms is $\{q_i^2 \to q_{i-5} \wedge q_{i+2}\}$.*

Proof. Suppose $2q_n = q_i + q_j$ and, without loss of generality, let $i > j$. We know $2q_n = q_n + q_n < q_n + q_{n+1} = q_{n+3}$, so $i < n + 3$.

If $i = n$, then $j = n$ gives us an illegal decomposition.

If $i < n$, then $j < n$, but the Fibonacci Quilt sequence is strictly increasing, so $2q_n \neq q_i + q_j$ for $i, j < n$.

So, $i = n + 1$ or $i = n + 2$. If $i = n + 2$, then we get the known solution $2q_n = q_{n-5} + q_{n+2}$. If $i = n + 2$, $j = n - 5$ is the unique solution to this addition. So, we must verify there are no legal decompositions for $i = n + 1$. If $j = n - 3$, then $q_i + q_j = q_{n+2} < q_{n+2} + q_{n-5} = 2q_n$, so $n - 2 \leq j \leq n$. $j \in \{n - 2, n\}$ gives an illegal decomposition with $i = n + 1$. So, the only possible case is $j = n - 1$. But, $2q_n < q_{n-9} + 2q_n = q_{n-1} + q_{n-4} + q_n = q_{n-1} + q_{n+1}$, by applying Rules (4e) and (2b). Thus, there is no value of j for $i = n + 1$, and $2q_n = q_{n-5} + q_{n+2}$ is the only legal decomposition using two terms. □

Now that we have established this, we may prove Theorem 1.8, starting with a few crucial lemmas. Our proof strategy is adapted from that used on the Zeckendorf Game [2, 1].

Lemma 2.2. *In one game of the Fibonacci Quilt Game, on some fixed integer n, Rule (2a), $\{q_1 \wedge q_5 \to q_2 \wedge q_4\}$, can be used at most once.*

This is a result of the restriction placed on Rule (2a), that it may only be used when there are no other possible moves. This is crucial in ensuring that the game terminates.

Proof. The trivial game $\{1^2 \to 2\}$ shows that we do not necessarily use this rule.

Now, we will consider a game where Rule (2a) has been applied once.

Let us begin before the rule has been applied. Recall that this rule may only be applied when there are no other possible moves to make. Thus, at the time the rule is applied, our unordered list must contain $\{q_1 \wedge q_5\}$. Furthermore, it cannot contain q_2, q_3, q_4, q_6, q_8, or q_9, because there is a rule in Definition 1.7 that applies to each of these and q_1 or q_5 *and* to use Rule (2a), no other moves may be possible. For example, if the list contained q_3, then the move $\{q_1 \wedge q_3 \to q_4\}$ could be applied, so Rule (2a) could not. A rule that could be applied before (2a) for each of these q_i is shown in Figure 3.

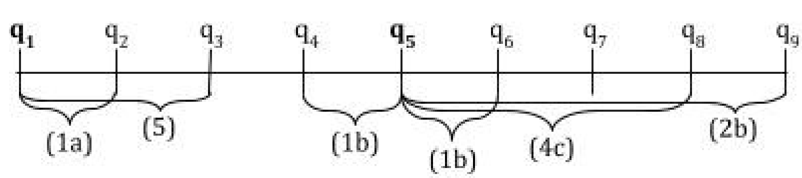

FIGURE 3. Rules from Definition 1.7 that combine q_1 or q_5 with each of the other q_i.

The first term that could possibly be in the list, besides q_1 and q_5, is q_7, because there is no Rule to combine it with q_1 or q_5 in Figure 3.

If $q_7 = 9$ is *not* in the unordered list, then the game terminates after the rule is applied:

$$\{q_1 \wedge q_5 \wedge q_k \wedge \cdots \to q_2 \wedge q_4 \wedge q_k \wedge \cdots\},$$

where q_k is the smallest possible next term, $k \geq 10$. There can be no possible moves within $\{q_k \wedge \cdots\}$, or else we would not have been able to apply this rule, so we are finished.

