EXTENDED GIBONACCI SUMS OF POLYNOMIAL PRODUCTS OF ORDER 3

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Abstract. We explore a gibonacci sum of polynomial products of order 3 and its Pell, Jacobsthal, Vieta, and Chebyshev implications; and confirm the gibonacci and Jacobsthal versions using graph-theoretic tools.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary complex variable, $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials, and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Then $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 7, 9].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. The *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [7, 8].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial* [3, 7, 10]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$.

Let $a(x) = x$ and $b(x) = -1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = V_n(x)$, the nth *Vieta polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = v_n(x)$, the nth *Vieta-Lucas polynomial* [4, 7, 10].

Finally, let $a(x) = 2x$ and $b(x) = -1$. When $z_0(x) = 1$ and $z_1(x) = x$, $z_n(x) = T_n(x)$, the nth *Chebyshev polynomial of the first kind*; and when $z_0(x) = 1$ and $z_1(x) = 2x$, $z_n(x) = U_n(x)$, the nth *Chebyshev polynomial of the second kind* [4, 7, 10].

1.1. Links Among the Subfamilies.

The gibonacci, Jacobsthal, Vieta, and Chebyshev subfamilies are closely related as Table 1 shows, where $i = \sqrt{-1}$ [4, 10, 13].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is no ambiguity; so z_n will mean $z_n(x)$. We let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $c_n = J_n(x)$ or $j_n(x)$, $d_n = V_n$ or v_n, and $e_n = T_n$ or U_n. Correspondingly, let $G_n = F_n$ or L_n, $B_n = P_n$ or Q_n, and $C_n = J_n$ or j_n.

Table 1. Relationships Among the Subfamilies

$$\begin{array}{rclrcl} J_n(x) &=& x^{(n-1)/2} f_n(1/\sqrt{x}) & j_n(x) &=& x^{n/2} l_n(1/\sqrt{x}) \\ V_n(x) &=& i^{n-1} f_n(-ix) & v_n(x) &=& i^n l_n(-ix) \\ V_n(2x) &=& U_{n-1}(x) & v_n(2x) &=& 2T_n(x) \end{array}$$

A *gibonacci polynomial product of order* m is a product of gibonacci polynomials g_{n+k} of the form $\prod_k g_{n+k}^{s_j}$, where k is an integer and $\sum_{s_j \geq 1} s_j = m$ [5, 12]. For example, the Fibonacci polynomial products f_{n+2}^3, $f_{n+2}^2 f_n$, $f_{n+2} f_n^2$, $f_{n+2} f_n f_{n-2}$, f_n^3, and $f_n^2 f_{n-2}$ are all of order 3, where as $f_{n+2} f_n^3 f_{n-2}$ is of order 5.

2. A Gibonacci Sum oF Polynomial Products of Order 3

The next theorem explores a sum of gibonacci polynomial products of order 3, and lays the foundation for the discourse. The proof hinges on the *addition formula* [7] for gibonacci polynomials g_n:

$$g_{m+n} = f_{m+1} g_n + f_m g_{n-1}.$$

Theorem 2.1. *Let $g_n = f_n$ or l_n, and r, s, and t be positive integers. Then,*

$$x g_{r+s+t} = f_{r+1} f_{s+1} g_{t+1} + x f_r f_s g_t - f_{r-1} f_{s-1} g_{t-1}. \tag{2.1}$$

Proof. Let $g_n = l_n$. We have

$$\begin{aligned}
x f_{r+1} l_{s+t} &= x f_{r+1}(f_{s+1} l_t + f_s l_{t-1}) \\
&= f_{r+1} f_{s+1}(l_{t+1} - l_{t-1}) + x f_{r+1} f_s l_{t-1} \\
&= f_{r+1} f_{s+1} l_{t+1} - f_{r+1} f_{s+1} l_{t-1} + x f_{r+1} f_s l_{t-1}; \\
x f_r l_{s+t-1} &= x f_r l_{(s-1)+t} \\
&= x f_r (f_s l_t + f_{s-1} l_{t-1}) \\
&= x f_r f_s l_t + x f_r f_{s-1} l_{t-1} \\
&= x f_r f_s l_t + (f_{r+1} - f_{r-1}) f_{s-1} l_{t-1} \\
&= x f_r f_s l_t - f_{r-1} f_{s-1} l_{t-1} + f_{r+1} f_{s-1} l_{t-1}.
\end{aligned}$$

Then,

$$\begin{aligned}
x l_{r+s+t} &= x l_{r+(s+t)} \\
&= x(f_{r+1} l_{s+t} + f_r l_{s+t-1}) \\
&= (f_{r+1} f_{s+1} f_{t+1} + x f_r f_s l_t - f_{r-1} f_{s-1} l_{t-1}) - f_{r+1} f_{s+1} l_{t-1} + f_{r+1} l_{t-1}(x f_s + f_{s-1}) \\
&= f_{r+1} f_{s+1} l_{t+1} + x f_r f_s l_t - f_{r-1} f_{s-1} l_{t-1},
\end{aligned}$$

as desired.

