FOCUSING SEQUENCES AND SELF-SIMILARITY

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ABSTRACT. In this paper, we prove that the variant semi-Fibonacci sequence satisfies a p-similarity property for a number of different primes p. The case $p = 3$ explains an empirical observation of Neil Sloane about the growth rate of the sequence. Taking $p = 3$ and $p = 5$ together, we show that the sequence is aperiodic modulo any integer $n > 2$.

1. INTRODUCTION

There have been a number of papers devoted to integer sequences that are self-similar or fractal in nature (see [2, 3, 4] and the references within). In addition, Michael Gilleland has compiled a list of self-similar sequences in the Online Encyclopedia of Integer Sequences (OEIS) [1]. There is no consensus about terminology so, I introduce the following definitions.

Definition 1.1. *S is m-self-similar if for all $n > 0$, $S(mn) = S(n)$.*

Definition 1.2. *S is m-similar if for every N, there exists an $n \geq 0$ such that for k, $0 < k < N$, $S(n + k) = mS(k)$.*

For instance, the Thue-Morse sequence (A010060 in the OEIS) is 2-self-similar, the number of integer-sided right triangles with hypotenuse n (A046080) is p-self-similar for all $p \not\equiv 1 \pmod 4$, and Gould's sequence (A001316) is 2-self-similar and 2-similar. In this paper, we will examine a sequence that is p-similar for many different values of p. This self-similarity, however, is far from obvious, and we do not know for which values of p the sequence is p-similar (though we suspect the answer is all of them).

The semi-Fibonacci sequence (A030067) is defined by

$$a(1) = 1,\ a(2n) = a(n),\ a(2n + 1) = a(2n - 1) + a(2n).$$

Now, consider the sequence defined by

$$b(1) = 1,\ b(2n) = b(n),$$

$$b(2n + 1) = \begin{cases} b(2n - 1) - b(2n), & \text{if } b(2n - 1) - b(2n) > 0; \\ b(2n - 1) + b(2n), & \text{otherwise.} \end{cases}$$

This is called the variant semi-Fibonacci sequence (A109671) due to Eric Angelini. The OEIS has a slightly different definition, namely that $b(2n+1)$ is the smallest positive number such that $|b(2n + 1) - b(2n - 1)| = b(n)$. Hence, $b(2n + 1) = b(2n - 1) - b(n)$, if that is positive; and otherwise, $b(2n + 1) = b(2n - 1) + b(n)$. Because $b(n) = b(2n)$, these definitions are equivalent and for our purposes, it is more convenient to use the former.

Note that a and b are 2-self-similar by definition. The sequence a has another kind of self-similarity — it remains the same when the first occurrence of every number is removed — but this will not be discussed further. As for b, here are its first 31 terms:

$$1\ 1\ 2\ 1\ 1\ 2\ \underline{3}\ 1\ 2\ 1\ 1\ 2\ \underline{3\ 3\ 6}\ 1\ 5\ 2\ 3\ 1\ 2\ 1\ 1\ 2\ \underline{3\ 3\ 6\ 3\ 3\ 6\ 9}.$$

The underlined segments hint at the 3-similarity of b. The proof of 3-similarity is inductive, but the key difficulty is to prove that the first term of each segment is indeed a 3 (the same goes for general p). This is hard because the terms of b between segments behave in an unpredictable way. Right before a segment begins, b becomes predictable due to a focusing effect.

Using p-similarity for two different primes p (we choose 3 and 5 in Section 4 but any pair of twin primes for which b is p-similar would work), we prove that the generating function of b is not rational and that b is aperiodic modulo n for all $n > 2$. We also examine the growth rate of b and the set of numbers appearing in b. Finally, we prove part of a conjecture of Neil Sloane.