If $q_7 = 9$ is in the unordered list, then the next moves are:

$$\{q_1 \wedge q_5 \wedge q_7 \wedge q_\ell \wedge \cdots \to q_2 \wedge q_4 \wedge q_9 \wedge q_\ell \wedge \cdots \to q_1 \wedge q_2 \wedge q_8 \wedge q_\ell \wedge \cdots \to q_3 \wedge q_8 \wedge q_\ell \wedge \cdots\},$$

where q_ℓ is the smallest possible next term, $\ell \geq 12$. If q_{12} is not in the list, then the game is over.

If q_{12} is present, we are in the situation where we have $q_{i-4} \wedge q_i$ followed by a legal decomposition. Having q_i, there are three possible next terms: q_{i+2}, q_{i+5}, or q_j for some $j > i+5$. The possible games for each of these are illustrated in Figure 4.

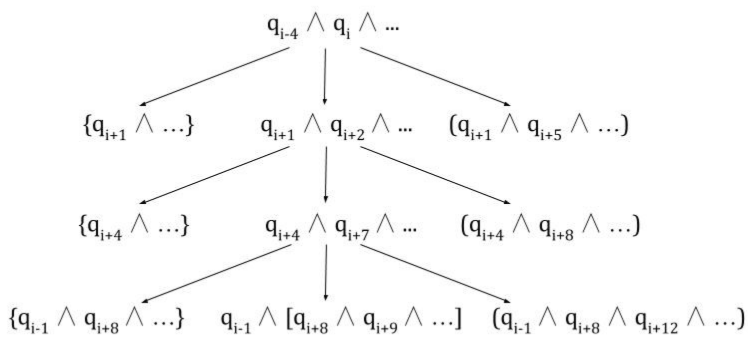

FIGURE 4. All possible moves, with terminating positions in { }, positions that return to the root of the tree in (), and positions that return to $\{q_{i+1} \wedge q_{i+2} \wedge \cdots\}$ in [].

If the next term is q_j for some $j > i + 5$, the game terminates immediately.

If the next term is q_{i+5}, then we reach $\{q_{i+1} \wedge q_{i+5} \wedge \cdots\}$. Let $k = i + 5$, then this can be rewritten as $\{q_{k-4} \wedge q_k \wedge \cdots\}$, and the possible games will follow the same possibilities as the root of our tree.

If the next term is q_{i+2}, then we reach $\{q_{i+1} \wedge q_{i+2} \wedge \cdots\}$. Again we have to consider the next possible term, there are three possibilities: q_{i+7}, q_{i+8}, or q_j for some $j > i + 8$. Note that q_{i+4} is not possible here, although it is an acceptable distance from q_{i+2}, because we know q_i was present and there could be no possible moves to begin with.

If the next terms are q_{i+2} and q_j for some $j > i + 8$, then the game terminates immediately.

If the next terms are q_{i+2} and q_{i+8}, then we reach $\{q_{i+4} \wedge q_{i+8} \wedge \cdots\}$. Let $l = i + 8$, then this can be rewritten as $\{q_{l-4} \wedge q_l \wedge \cdots\}$, and the possible games will follow the same possibilities as the root of our tree.

If the next terms are q_{i+2} and q_{i+7}, then we must again consider the next possible term. If it is q_j for some $j > i + 12$, we reach $\{q_{i-1} \wedge q_{i+8} \wedge q_j \wedge \cdots\}$ and the game terminates. The other two possibilities are q_{i+9} and q_{i+12}.

If the next terms are q_{i+2}, q_{i+7}, and q_{i+12}, then we reach $\{q_{i-1} \wedge q_{i+8} \wedge q_{i+12} \wedge \cdots\}$. Let $t = i+12$, then this can be rewritten as $\{q_{t-4} \wedge q_t \wedge \cdots\}$, and the possible games will follow the same possibilities as the root of our tree.

If the next terms are q_{i+2}, q_{i+7}, and q_{i+9}, then we reach $\{q_{i-1} \wedge q_{i+8} \wedge q_{i+9} \wedge \cdots\}$. Let $s = i+7$, then this can be rewritten as $\{q_{s+1} \wedge q_{s+2} \wedge \cdots\}$, and the possible games will follow the same possibilities as $\{q_{i+1} \wedge q_{i+2} \wedge \cdots\}$.