The case $g_n = f_n$ follows similarly (or by simply changing l_n into f_n in the above case). □

In particular, we have

$$\begin{aligned}
x g_{2m+n} &= f_{m+1}^2 g_{n+1} + x f_m^2 g_n - f_{m-1}^2 g_{n-1}; \\
x g_{3n} &= f_{n+1}^2 g_{n+1} + x f_n^2 g_n - f_{n-1}^2 g_{n-1}; && (2.2) \\
G_{r+s+t} &= F_{r+1} F_{s+1} G_{t+1} + F_r F_s G_t - F_{r-1} F_{s-1} G_{t-1}; && (2.3) \\
G_{2m+n} &= F_{m+1}^2 G_{n+1} + x F_m^2 G_n - F_{m-1}^2 G_{n-1}; \\
G_{3n} &= F_{n+1}^2 G_{n+1} + x F_n^2 G_n - F_{n-1}^2 G_{n-1}.
\end{aligned}$$

Identity (2.3) with $G_n = F_n$ appears in [6].

It follows from equation (2.2) that [9]

$$\begin{aligned} xf_{3n} &= f_{n+1}^3 + xf_n^3 - f_{n-1}^3; \\ xl_{3n} &= f_{n+1}^2 l_{n+1} + xf_n^2 l_n - f_{n-1}^2 l_{n-1} \\ &= f_{n+1} f_{2n+2} + xf_n f_{2n} - f_{n-1} f_{2n-2}, \end{aligned} \tag{2.4}$$

where we have used $f_{2n} = f_n l_n$.

Using the identity $l_n^2 - \Delta^2 f_n^2 = 4(-1)^n$ [7], we can rewrite equation (2.4) in a more familiar form [9], where $\Delta^2 = x^2 + 4$:

$$\begin{aligned} x\Delta^2 l_{3n} &= \left(\Delta^2 f_{n+1}^2\right) l_{n+1} + \left(\Delta^2 f_n^2\right) xl_n - \left(\Delta^2 f_{n-1}^2\right) l_{n-1} \\ &= \left[l_{n+1}^2 + 4(-1)^n\right] l_{n+1} + \left[l_n^2 - 4(-1)^n\right] xl_n - \left[l_{n-1}^2 + 4(-1)^n\right] l_{n-1} \\ &= l_{n+1}^3 + xl_n^3 - l_{n-1}^3 + 4(-1)^n(l_{n+1} - xl_n - l_{n-1}) \\ &= l_{n+1}^3 + xl_n^3 - l_{n-1}^3, \end{aligned}$$

as desired.

Thus [9],

$$g_{n+1}^3 + xg_n^3 - g_{n-1}^3 = \begin{cases} xf_{3n}, & \text{if } g_n = f_n; \\ x\Delta^2 l_{3n}, & \text{otherwise.} \end{cases}$$

Next, we confirm identity (2.1) using graph-theoretic tools.

2.1. **Graph-theoretic Confirmation.** Consider the *weighted digraph* D_1 in Figure 1 with vertices v_1 and v_2. It follows by induction from its *weighted adjacency matrix* $Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ that

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \geq 1$ [11]. The ijth entry of Q^n gives the sum of the weights of all walks of length n from v_i to v_j in the weighted digraph D_1, where $1 \leq i, j \leq n$. The sum of the weights of closed walks of length n originating at v_1 is f_{n+1} and that of those originating at v_2 is f_{n-1}. So, the sum of all closed walks of length n in the digraph is $f_{n+1} + f_{n-1} = l_n$. Because $f_{n+1} = xf_n + f_{n-1}$, it follows that the sum of the weights of closed walks of length n originating at v_1 and beginning with a loop is xf_n.