2. Preliminary Results

Lemma 2.1. *If S is m_1-self-similar and m_2-self-similar, then S is m_1m_2-self-similar. If S is m_1-similar and m_2-similar, then S is m_1m_2-similar.*

Proof. For the first part, we have $S(m_1m_2n) = S(m_2n) = S(n)$. For the second part, let n_2 satisfy $S(n_2 + k) = m_2S(k)$ for $0 < k < N$ and n_1 satisfy $S(n_1 + k) = m_1S(k)$ for $0 < k < n_2 + N$. Then,

$$S(n_1 + n_2 + k) = m_1S(n_2 + k) = m_1m_2S(k)$$

for $0 < k < N$. □

Remark 2.2. *Every sequence S is 1-self-similar and 1-similar and so, the set of n for which S is n-self-similar is a multiplicative monoid, as is the set of n for which S is n-similar.*

Proposition 2.3. *For all $n \geq 0$,*

$$b(3n + 1) \equiv b(3n + 2) \equiv b(3n + 3) + 1 \equiv 1 \pmod 2.$$

Proof. We use induction. Suppose that for all m, $0 \leq m < n$, $b(3m + 1) \equiv b(3m + 2) \equiv 1 \pmod 2$, and $b(3m + 3) \equiv 0 \pmod 2$. We now divide the proof into cases.

Case 1: n is odd. We can then write n as $2m + 1$. Then,

$$b(3n + 1) = b(3(2m + 1) + 1) = b(6m + 4) = b(3m + 2),$$

which is odd, and

$$b(3n + 3) = b(3(2m + 1) + 3) = b(6m + 6) = b(3m + 3)$$

is even. Also, $b(3n)$ is even, so $b(3n + 2) = b(3n) \pm b(3n + 1)$ is odd.

Case 2: n is even. We can then write n as $2m$. Then,

$$b(3n + 2) = b(3(2m) + 2) = b(6m + 2) = b(3m + 1),$$

which is odd. Because $b(3n-1)$ is odd and $b(3n)$ is even, $b(3n+1) = b(3n-1) \pm b(3n)$ is odd. Finally, $b(3n + 3) = b(3n + 1) \pm b(3n + 2)$ is even.

The base case is $n = 0$. Then, $b(1) = b(2) = 1$ and $b(3) = 2$. □

Remark 2.4. *By parity, there is no n such that $b(n) = b(n + 1) = b(n + 2)$.*

Remark 2.5. *The sequence b is not n-similar for any even n as otherwise, b would contain arbitrarily long strings of even numbers, whereas we know that b does not even have two consecutive even numbers.*

3. Focusing

Definition 3.1. *A* focusing sequence S *is a sequence of positive integers such that for every* $n > 0$, $2S(n) - 2 \leq \sum_{i=1}^{n-1} S(i)$.

Remark 3.2. *The maximal focusing sequence is described by the recurrence*

$$S(n) = \lfloor \frac{2 + \sum_{i=1}^{n-1} S(i)}{2} \rfloor = \lceil \frac{1 + \sum_{i=1}^{n-1} S(i)}{2} \rceil,$$

which is the definition of A005428 in the OEIS.

Remark 3.3. *If* S *is a focusing sequence,* $0 < S(1) \leq \frac{\sum_{i=1}^{0} S(i)+2}{2} = 1$ *so* $S(1) = 1$*. Similarly,* $0 < S(2) \leq \frac{\sum_{i=1}^{1} S(i)+2}{2} = \frac{3}{2}$ *so* $S(2) = 1$ *as well.*

Lemma 3.4. *Let* S *be a focusing sequence. Let* T *satisfy* $T(1) = 1$*,* $T(2n) = S(n)$*, and* $T(2n+1) = T(2n-1) \pm T(2n) > 0$*. Then,* T *is a focusing sequence.*

Proof. Let $n > 2$. Suppose that for all $0 < m < n$,

$$2T(m) - 2 \leq \sum_{i=1}^{m-1} T(i).$$

We divide the proof into cases.

Case 1: n is even. Then,

$$2T(n) - 2 = 2S\left(\frac{n}{2}\right) - 2 \leq \sum_{i=1}^{\frac{n}{2}-1} S(i) = \sum_{i=1}^{\frac{n}{2}-1} T(2i) < \sum_{i=1}^{n-1} T(i).$$