Note in the last two cases that we have an additional term q_{i-1}. However, following the tree, the next smallest terms that could be created are q_{i+11} and q_{i+6}, respectively, and there are no rules that combine q_{i-1} with anything this large.

Because there are a finite number of terms in the unordered list to begin the game, and each time through, the tree reduces this number by at least one, we know that we must terminate at some point. Throughout the tree, the smallest possible term we could create is $q_{i-1} = q_{11}$, which cannot be combined with any smaller terms, and thus, we can never create another q_5, and the rule cannot be applied again. □

Knowing that we may only apply this rule at most once, we now must ensure that the game takes on only a finite number of moves before and potentially after this rule is applied. We do this by introducing a quantity that is almost a monovariant.

Lemma 2.3. *The sum of the square roots of the indices of the* q_i *in the unordered list on any given turn, besides Rule (2a) of* $\{q_1 \wedge q_5 \to q_2 \wedge q_4\}$*, is a strictly decreasing monovariant; however, Rule (2a) can be used at most once and thus, this quantity is effectively a monovariant.*

Proof. When considering this monovariant, we must only consider the terms directly affected by the move, because all other terms will contribute the same to the sum before and after the move. For clarity, we will show the monovariant in the same order as Definition 1.7. The contribution of the terms directly affected by the move is always smaller after the move is applied.

(1) **Combining Consecutive Terms:**
  (a) $\{q_1 \wedge q_2 \to q_3\}$: $\sqrt{3} - \sqrt{2} - 1 < 0$.
  (b) $i \geq 2$ and $\{q_i \wedge q_{i+1} \to q_{i+3}\}$: $\sqrt{i+3} - \sqrt{i+1} - \sqrt{i} < 0$.
(2) **Combining** q_i **and** q_{i+4}
  (a) $\{q_1 \wedge q_5 \to q_2 \wedge q_4\}$: this rule is not included in this lemma.
  (b) $i \geq 2$ and $\{q_i \wedge q_{i+4} \to q_{i+5}\}$: $\sqrt{i+5} - \sqrt{i+4} - \sqrt{i} < 0$.
(3) **Combining** $2q_i$
  (a) $\{q_1^2 \to q_2\}$: $\sqrt{2} - 2 < 0$.
  (b) $\{q_2^2 \to q_4\}$: $2 - 2\sqrt{2} < 0$.
  (c) $\{q_3^2 \to q_2 \wedge q_4\}$: $2 + \sqrt{2} - 2\sqrt{3} < 0$.
  (d) $\{q_4^2 \to q_1 \wedge q_6\}$: $\sqrt{6} + 1 - 4 < 0$.
  $\{q_4^2 \to q_3 \wedge q_5\}$: $\sqrt{5} + \sqrt{3} - 4 < 0$.
  (e) $\{q_5^2 \to q_1 \wedge q_7\}$: $\sqrt{7} + 1 - 2\sqrt{5} < 0$.
  (f) $\{q_6^2 \to q_2 \wedge q_8\}$: $\sqrt{8} + \sqrt{2} - 2\sqrt{6} < 0$.
  $\{q_6^2 \to q_3 \wedge q_7\}$: $\sqrt{7} + \sqrt{3} - 2\sqrt{6} < 0$.
  (g) if $i \geq 7$, $\{q_i^2 \to q_{i-5} \wedge q_{i+2}\}$: $\sqrt{i+2} + \sqrt{i-5} - 2\sqrt{i} < 0$.
(4) **Combining** q_i **and** q_{i+3}
  (a) $i = 1,\ 2$ and $\{q_i \wedge q_{i+3} \to q_{i+4}\}$: $\sqrt{i+4} - \sqrt{i+3} - \sqrt{i} < 0$.
  (b) $\{q_3 \wedge q_6 \to q_1 \wedge q_7\}$: $\sqrt{7} + 1 - \sqrt{6} - \sqrt{3} < 0$.
  (c) $i = 4,\ 5$ and $\{q_i \wedge q_{i+3} \to q_1 \wedge q_{i+4}\}$: $\sqrt{i+4} + 1 - \sqrt{i+3} - \sqrt{i} < 0$.
  (d) $\{q_6 \wedge q_9 \to q_2 \wedge q_{10}\}$: $\sqrt{10} + \sqrt{2} - 3 - \sqrt{6} < 0$.
  (e) $i \geq 7$ and $\{q_i \wedge q_{i+3} \to q_{i-5} \wedge q_{i+4}\}$: $\sqrt{i+4} + \sqrt{i-5} - \sqrt{i+3} - \sqrt{i} < 0$.
(5) $\{q_1 \wedge q_3 \to q_4\}$: $2 - 1 - \sqrt{3} < 0$.