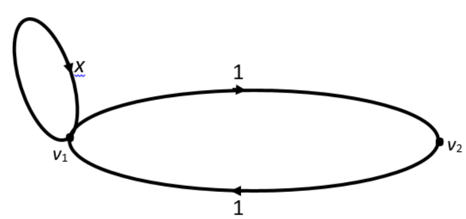

FIGURE 1. Weighted Fibonacci Digraph D_1

With this brief background, we are now ready for the graph-theoretic proof.

Proof.
Part 1. Suppose $g_n = f_n$. The sum S of the weights of closed walks v of length $r+s+t-1$ originating at v_1 is f_{r+s+t}; so $xS = xf_{r+s+t}$. (This is indeed the sum of the weights of closed walks of length $r+s+t$ originating at v_1 and beginning with a loop.)

We will now compute xS in a different way.

Case 1. Suppose v begins with a loop. Using the addition formula, the sum of the weights of such walks is

$$\begin{aligned} xf_{r+s+t-1} &= xf_{r+(s+t-1)} \\ &= xf_{r+1}f_{s+t-1} + xf_r f_{s+t-2}. \end{aligned}$$

Case 2. Suppose v does not begin with a loop. The sum of the weights of such walks is

$$\begin{aligned} 1 \cdot 1 \cdot f_{r+s+t-2} &= f_{r+(s+t-2)} \\ &= f_{r+1}f_{s+t-2} + f_r f_{s+t-3}. \end{aligned}$$

Combining the two cases, we have

$$\begin{aligned} S &= (xf_{r+1}f_{s+t-1} + xf_r f_{s+t-2}) + (f_{r+1}f_{s+t-2} + f_r f_{s+t-3}) \\ &= f_{r+1}(xf_{s+t-1} + f_{s+t-2}) + f_r(xf_{s+t-2} + f_{s+t-3}) \\ &= f_{r+1}f_{s+t} + f_r f_{s+t-1}; \\ xS &= xf_{r+1}f_{s+t} + xf_r f_{s+t-1}. \end{aligned}$$

Notice that

$$\begin{aligned} xf_{r+1}f_{s+t} &= xf_{r+1}(f_{s+1}f_t + f_s f_{t-1}) \\ &= f_{r+1}f_{s+1}(f_{t+1} - f_{t-1}) + xf_{r+1}f_s f_{t-1} \\ &= f_{r+1}f_{s+1}f_{t+1} - f_{r+1}f_{s+1}f_{t-1} + xf_{r+1}f_s f_{t-1}; \\ xf_r f_{s+t-1} &= xf_r(f_s f_t + f_{s-1}f_{t-1}) \\ &= xf_r f_s f_t + xf_r f_{s-1}f_{t-1} \\ &= xf_r f_s f_t + (f_{r+1} - f_{r-1})f_{s-1}f_{t-1} \\ &= xf_r f_s f_t - f_{r-1}f_{s-1}f_{t-1} + f_{r+1}f_{s-1}f_{t-1}. \end{aligned}$$

Thus,

$$\begin{aligned} xS &= (f_{r+1}f_{s+1}f_{t+1} + xf_r f_s f_t - f_{r-1}f_{s-1}f_{t-1}) - f_{r+1}f_{s+1}f_{t-1} + f_{r+1}f_{t-1}(xf_s + f_{s-1}) \\ &= f_{r+1}f_{s+1}f_{t+1} + xf_r f_s f_t - f_{r-1}f_{s-1}f_{t-1}. \end{aligned}$$

Equating the two values of xS, we get the desired result, as expected.

Part 2. Suppose $g_n = l_n$. The sum S of the weights of all closed walks of length $r+s+t$ in the digraph is l_{r+s+t}. Then, $xS = xl_{r+s+t}$.

We will now compute xS in a different way. The sum of the weights of closed walks of length $r+s+t$ originating at v_1 is $f_{r+s+t+1}$, and those originating at v_2 is $f_{r+s+t-1}$. So, $S = f_{r+s+t+1} + f_{r+s+t-1}$.