Case 2: n is odd and $T(n) = T(n-2) - T(n-1)$. Then, $T(n-2) > T(n)$ so we have

$$2T(n) - 2 < 2T(n-2) - 2 \leq \sum_{i=1}^{n-3} T(i) < \sum_{i=1}^{n-1} T(i).$$

Case 3: n is odd, $T(n) = T(n-2) + T(n-1)$, and $T(n-2) \geq T(n-1)$. Then,

$$\sum_{i=1}^{n-3} T(i) \geq 2T(n-2) - 2$$

so

$$\begin{aligned}\sum_{i=1}^{n-1} T(i) &\geq 2T(n-2) - 2 + T(n-2) + T(n-1)\\ &\geq 2T(n-2) + 2T(n-1) - 2 = 2T(n) - 2.\end{aligned}$$

Case 4: n is odd, $T(n) = T(n-2) + T(n-1)$, and $T(n-2) < T(n-1)$. Then,

$$2T(n-1) - 2 = 2S\left(\frac{n-1}{2}\right) - 2 \leq \sum_{i=1}^{\frac{n-1}{2}-1} S(i)$$

so

$$\sum_{i=1}^{n-1} T(i) \geq T(n-1) + T(n-2) + \sum_{i=1}^{\frac{n-1}{2}-1} T(2i)$$
$$= T(n) + \sum_{i=1}^{\frac{n-1}{2}-1} S(i) \geq T(n) + 2T(n-1) - 2$$
$$> T(n) + T(n-1) + T(n-2) - 2 = 2T(n) - 2.$$

Since

$$2T(1) - 2 = 0 \leq \sum_{i=1}^{0} T(i)$$

and

$$2T(2) - 2 = 2S(1) - 2 = 0 \leq \sum_{i=1}^{1} T(i)$$

by Remark 3.3, the result follows by induction. □

Theorem 3.5. *For some* $n \equiv 0 \pmod 3$, *let* S *be defined by* $S(k) = b(n-k)$ *for all* k, $0 < k < n$. *Let* T *be defined as* $T(k) = b(2n-k)$ *for all* k, $0 < k < 2n$. *If* S *is a focusing sequence, then so is* T.

Proof. Set $b'(1) = 1$, and for $k > 1$, recursively define

$$b'(k) = \begin{cases} b'(k-1) - S(n+1-k), & \text{if } b'(k-1) - S(n+1-k) > 0; \\ b'(k-1) + S(n+1-k), & \text{otherwise.} \end{cases}$$

We now show that for all $0 < k \leq n$, $b'(k) = b(2k-1)$. Assume that $b'(k-1) = b(2(k-1)-1)$ for some $k > 1$. By the definition of S,

$$S(n+1-k) = b(k-1) = b(2k-2)$$

so

$$b'(k) = \begin{cases} b(2(k-1)-1) - b(2(k-1)), & \text{if } b(2(k-1)-1) - b(2(k-1)) > 0; \\ b(2(k-1)-1) + b(2(k-1)), & \text{otherwise.} \end{cases}$$
$$= b(2(k-1)+1) = b(2k-1).$$

The base case is $k = 1$. Then, $b'(1) = 1 = b(1) = b(2k-1)$. Therefore for all $k > 0$, $b'(k) = b(2k-1)$.

Next, we prove that

$$b'(k) \leq \sum_{i=1}^{n-k} S(i) + 2$$

for $0 < k < n+1$. Assume that for $1 < k < n+1$,

$$b'(k-1) \leq \sum_{i=1}^{n+1-k} S(i) + 2.$$

If

$$b'(k) = b'(k-1) + S(n+1-k),$$

then

$$S(n+1-k) \geq b'(k-1)$$

and so

$$b'(k) \leq 2S(n+1-k) \leq \sum_{i=1}^{n-k} S(i) + 2$$

because S is a focusing sequence. If

$$b'(k) = b'(k-1) - S(n+1-k),$$

then

$$b'(k) = b'(k-1) - S(n+1-k) \leq \sum_{i=1}^{n+1-k} S(i) + 2 - S(n+1-k) = \sum_{i=1}^{n-k} S(i) + 2.$$