For the values of i on which these rules apply, each of the resulting expressions is negative. Thus, the sum of the square roots of the indices of the terms decreases on each move and is a monovariant. □

With these two lemmas, we can now prove Theorem 1.8.

Proof. From Lemma 2.2, we know that the rule $\{q_1 \wedge q_5 \to q_2 \wedge q_4\}$ is used either once or not at all; thus, we can consider these two cases. In the first case, we consider two different subgames: the game before applying this rule and the game after. For the second case, we consider the whole game.

We can see that each of these games is finite using the monovariant from Lemma 2.3. At the beginning of the game, the sum of the square roots of the indices is $\sqrt{n}$, and with each move, this value is decreasing. This means that the sum cannot be the same for two different turns and thus, there will be no repeat turns. Because on each turn the unordered list is essentially a partition of n, of which there are finitely many, the game must terminate in finitely many moves. Similarly, for a game after applying $\{q_1 \wedge q_5 \to q_2 \wedge q_4\}$, we begin with some monovariant value less than $\sqrt{n}$ and the argument continues as before.

When the game terminates, it must be at an FQ-legal decomposition of n, because there is a rule in Definition 1.7 corresponding to each illegal distance in Definition 1.4. Thus, if the decomposition was not an FQ-legal decomposition, there would be a rule that could be applied and the game would not be over. □

Now that we know the game terminates, we want to make sure that it is interesting to play; that is, that the players have choices to make and either player could win.

Lemma 2.4. *Given any positive integer n such that $n > 3$, there are at least two distinct sequences of moves $M = \{m_i\}$, where the application of each set of moves to the initial set, denoted by $M(\{q_1\}^n)$, leads to an FQ-legal decomposition of n.*

Proof. To show this, we must only show that there are two distinct games on $n = 4$; for $n > 4$, starting with these moves would create two distinct games. There are exactly two distinct games on $n = 4$, they are:

$$M_1 = \{\{q_1^2 \to q_2\}, \{q_1 \wedge q_2 \to q_3\}, \{q_1 \wedge q_3 \to q_4\}\},$$
$$M_2 = \{\{q_1^2 \to q_2\}, \{q_1^2 \to q_2\}, \{q_2^2 \to q_4\}\}.$$

Thus, there are distinct games for $n > 3$. □

For $n \leq 3$, there is only one unique game. For $n = 4$, there are two unique games, and for $n = 5$, there are four games; however, all games have the same length. Games begin to vary in length at $n = 6$.

Corollary 2.5. *For all $n > 5$, there are at least two games with different numbers of moves. Further, there is always a game with an odd number of moves and one with an even number of moves.*

Proof. There are two distinct games, one of odd length and one of even length for $n = 6$. A game of odd length on $n = 6$ is

$$\{\{q_1^2 \to q_2\}, \{q_1 \wedge q_2 \to q_3\}, \{q_1 \wedge q_3 \to q_4\}, \{q_1 \wedge q_4 \to q_5\}, \{q_1 \wedge q_5 \to q_2 \wedge q_4\}\}.$$

A game of even length on $n = 6$ is

$$\{\{q_1^2 \to q_2\}, \{q_1^2 \to q_2\}, \{q_1 \wedge q_2 \to q_3\}, \{q_1 \wedge q_3 \to q_4\}\}.$$

For $n \geq 7$, it is enough to show that there are two distinct games, one of odd length and one of even length for $n = 7$. For $n > 7$, we know there is some M' that takes $\{q_1^{n-7} \wedge q_6\}$ to a FQ-legal decomposition of n. If the length of M' is even, combine it with the even game on $n = 7$ to get an even length game, and the odd game on $n = 7$ to get an odd game, and similarly, if the length of M' is odd.

