INTEGRALS OF FIBONACCI POLYNOMIALS AND THEIR VALUATIONS

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ABSTRACT. The p-adic valuation of an integer x is the highest power of the prime p dividing x. This work discusses the p-adic valuation of a sequence of numbers $\{e_n\}$, defined in terms of the Fibonacci polynomials $F_n(x)$.

1. INTRODUCTION

The Fibonacci numbers F_n are defined by the recurrence $F_n = F_{n-1} + F_{n-2}$, with initial conditions $F_0 = 0$ and $F_1 = 1$. The extension of this sequence to polynomials, the so-called *Fibonacci polynomials* are defined by

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad \text{for } n \geq 3, \tag{1.1}$$

with initial conditions $F_1(x) = 1$ and $F_2(x) = x$. These are polynomials with positive integer coefficients.

Divisibility properties of F_n appear in Lengyel [4]. For a prime p, the highest power of p that divides $n \in \mathbb{N}$ is called the p-adic valuation of n and it is denoted by $\nu_p(n)$. The valuations of the Fibonacci sequence are expressed in terms of $\alpha(k)$, the smallest value of $n \geq 1$ such that k divides F_n. This value is connected to $\pi(k)$, the smallest period of the sequence $F_n \bmod k$. For instance, it is known that $\alpha(k)$ divides $\pi(k)$ [8, Theorem 3].

Theorem 1.1 (Lengyel, [4])**.** *For $n \geq 1$,*

$$\nu_2(F_n) = \begin{cases} \nu_2(n) + 2, & \text{if } n \equiv 0 \bmod 6; \\ 1, & \text{if } n \equiv 3 \bmod 6; \\ 0, & \text{if } n \equiv 1, 2, 4, 5 \bmod 6, \end{cases} \tag{1.2}$$

$\nu_5(F_n) = \nu_5(n)$, *and for a prime* $p \neq 2, 5$,

$$\nu_p(F_n) = \begin{cases} \nu_p(n) + \nu_p(F_{\alpha(p)}), & \text{if } \alpha(p) \mid n; \\ 0, & \text{if } \alpha(p) \nmid n. \end{cases} \tag{1.3}$$

Arithmetic properties of $\nu_p(F_n)$ have been described in [5] in terms of *regular sequences*, a concept introduced by Allouche and Shallit [2, 1]. For $k \geq 2$, the *k-kernel* of a sequence $\{a(n)\}_{n \geq 0}$ is the set of subsequences

$$\{\{a(k^e n + i)\}_{n \geq 0} : \ e \geq 0, \, 0 \leq i \leq k^e - 1\}. \tag{1.4}$$

A sequence is *k-regular* if the $\mathbb{Z}$-module generated by its k-kernel is finitely generated. The *rank* of this Z-module is called the *rank of the sequence*. An example of the results in [5] is Theorem 1.2.

Theorem 1.2. *Let p be a prime $p \equiv 1, 4 \bmod 5$ such that $\nu_p(F_{\alpha(p)}) = 1$. Then, $\{\nu_p(F_{n+1})\}_{n \geq 0}$ is a p-regular sequence of rank at most p. Moreover, for $p \neq 2, 5$, the rank is conjectured to be $\alpha(p) + 1$.*

The goal of this note is to analyze a related sequence of numbers, defined by

$$e_n = \int_0^\infty F_n(x)e^{-x}\,dx. \tag{1.5}$$

The first few values $\{1, 1, 3, 8, 31, 147, 853, 5824\}$ are found in the site OEIS (developed by N. Sloane) as entry A003470.

The central question discussed here concerns the sequence of p-adic valuations $\{\nu_p(e_n) : n \in \mathbb{N}\}$. Here, p is a prime and $\nu_p(x)$ is the highest power of p dividing the integer x. The analysis of $\nu_p(e_n)$ proceeds as follows: for $n \in \mathbb{N}$, consider first the sequence $\{e_n\}$ modulo the prime p. If $e_n \not\equiv 0 \bmod p$, then $\nu_p(e_n) = 0$. It is shown that $\{e_n \bmod p\}$ is periodic with period length $2p$. Therefore, as a first step, one only needs to compute the values $\{e_1 \bmod p, \dots, e_{2p} \bmod p\}$. The last term is special and $e_{2p} \equiv 0 \bmod p$ for every prime p. The valuation $\nu_p(e_{2p})$ is 1 for almost every prime. The condition $\sum_{j=0}^{p-1}(-1)^j j!^2 \not\equiv 0 \bmod p$ guarantees this. Among the first 50000 odd primes, $p_{25} = 97$ is the only case where this condition fails. For this prime, $\nu_{97}(e_{194}) = 2$. Now, consider the remaining indices $j \in \{1, 2, \dots, 2p-1\}$ and say that j is a *root* if $e_j \equiv 0 \bmod p$. Assume j_1 is such a root, then if $n \equiv j_1 \bmod p$, it follows that $\nu_p(e_n) \geq 1$, because $e_n \equiv 0 \bmod p$. In the second step, consider the p indices $n \equiv j_1 \bmod p^2$. Those indices n with $e_n \not\equiv 0 \bmod p^2$ satisfy $e_n \equiv 0 \bmod p$ and $e_n \not\equiv 0 \bmod p^2$; therefore, $\nu_p(e_n) = 2$. This process is continued for higher powers of the prime p. The main result is that at every level of a branch, there is a single vertex where the valuation is not determined. Moreover, this property is determined by a single congruence modulo p. The number of branches modulo p remains an open question.