The base case is $k = 1$. Since

$$b'(1) = 1 < \sum_{i=1}^{n-1} S(i) + 2,$$

the proposition holds for all $k < n+1$ by induction. In particular,

$$b'(n) \leq \sum_{i=1}^{0} S(i) + 2 = 2.$$

Since $b'(n) = b(2n-1)$ and $2n-1 \equiv 2 \pmod 3$, $b'(n)$ is odd by Proposition 2.3. Thus, $T(1) = b'(n) = 1$. Also,

$$T(2k) = b(2(n-k)) = b(n-k) = S(k)$$

and

$$T(2k-1) = b(2(n-k)+1) = b(2(n-k)-1) \pm b(2(n-k)) = T(2k+1) \pm T(2k),$$

so

$$T(2k+1) = T(2k-1) \pm T(2k).$$

Thus by Lemma 3.4, T is a focusing sequence. □

Corollary 3.6. *Let S be a focusing sequence defined by $S(k) = b(n-k)$ for all k, $0 < k < n$, for some $n \equiv 0 \pmod 3$. For $j > 0$, let $n_j = 2^j n$ and let T_j be defined as $T_j(k) = b(n_j - k)$ for all k, $0 < k < n_j$. Then, T_j is a focusing sequence.*

Proof. Because $n \equiv 0 \pmod 3$, $n_j = 2^j n \equiv 0 \pmod 3$. The corollary follows by repeated application of Theorem 3.5. □

Definition 3.7. *For prime p, $B(p)$, when it exists, is the smallest positive integer such that*

1. *The sequence starting with $b(B(p)-1)$ and going backward to $b(1)$ is a focusing sequence.*
2. *$b(B(p)) = p-1$, $b(B(p)+1) = p$, $b(B(p)+2) = p$, and $b(B(p)+3) = 2p$.*

Remark 3.8. *$B(2)$ does not exist, because if it did, then using Remark 3.3,*

$$b(B(2)-2) = b(B(2)-1) = b(B(2)) = 1.$$

That, however, would violate Remark 2.4. It would also violate Remark 2.5.

Remark 3.9. *Since $p \neq p-1 \pm p$, $B(p) \equiv 0 \pmod 2$. By Remark 3.8, p is odd. Thus, $b(B(p)) \equiv 0 \pmod 2$ so $B(p) \equiv 0 \pmod 3$.*

Remark 3.10. *For $j \geq 0$, $b(2^j B(p)) = b(2^{j-1}B(p)) = \cdots = b(2^0 B(p)) = p - 1$.*

Lemma 3.11. *For $j \geq -1$, $b(2^j B(p) + 1) = p$.*

Proof. Let $j > -1$. By Corollary 3.6 (which we can use because $B(p)$ is divisible by 3 by Remark 3.9), $b(2^j B(p) - 1)$ is the first term of a focusing sequence and therefore, equal to 1. By Remark 3.10, $b(2^j B(p)) = p - 1$. Since $1 - (p-1) = 2 - p \leq 0$, we have $b(2^j B(p) + 1) = p$.

For $j = -1$, $2^j B(p) + 1$ is an integer because $B(p)$ is even by Remark 3.9. Thus, $b(\frac{B(p)}{2} + 1) = b(B(p) + 2) = p$. □

Lemma 3.12. *Let $j > -1$ and let p be such that $B(p)$ exists. Assume that $b(2^{j-1}B(p) + k) = pb(k)$ for all $0 < k < 2^{j+1}$. For all $0 < k < 2^{j+2}$, $b(2^j B(p) + k) = pb(k)$.*

Proof. Let $c(k) = \frac{b(2^j B(p)+k)}{p}$ for $0 < k < 2^{j+2}$. Suppose there exists $0 < j' < j + 2$ such that for all $k < 2^{j'}$, $c(k) = b(k)$. Then for all $0 < k < 2^{j'}$,

$$c(2k) = \frac{b(2^j B(p) + 2k)}{p} = \frac{b(2^{j-1}B(p) + k)}{p} = \frac{pb(k)}{p} = b(k) = c(k)$$

and

$$\begin{aligned} c(2k+1) &= \frac{b(2^j B(p) + 2k + 1)}{p} \\ &= \begin{cases} \frac{b(2^j B(p)+2k-1) - b(2^j B(p)+2k)}{p}, & \text{if } \frac{b(2^j B(p)+2k-1) - b(2^j B(p)+2k)}{p} > 0; \\ \frac{b(2^j B(p)+2k-1) + b(2^j B(p)+2k)}{p}, & \text{otherwise.} \end{cases} \\ &= \begin{cases} c(2k-1) - c(2k), & \text{if } c(2k-1) - c(2k) > 0; \\ c(2k-1) + c(2k), & \text{otherwise.} \end{cases} \end{aligned}$$