By identity (2.1) with $g_n = l_n$, we then have

$$\begin{aligned} xS &= xf_{r+s+(t+1)} + xf_{r+s+(t-1)} \\ &= (f_{r+1}f_{s+1}f_{t+2} + xf_r f_s f_{t+1} - f_{r-1}f_{s-1}f_t) + (f_{r+1}f_{s+1}f_t + xf_r f_s f_{t-1} - f_{r-1}f_{s-1}f_{t-2}) \\ &= f_{r+1}f_{s+1}(f_{t+2} + f_t) + xf_r f_s(f_{t+1} + f_{t-1}) - f_{r-1}f_{s-1}(f_t + f_{t-2}) \\ &= f_{r+1}f_{s+1}l_{t+1} + xf_r f_s l_t - f_{r-1}f_{s-1}l_{t-1}. \end{aligned}$$

Equating the two values of xS yields the desired result. □

3. Pell Implications

Because $p_n(x) = f_n(2x)$ and $q_n(x) = l(2x)$, it follows that identity (2.1) has Pell consequences:

$$\begin{aligned} 2xb_{r+s+t} &= p_{r+1}p_{s+1}b_{t+1} + 2xp_r p_s b_t - p_{r-1}p_{s-1}b_{t-1}; \\ 2B_{r+s+t} &= P_{r+1}P_{s+1}B_{t+1} + 2P_r P_s B_t - P_{r-1}P_{s-1}B_{t-1}; \\ 2B_{3n} &= P_{n+1}^2 B_{n+1} + 2P_n^2 B_n - P_{n-1}^2 B_{n-1}. \end{aligned} \tag{3.1}$$

Because $Q_n^2 - 2P_n^2 = (-1)^n$ [7, 8], identity (3.1) can be rewritten as [9]

$$B_{n+1}^3 + 2B_n^3 - B_{n-1}^3 = \begin{cases} 2B_{3n}, & \text{if } B_n = P_n; \\ 4B_{3n}, & \text{otherwise.} \end{cases}$$

Next, we pursue the consequences of identity (2.1) to the Jacobsthal subfamily.

4. Jacobsthal Implications

Let $g_n = f_n$. Replace x with $1/\sqrt{x}$ in equation (2.1) and multiply the resulting equation with $x^{(r+s+t)/2}$. This yields

$$\begin{aligned} x^{(r+s+t)/2} f_{r+s+t} &= \left(x^{r/2} f_{r+1}\right)\left(x^{s/2} f_{s+1}\right)\left(x^{t/2} f_{t+1}\right) \\ &\quad + x\left[x^{(r-1)/2} f_r\right]\left[x^{(s-1)/2} f_s\right]\left[x^{(t-1)/2} f_t\right] \\ &\quad - x^3\left[x^{(r-2)/2} f_{r-1}\right]\left[x^{(s-2)/2} f_{s-1}\right]\left[x^{(t-2)/2} f_{t-1}\right]; \\ J_{r+s+t}(x) &= J_{r+1}(x)J_{s+1}(x)J_{t+1}(x) + xJ_r(x)J_s(x)J_t(x) - x^3 J_{r-1}(x)J_{s-1}(x)J_{t-1}(x), \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$.

On the other hand, let $g_n = l_n$. Replacing x with $1/\sqrt{x}$ in equation (2.1) and multiplying the resulting equation with $x^{(r+s+t)/2}$ yields

$$j_{r+s+t}(x) = J_{r+1}(x)J_{s+1}(x)j_{t+1}(x) + xJ_r(x)J_s(x)j_t(x) - x^3 J_{r-1}(x)J_{s-1}(x)j_{t-1}(x).$$

Combining the two cases, we get

$$c_{r+s+t} = J_{r+1}(x)J_{s+1}(x)c_{t+1} + xJ_r(x)J_s(x)c_t - x^3 J_{r-1}(x)J_{s-1}(x)c_{t-1}. \tag{4.1}$$

In particular, we have

$$\begin{aligned} c_{2m+n} &= J_{m+1}^2(x)c_{n+1} + xJ_m^2(x)c_n - x^3 J_{m-1}^2(x)c_{n-1}; \\ c_{3n} &= J_{n+1}^2(x)c_{n+1} + xJ_n^2(x)c_n - x^3 J_{n-1}^2(x)c_{n-1}; \\ C_{r+s+t} &= J_{r+1}J_{s+1}C_{t+1} + 2J_r J_s C_t - 8J_{r-1}J_{s-1}C_{t-1}; \\ C_{2m+n} &= J_{m+1}^2 C_{n+1} + 2J_m^2 C_n - 8J_{m-1}^2 C_{n-1}; \\ C_{3n} &= J_{n+1}^2 C_{n+1} + 2J_n^2 C_n - 8J_{n-1}^2 C_{n-1}. \end{aligned} \tag{4.2}$$

Using $j_n^2 - 9J_n^2 = 4(-2)^n$ [2, 7], we can rewrite identity (4.2) as follows [10]:

$$C_{n+1}^3 + 2C_n^3 - 8C_{n-1}^3 = \begin{cases} C_{3n}, & \text{if } C_n = J_n; \\ 9C_{3n}, & \text{otherwise.} \end{cases}$$

Next, we present a graph-theoretic confirmation of identity (4.1).