A game of odd length on $n = 7$ is

$$\{\{q_1^2 \to q_2\}, \{q_1 \wedge q_2 \to q_3\}, \{q_1^2 \to q_2\}, \{q_1 \wedge q_2 \to q_3\}, \{q_3^2 \to q_2 \wedge q_4\}, \{q_1 \wedge q_2 \to q_3\}, \{q_3 \wedge q_4 \to q_6\}\}.$$

A game of even length on $n = 7$ is

$$\{\{q_1^2 \to q_2\}, \{q_1 \wedge q_2 \to q_3\}, \{q_1^2 \to q_2\}, \{q_1^2 \to q_2\}, \{q_2^2 \to q_4\}, \{q_3 \wedge q_4 \to q_6\}\}.$$

Thus, there are game of even and odd length for all $n \geq 6$. □

In the next section, we will explore the behavior of the game length more.

2.2. **Game Length.** For some n, FQ-legal decompositions are not unique. The smallest example of this is $n = 8 = 1 + 7 = 3 + 5$. From this, we can define two values for n. Let $L(n)$ be the *maximum* number of terms in an FQ-legal decomposition of n, and let $l(n)$ be the *minimum* number of terms in an FQ-legal decomposition of n. For $n = 8$, we see that $L(8) = l(8) = 2$; however, they are not always equal. For example $50 = 49 + 1 = 2 + 4 + 16 + 28$, so $l(50) = 2$, but $L(50) = 4$. Theorem 1.9 says that the shortest possible game on n is achieved in $n - L(n)$ moves.

Proof of Theorem 1.9. Note that this is trivially true for $n = 1$. It takes $0 = 1 - 1$ moves to complete the game on 1.

Assume that the shortest game on i, for $1 \leq i \leq n-1$, is achieved in $i - L(i)$ moves. Then, consider the shortest possible game on n.

If n is in the Fibonacci Quilt Sequence, denote it q_j. One can quickly verify, for $j < 5$, that the lower bound holds. For $j \geq 5$, $q_j = q_{j-2} + q_{j-3}$. To reach the right side, it would take $(q_{j-2} - 1) + (q_{j-3} - 1) = q_j - 2$ moves, using one additional move to combine q_{j-2} and q_{j-3} gives us q_j in $q_j - 1$ moves.

If n is not in the Fibonacci Quilt Sequence, then write it in an FQ-legal decomposition using the maximum possible number of terms: $n = q_{\ell_1} + q_{\ell_2} + \cdots + q_{\ell_{L(n)}}$. To reach the right side, it would take $(q_{\ell_1} - 1) + (q_{\ell_2} - 1) + \cdots + (q_{\ell_{L(n)}} - 1) = (q_{\ell_1} + q_{\ell_2} + \cdots + q_{\ell_{L(n)}}) - L(n) = n - L(n)$ moves.

To see why the game would not terminate in fewer moves, note that every move can reduce the total number of terms in the unordered list by at most 1. Thus, after $n - L(n) - 1$ moves, we would still have at least $n - (n - L(n) - 1) = L(n) + 1$ terms, which cannot be an FQ-legal decomposition as $L(n)$ is the maximum. □

From this theorem, we see that we must be able to play the game without using any of the rules that take two terms to two terms, because we must remove one term on each turn to reach the lower bound.

Corollary 2.6. *It is possible, for any n, to play the Fibonacci Quilt Game without using Rules (2a), (3c–g), (4b–e).*

We have obtained this lower bound for many values of n, but an algorithm to reach the lower bound for all n is still unknown.

For small values of n, it is clear that the lower bound will not be reached in a large number of possible games. To better understand the length of an average game, we used Mathematica code to simulate completely random games, where every possible move on each turn was equally likely. We then looked at the distribution of random games as n increased, leading us to Conjecture 1.10, that the distribution of these random games will approach a Gaussian curve as n approaches infinity.