By Lemma 3.11, $c(1) = 1$. Therefore, c satisfies the same recursive property of b. Thus, $c(k) = b(k)$ for all $k < 2^{j'+1}$. Because $j' = 1$ satisfies the required property, the lemma follows by induction. □

Lemma 3.13. *For all $j \geq -1$, $0 < k < 2^{j+2}$, and p for which $B(p)$ exists, $b(2^j B(p) + k) = pb(k)$.*

Proof. The base case is $j = -1$. Then $0 < k < 2$, so $k = 1$. Indeed, $b(2^j B(p) + 1) = p = pb(k)$ by Lemma 3.11. The result follows by induction, using Lemma 3.12. □

Corollary 3.14. *If $B(p)$ exists, then b is p-similar.*

Proof. Given N, take $j = \lceil \log_2 N \rceil - 2$ and $n = 2^j B(p)$. When $k < N$, $k < 2^{\lceil \log_2 N \rceil} = 2^{j+2}$. Hence, $b(n + k) = pb(k)$ by Lemma 3.13. □

Remark 3.15. *We note that $B(3) = 12$. By Lemma 3.13, $b(12 \cdot 2^n + m) \equiv 0 \pmod 3$ for all $0 < m < 2^{n+2}$. Thus, $b(3 \cdot 2^n + m) \equiv 0 \pmod 3$ for all $0 < m < 2^n$. For $p \neq 3$, $b(B(p) + 1) = p \not\equiv 0 \pmod 3$, so $2 \cdot 2^k - 2 < B(p) < 3 \cdot 2^k$ for some k. Additionally, $B(p) \equiv 0 \pmod 6$. Thus, we can refine our bound to $2 \cdot 2^k < B(p) \leq 3 \cdot 2^k$ for all p, including 3. Clearly $k = \lceil \log_2 B(p) \rceil - 2$. It is easy to check that there is no $n \leq 8$ such that*

$$b(n) = b(n+1) - 1 = b(n+2) - 1 = \frac{b(n+3) - 2}{2}.$$

Therefore, $B(p) > 8$ and $k \geq 2$.

Theorem 3.16. *For all $N \geq -1$, $0 < m < 2^{N+2}$, $n \geq 0$, and p for which $B(p)$ exists,*

$$b(2^N\Big(\frac{B(p)(2^{n(\lceil\log_2 B(p)\rceil-2)}-1)}{2^{\lceil\log_2 B(p)\rceil-2}-1}\Big)+m) = p^n b(m).$$

Proof. Let $k = \lceil\log_2 B(p)\rceil - 2$. By Remark 3.15, $k \geq 2$ and $B(p) \leq 3 \cdot 2^k$. Therefore,

$$\begin{aligned}
\sum_{i=0}^{j-1} 2^{ki}B(p) &\leq 3 \cdot \sum_{i=0}^{j-1} 2^{k(i+1)} = 3 \cdot 2^{kj} \sum_{i=0}^{j-1} \Big(\frac{1}{2^k}\Big)^i \\
&= 3 \cdot (2^{kj} \sum_{i=0}^{\infty} \Big(\frac{1}{2^k}\Big)^i - 2^{kj} \sum_{i=j}^{\infty} \Big(\frac{1}{2^k}\Big)^i) \\
&\leq 3 \cdot (2^{kj} \sum_{i=0}^{\infty} \Big(\frac{1}{4}\Big)^i - 2^{kj} \sum_{i=j}^{\infty} \Big(\frac{1}{2^k}\Big)^i) \\
&< 3 \cdot (2^{kj} \sum_{i=0}^{\infty} \Big(\frac{1}{4}\Big)^i - 2^{kj} \Big(\frac{1}{2^k}\Big)^j) = 3 \cdot (2^{kj}\frac{4}{3} - 1) = 2^{kj+2} - 3.
\end{aligned}$$