4.1. **Graph-theoretic Proof.** Consider the weighted digraph D_2 in Figure 2 with vertices v_1 and v_2. It follows from its weighted adjacency matrix $M = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix}$ that

$$M^n = \begin{bmatrix} J_{n+1}(x) & xJ_n(x) \\ J_n(x) & xJ_{n-1}(x) \end{bmatrix}.$$

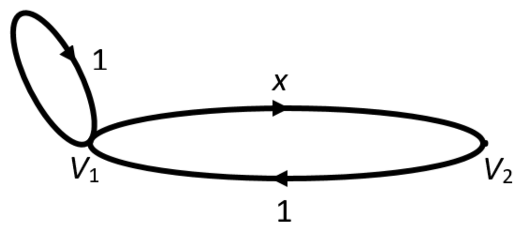

FIGURE 2. Jacobsthal Digraph D_2

The sum of the closed walks of length n from v_1 to itself is $J_{n+1}(x)$, and that from v_2 to itself is $xJ_{n-1}(x)$. Consequently, the sum of the weights of all closed walks of length n is $J_{n+1}(x) + xJ_{n-1}(x) = j_n(x)$ [7]. These facts play a central role in the graph-theoretic proof.

Proof. (In the interest of brevity and clarity, we omit the argument in the functional notation, when there is *no* confusion.)

Part 1. Suppose $c_n = J_n(x)$. The sum S of the weights of closed walks w of length $r+s+t-1$ that originate at v_1 is J_{r+s+t}. We will now compute S in a different way.

Case 1. Suppose w begins with a loop. The sum of the weights of such walks is

$$\begin{aligned} 1 \cdot J_{r+s+t-1} &= J_{r+(s+t-1)} \\ &= J_{r+1}J_{s+t-1} + xJ_rJ_{s+t-2}. \end{aligned}$$

Case 2. Suppose w does not begin with a loop. The sum of the weights of such walks is

$$\begin{aligned} x \cdot 1 \cdot J_{r+s+t-2} &= xJ_{r+(s+t-2)} \\ &= x(J_{r+1}J_{s+t-2} + xJ_rJ_{s+t-3}) \\ &= xJ_{r+1}J_{s+t-2} + x^2J_rJ_{s+t-3}. \end{aligned}$$

Thus,

$$\begin{aligned} S &= (J_{r+1}J_{s+t-1} + xJ_rJ_{s+t-2}) + (xJ_{r+1}J_{s+t-2} + x^2J_rJ_{s+t-3}) \\ &= J_{r+1}(J_{s+t-1} + xJ_{s+t-2}) + xJ_r(J_{s+t-2} + xJ_{s+t-3}) \\ &= J_{r+1}J_{s+t} + xJ_rJ_{s+t-1}. \end{aligned}$$

Notice that

$$\begin{aligned} J_{r+1}J_{s+t} &= J_{r+1}(J_{s+t}J_t + xJ_{r+1}J_sJ_{t-1}) \\ &= J_{r+1}J_{s+1}(J_{t+1} - xJ_{t-1}) + xJ_{r+1}J_sJ_{t-1} \\ &= J_{r+1}J_{s+1}J_{t+1} - xJ_{r+1}J_{s+1}J_{t-1} + xJ_{r+1}J_sJ_{t-1}; \\ xJ_rJ_{s+t-1} &= xJ_r(J_sJ_t + xJ_{s-1}J_{t-1}) \\ &= xJ_rJ_sJ_t + x^2J_rJ_{s-1}J_{t-1} \\ &= xJ_rJ_sJ_t + x^2(J_{r+1} - xJ_{r-1})J_{s-1}J_{t-1}) \\ &= xJ_rJ_sJ_t - x^3J_{r-1}J_{s-1}J_{t-1} + x^2J_{r+1}J_{s-1}J_{t-1}. \end{aligned}$$

Thus,

$$\begin{aligned} S &= (J_{r+1}J_{s+1}J_{t+1} + xJ_rJ_sJ_t - x^3J_{r-1}J_{s-1}J_{t-1}) - xJ_{r+1}J_{s+1}J_{t-1} + xJ_{r+1}J_{t-1}(J_s + xJ_{s-1}) \\ &= J_{r+1}J_{s+1}J_{t+1} + xJ_rJ_sJ_t - x^3J_{r-1}J_{s-1}J_{t-1}. \end{aligned}$$

Equating the two values of S gives the desired result.