We ran $10,000$ simulations of a random game, and plotted the distribution of game lengths. For small values of n, the Gaussian does not fit as well. Figure 5 shows the distribution of random games for $n = 20$.

As we increase n to 200 in Figure 6, we see that the Gaussian curve fits better.

We also looked at the moments of these distributions compared to those with the same mean and standard deviation, with the differences of these values shown in Figure 7 (note that, because we are using the best fit Gaussian, there is no error in the mean or second moment).

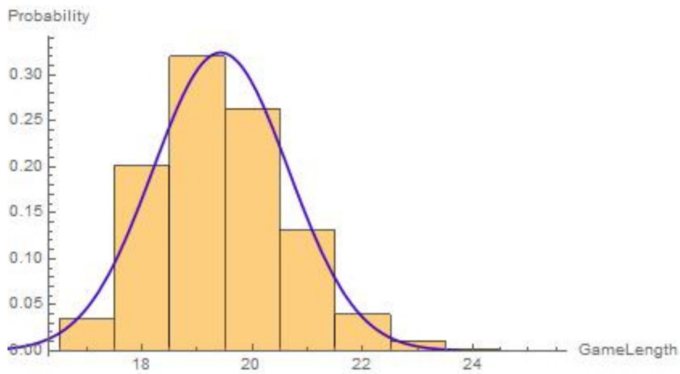

FIGURE 5. The distribution of game lengths of 10,000 random games on $n = 20$.

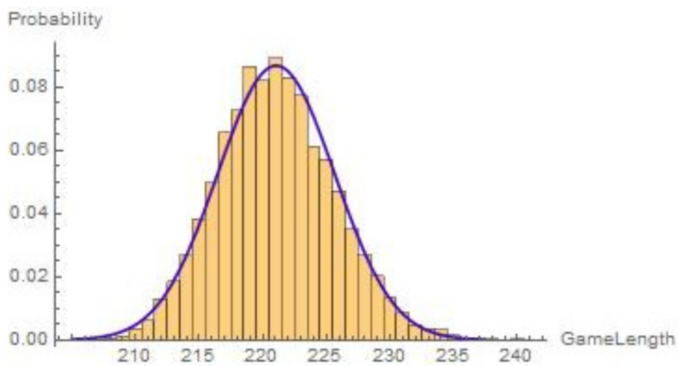

FIGURE 6. The distribution of game lengths of 10,000 random games on $n = 200$.

n	2^{nd} Moment Difference	4^{th} Moment Difference	6^{th} Moment Difference
20	0	0.044176	0.219217
60	0	0.009249	0.046575
200	0	0.000008	0.004052

FIGURE 7. The percent difference between the moments of the distribution and the moments of the Gaussian curve with the same mean and standard deviation.

Thus, a Gaussian curve appears to fit the randomly simulated games well. From this, we can also see that in a random game, either player has an equal chance of winning, so the game is fair.

3. FUTURE WORK

There are many questions about this game that can still be asked. It is known that if $n \neq 2$, then Player 2 has a winning strategy in the original Zeckendorf game. The proof techniques do not easily generalize to the Fibonacci Quilt Game because of the odd behavior of the quilt at its center, which necessitates a significantly larger set of strategies to investigate. Does Player 2 still have a winning strategy for the Fibonacci Quilt Game? If so, what is it? Note, we do not know the answer to the second question for the original game; the proof that Player 2 has a winning strategy is nonconstructive.

Other questions arise from bounds on game length. Is there one algorithm that reaches the lower bound for all values of n? Is there a reasonable upper bound on the length of a game? Simulations have never given a game of length close to or longer than $2n$, and numerical exploration of small n (up to 120) suggest that the mean and maximum length grow linearly.

Another question, asked by Dylan King at a presentation of this work, relates to $L(n)$ and $l(n)$. We give an example where $L(n) - l(n) = 0$, and another where $L(n) - l(n) = 2$, but can the distance between these two values grow arbitrarily large?

Lastly, like the Zeckendorf Game, this game has been constructed for two players, but one could also study how the behavior of this game changes if it was constructed to be played by more people at once. Who has a winning strategy (as a function of n and the number of people)?

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