That means that $\sum_{i=0}^{j-1} 2^{ki}B(p) \leq 2^{kj+2} - 4$. It follows that

$$2^N \sum_{i=0}^{j-1} 2^{ki}B(p) + m < 2^N\Big(\sum_{i=0}^{j-1} 2^{ki}B(p) + 4\Big) \leq 2^N(2^{kj+2} - 4 + 4) = 2^N 2^{kj+2}.$$

Applying Lemma 3.13 repeatedly,

$$\begin{aligned}
b(2^N\Big(\frac{B(p)(2^{n(\lceil\log_2 B(p)\rceil-2)}-1)}{2^{\lceil\log_2 B(p)\rceil-2}-1}\Big)+m) &= b(2^N 2^{k(n-1)}B(p) + 2^N \sum_{i=0}^{n-2} 2^{ki}B(p) + m) \\
&= pb(2^N 2^{k(n-2)}B(p) + 2^N \sum_{i=0}^{n-3} 2^{ki}B(p) + m) \\
&= p^2 b(2^N 2^{k(n-3)}B(p) + 2^N \sum_{i=0}^{n-4} 2^{ki}B(p) + m) \\
&= \cdots \\
&= p^{n-1} b(2^N B(p) + m) = p^n b(m).
\end{aligned}$$

□

A natural question to ask is when $B(p)$ exists. A computer search up to 1,000,000,000 yields the following results (plugging in $N = -1$ and $m = 1$ to Theorem 3.16 to get the formula).

p	$B(p)$	Formula for p^n	Formula's Growth Rate
3	12	$b\left(\frac{6\cdot(2^{2n}-1)}{3}+1\right)=3^n$	$n^{\frac{\log 3}{\log 4}}$
5	2268	$b\left(\frac{1134\cdot(2^{10n}-1)}{1023}+1\right)=5^n$	$n^{\frac{\log 5}{\log 1024}}$
7	20340	$b\left(\frac{10170\cdot(2^{13n}-1)}{8191}+1\right)=7^n$	$n^{\frac{\log 7}{\log 8192}}$
11	302688	$b\left(\frac{151344\cdot(2^{17n}-1)}{131071}+1\right)=11^n$	$n^{\frac{\log 11}{\log 131072}}$
13	283776	$b\left(\frac{141888\cdot(2^{17n}-1)}{131071}+1\right)=13^n$	$n^{\frac{\log 13}{\log 131072}}$
17	1074128	$b\left(\frac{537064\cdot(2^{19n}-1)}{524287}+1\right)=17^n$	$n^{\frac{\log 17}{\log 524288}}$
19	672960	$b\left(\frac{336480\cdot(2^{18n}-1)}{262143}+1\right)=19^n$	$n^{\frac{\log 19}{\log 262144}}$
23	263280	$b\left(\frac{131640\cdot(2^{17n}-1)}{131071}+1\right)=23^n$	$n^{\frac{\log 23}{\log 131072}}$
29	22051824	$b\left(\frac{11025912\cdot(2^{23n}-1)}{8388607}+1\right)=29^n$	$n^{\frac{\log 29}{\log 8388608}}$
31	2748912	$b\left(\frac{1374456\cdot(2^{20n}-1)}{1048575}+1\right)=31^n$	$n^{\frac{\log 31}{\log 1048576}}$
37	67416576	$b\left(\frac{33708288\cdot(2^{25n}-1)}{33554431}+1\right)=37^n$	$n^{\frac{\log 37}{\log 33554432}}$
41	36846720	$b\left(\frac{18423360\cdot(2^{24n}-1)}{16777215}+1\right)=41^n$	$n^{\frac{\log 41}{\log 16777216}}$
43	166979328	$b\left(\frac{83489664\cdot(2^{26n}-1)}{67108863}+1\right)=43^n$	$n^{\frac{\log 43}{\log 67108864}}$
47	163571136	$b\left(\frac{81785568\cdot(2^{26n}-1)}{67108863}+1\right)=47^n$	$n^{\frac{\log 47}{\log 67108864}}$
53	89536512	$b\left(\frac{44768256\cdot(2^{25n}-1)}{33554431}+1\right)=53^n$	$n^{\frac{\log 53}{\log 33554432}}$
59	269850624	$b\left(\frac{134925312\cdot(2^{27n}-1)}{134217727}+1\right)=59^n$	$n^{\frac{\log 59}{\log 134217728}}$
61	274435008	$b\left(\frac{137217504\cdot(2^{27n}-1)}{134217727}+1\right)=61^n$	$n^{\frac{\log 61}{\log 134217728}}$
71	569617920	$b\left(\frac{284808960\cdot(2^{28n}-1)}{268435455}+1\right)=71^n$	$n^{\frac{\log 71}{\log 268435456}}$
89	703549056	$b\left(\frac{351774528\cdot(2^{28n}-1)}{268435455}+1\right)=89^n$	$n^{\frac{\log 89}{\log 268435456}}$
97	272467968	$b\left(\frac{136233984\cdot(2^{27n}-1)}{134217727}+1\right)=97^n$	$n^{\frac{\log 97}{\log 134217728}}$
101	22129536	$b\left(\frac{11064768\cdot(2^{23n}-1)}{8388607}+1\right)=101^n$	$n^{\frac{\log 101}{\log 8388608}}$
137	551375712	$b\left(\frac{275687856\cdot(2^{28n}-1)}{268435455}+1\right)=137^n$	$n^{\frac{\log 137}{\log 268435456}}$