Part 2. Suppose $c_n = j_n(x)$. The sum S of the weights of all closed walks of length $r+s+t$ in the digraph is j_{r+s+t}.

We will now compute S in a different way. The sum S of the weights of closed walks of length $r+s+t-1$ originating at v_1 is $J_{r+s+t+1}$, and that of those originating at v_2 is $xJ_{r+s+t-1}$. Then, by identity (4.1) with $c_n = J_n(x)$, we have

$$\begin{aligned} S &= J_{r+s+(t+1)} + xJ_{r+s+(t-1)} \\ &= (J_{r+1}J_{s+1}J_{t+2} + xJ_rJ_sJ_{t+1} - x^3J_{r-1}J_{s-1}J_t) \\ &\quad + x(J_{r+1}J_{s+1}J_t + xJ_rJ_sJ_{t-1} - x^3J_{r-1}J_{s-1}J_{t-2}) \\ &= J_{r+1}J_{s+1}(J_{t+2} + xJ_t) + xJ_rJ_s(J_{t+1} + xJ_{t-1}) - x^3J_{r-1}J_{s-1}(J_t + xJ_{t-2}) \\ &= J_{r+1}J_{s+1}j_{t+1} + xJ_rJ_sj_t - x^3J_{r-1}J_{s-1}j_{t-1}. \end{aligned}$$

This, coupled with the earlier value of S, yields the desired result. □

Finally, we explore the Vieta and Chebyshev consequences of identity (2.1).

5. Vieta and Chebyshev Implications

Using the gibonacci-Vieta and Vieta-Chebyshev relationships in Table 1, we can extract the Vieta and Chebyshev counterparts of identity (2.1); in the interest of brevity, we omit the basic algebra:

$$V_{r+1}V_{s+1}d_{t+1} - xV_r(x)V_s(x)d_t + V_{r-1}(x)V_{s-1}(x)d_{t-1} = \begin{cases} xd_{r+s+t}, & \text{if } d_n = V_n; \\ xd_{r+s+t}, & \text{otherwise;} \end{cases}$$

$$U_{r+1}U_{s+1}e_{t+1} - 2xU_rU_se_t + U_{r-1}U_{s-1}e_{t-1} = \begin{cases} 2xe_{r+s+t+2}, & \text{if } e_n = U_n; \\ 2xe_{r+s+t+2}, & \text{otherwise.} \end{cases}$$

6. Ackowledgment

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References

[1] M. Bicknell, *A Primer for the Fibonacci numbers: Part VII*, The Fibonacci Quarterly, **8.4** (1970), 407–420.
[2] A. F. Horadam, *Jacobsthal representation numbers*, The Fibonacci Quarterly, **34.1** (1996), 40–44.
[3] A. F. Horadam, *Jacobsthal representation polynomials*, The Fibonacci Quarterly, **35.2** (1997), 137–148.
[4] A. F. Horadam, *Vieta polynomials*, The Fibonacci Quarterly, **40.3** (2002), 223–232.
[5] T. Koshy, *Differences of Gibonacci polynomial products of orders 2, 3, and 4*, The Fibonacci Quarterly, **56.3** (2018), 212–220.
[6] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Volume I, second edition, Wiley, Hoboken, New Jersey, 2018.
[7] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Volume II, Wiley, Hoboken, New Jersey, 2019.
[8] T. Koshy, *Pell and Pell-Lucas Numbers with Applications*, Springer, New York, 2014.
[9] T. Koshy, *Polynomial extensions of the Lucas and Ginsburg identities*, The Fibonacci Quarterly, **52.2** (2014), 141–147.

[10] T. Koshy, *Polynomial extensions of the Lucas and Ginsburg identities revisited*, The Fibonacci Quarterly, **55.2** (2017), 147–151.

[11] T. Koshy, *A Recurrence for Gibonacci cubes with graph-theoretic confirmations*, The Fibonacci Quarterly, **57.2** (2019), 139–147.

[12] R. S. Melham, *A Fibonacci identity in the spirit of Simson and Gelin-Cesàro*, The Fibonacci Quarterly, **41.2** (2003), 142–143.

[13] A. G. Shannon and A. F. Horadam, *Some relationships among Vieta, Morgan-Voyce and Jacobsthal polynomials*, *Applications of Fibonacci Numbers*, (ed. F. T. Howard), Kluwer, Dordrecht, **8** (1999), 307–323.

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