4. Applications

Proposition 4.1. *The generating function $f(x)=\sum_{i=1}^{\infty} b(i)x^i$ is not a rational function.*

Proof. Suppose $f(x)=\frac{P(x)}{Q(x)}$ for polynomials $P(x)$ and $Q(x)$. Let $Q(x)=c_0+c_1x+c_2x^2+\cdots+c_qx^q$ be of degree q and $P(x)$ be of degree p. Let j be such that $2^{j+2}>q$ and $12\cdot 2^j>p$. Let $n=12\cdot 2^j$. By Lemma 3.13, $b(n+k)=3b(k)$ for all k, $0<k<q<2^{j+2}$. Looking at the x^{n+q} term of the equation $f(x)Q(x)=P(x)$, we have

$$c_0b(n+q)+c_1b(n+q-1)+\cdots+c_qb(n)=0.$$

Then,

$$b(n) = -\frac{c_0}{c_q}b(n+q) - \cdots - \frac{c_{q-1}}{c_q}b(n+1) = 3(-\frac{c_0}{c_q}b(q) - \cdots - \frac{c_{q-1}}{c_q}b(1)).$$

Notice also that $b(n) = b(12) = b(B(3)) = 2$. Dividing both sides of the equation by 3, we have

$$\frac{2}{3} = -\frac{c_0}{c_q}b(q) - \cdots - \frac{c_{q-1}}{c_q}b(1).$$

Now, let $m = 2268 \cdot 2^j$. By Lemma 3.13, $b(m+k) = 5b(k)$ for all k, $0 < k < q < 2^{j+2}$. Looking at the x^{m+q} term of the equation $f(x)Q(x) = P(x)$, we have

$$c_0 b(m+q) + c_1 b(m+q-1) + \cdots + c_q b(m) = 0.$$

Then,

$$b(m) = -\frac{c_0}{c_q}b(m+q) - \cdots - \frac{c_{q-1}}{c_q}b(m+1) = 5(-\frac{c_0}{c_q}b(q) - \cdots - \frac{c_{q-1}}{c_q}b(1)).$$

Also, notice that $b(m) = b(2268) = b(B(5)) = 4$. Dividing both sides of the equation by 5, we have

$$\frac{4}{5} = -\frac{c_0}{c_q}b(q) - \cdots - \frac{c_{q-1}}{c_q}b(1) = \frac{2}{3},$$

which is a contradiction. □

Lemma 4.2. *For all $N \geq -1$, $n > 0$, and p for which $B(p)$ exists,*

$$b(2^N\Big(\frac{B(p)(2^{n(\lceil \log_2 B(p)\rceil - 2)} - 1)}{2^{\lceil \log_2 B(p)\rceil - 2} - 1}\Big)) = p^n - p^{n-1}.$$

Proof. Let $k = \lceil \log_2 B(p)\rceil - 2$. Since $B(p) \leq 3 \cdot 2^k < 2^{k+2}$,

$$\begin{aligned} b(2^N\Big(\frac{B(p)(2^{nk}-1)}{2^k-1}\Big)) &= b\Big(\frac{B(p)(2^{nk}-1)}{2^k-1}\Big) \\ &= b\Big(\frac{B(p)}{2^k-1}(2^k(2^{(n-1)k}-1) + 2^k - 1)\Big) \\ &= b\Big(2^k\frac{B(p)(2^{(n-1)k}-1)}{2^k-1} + B(p)\Big) = p^{n-1}b(B(p)) \\ &= p^n - p^{n-1} \end{aligned}$$

by Theorem 3.16. When $N = -1$, the same reasoning holds because $B(p)$ is even. □

Proposition 4.3. *If b is eventually periodic modulo $m > 1$, then $m = 2$.*

Proof. Suppose that b has minimum period k modulo m. We know that there exists some A such that for all $n \geq A$, $b(n+k) \equiv b(n) \pmod{m}$. Let

$$k_1 = 2^{A+k}\frac{12(2^{2\varphi(m)}-1)}{3}.$$

By Lemma 4.2 and Theorem 3.16 (plugging in $p = 3$, $N = A + k$, and $n = \varphi(m)$), $b(k_1) = 2 \cdot 3^{\varphi(m)-1}$ and for all $0 < j \leq k < 2^{A+k+2}$, we have $b(k_1 + j) = 3^{\varphi(m)}b(j)$. Let

$$k_2 = 2^{A+k}\frac{2268(2^{10\varphi(m)}-1)}{1023}.$$

By Lemma 4.2 and Theorem 3.16 (plugging in $p = 5$, $N = A + k$, and $n = \varphi(m)$), $b(k_2) = 4 \cdot 5^{\varphi(m)-1}$, and for all $0 < j \le k < 2^{A+k+2}$, we have $b(k_2+j) = 5^{\varphi(m)}b(n)$. By Euler's theorem,

$$b(k_1 + j) \equiv 3^{\varphi(m)}b(j) \equiv b(j) \equiv 5^{\varphi(m)}b(j) \equiv b(k_2 + j) \pmod{m}.$$

Setting $j = k$,

$$\begin{aligned} b(k_1 + k) \equiv b(k_1) \equiv 2 \cdot 3^{\varphi(m)-1} \equiv \frac{2}{3}3^{\varphi(m)} \equiv \frac{2}{3} \equiv b(k_2 + k) \\ \equiv b(k_2) \equiv 4 \cdot 5^{\varphi(m)-1} \equiv \frac{4}{5}5^{\varphi(m)} \equiv \frac{4}{5} \pmod{m} \end{aligned}$$

using b's periodicity. Cross-multiplying, we have $10 \equiv 12 \pmod{m}$. Hence, $m = 2$. □

Proposition 4.4. *For all $n > 0$,*

$$b(2^n - 1) = \begin{cases} 2 \cdot 3^{\frac{n-2}{2}}, & \text{if } n \text{ is even;} \\ 3^{\frac{n-1}{2}}, & \text{otherwise.} \end{cases}$$

Proof. Plugging in $N = -1$, $m = 1$, and $p = 3$ to Theorem 3.16 and simplifying, we get $b(2^{2n+1} - 1) = 3^n$. Plugging in $N = 0$, $m = 3$, and $p = 3$ to Theorem 3.16 and simplifying, we get $b(2^{2n+2} - 1) = 2 \cdot 3^n$. □

Neil Sloane conjectured in the OEIS [5] that the values at $2^n - 1$ are those given by Proposition 4.4 and that these are the record values of the sequence. The second part seems plausible based on how much quicker the rate of growth is for 3 than for the other primes. However, this remains open.

Do all positive integers appear in b? All the numbers from 1 to 10,000 appear among its first 1,000,000,000 terms. (The smallest that does not is 14,732.) Because b is p-similar for 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 71, 89, 97, 101, and 137, we know (by Lemma 2.1) that if n has a prime factorization consisting only of these primes, then n appears in b (b is n-similar). If $B(p)$ were proven to exist for all odd primes, all odd numbers would thus be guaranteed to appear in b. We may extend the definition of B to odd composite numbers. Just because b is n-similar does not mean that $B(n)$ exists. However, if $B(n)$ existed for all odd numbers, then $b(B(2m + 1)) = 2m$, so all positive integers would appear in b.

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