The coefficients $\{e_n\}$ are expressed from the well-known formula

$$F_n(x) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} x^{n-2j-1}, \tag{1.6}$$

which produces

$$e_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(n-j-1)!}{j!}. \tag{1.7}$$

This shows that e_n is actually a positive integer.

The first result is an alternative expression for e_n.

Proposition 1.3. The numbers e_n are given by

$$e_{2n+1} = \sum_{j=0}^{n} \frac{(2n-j)!}{j!} = \sum_{j=0}^{n} \frac{(n+j)!}{(n-j)!} \tag{1.8}$$

and

$$e_{2n} = \sum_{j=0}^{n-1} \frac{(2n-j-1)!}{j!} = \sum_{j=0}^{n-1} \frac{(n+j)!}{(n-j-1)!}. \tag{1.9}$$

Proof. The first expression in (1.8) and (1.9) comes from (1.7). To obtain the second formula, reverse the order of summation. □

Note 1.1. Observe that the term in e_{2n} is

$$\frac{(n+j)!}{(n-j-1)!} = (n+j)(n+j-1)\cdots(n-j+1)(n-j). \tag{1.10}$$

Combining the terms in (1.10), one from each end, gives the form

$$\frac{(n+j)!}{(n-j-1)!} = n\prod_{\ell=1}^{j}\left(n^2-\ell^2\right). \tag{1.11}$$

Similarly, the term in e_{2n+1} is written as

$$\frac{(n+j)!}{(n-j)!} = n(n+j)\prod_{\ell=1}^{j-1}\left(n^2-\ell^2\right). \tag{1.12}$$

The coefficients $\{e_n\}$ are now given in terms of the products $P(n,j)$ defined by $P(n,0)=1$ and $P(n,j)=\prod_{\ell=1}^{j}\left(n^2-\ell^2\right)$. Then,

$$e_{2n} = n\sum_{j=0}^{n-1}P(n,j) \tag{1.13}$$

and

$$e_{2n+1} = 1+n\sum_{j=1}^{n}(n+j)P(n,j-1). \tag{1.14}$$

Observe that $P(n,j)=0$ for $j \geq n$.

Note 1.2. Corollary 3.3 shows that $\{e_n \bmod m\}$ is a periodic sequence with fundamental period of length $2m$. In particular, for p prime, the sequence has period $2p$. The analysis below will determine properties of the valuations $\nu_p(e_n)$ when $n \not\equiv 0 \bmod p$. The special case of $\nu_p(e_{2p})$ is discussed first.

The coefficient e_{2p} is given by

$$e_{2p} = \sum_{j=0}^{p-1}\frac{(p+j)!}{(p-j-1)!}. \tag{1.15}$$

The general term in the sum is written as

$$\frac{(p+j)!}{(p-j-1)!} = (p+j)(p+j-1)\cdots(p+1)p(p-1)\cdots(p-j) \tag{1.16}$$

and this gives

$$\frac{e_{2p}}{p} = \sum_{j=0}^{p-1}(p+j)(p+j-1)\cdots(p+1)\times(p-1)\cdots(p-j). \tag{1.17}$$

Now, consider the right side modulo p to obtain Proposition 1.4.

Proposition 1.4. Let p be a prime. Assume

$$\beta(p) = \sum_{j=0}^{p-1}(-1)^j j!^2 \bmod p \tag{1.18}$$

is nonzero modulo p. Then $\nu_p(e_{2p}) = 1$. Among the first 50000 odd primes, this conditions fails only for $p = 97$. In this case, $\nu_{97}(e_{194}) = 2$.

Section 2 presents a variety of recurrences satisfied by $\{e_n\}$. Section 3 presents analytic formulas for the valuations $\nu_p(e_n)$ for small primes p. These are defined as the highest power of p dividing e_n. There is an interesting representation of $\{\nu_p(e_n)\}$, appearing for the first time at $p = 7$. This is discussed in detail in Section 3.

2. Recurrences

This section presents some recurrences satisfied by the numbers e_n. Applications to modular properties of these numbers are discussed in later sections.

Theorem 2.1. *The sequence $\{e_n\}$ satisfies the recurrence*

$$e_n = (n-1)e_{n-1} - e_{n-2} + 1 - (-1)^n, \tag{2.1}$$

with initial conditions $e_1 = e_2 = 1$.

Proof. The proof is done by a small variation of the WZ-method [6, 7]. Define

$$F(n,k) = \frac{(n-k-1)!}{k!} \quad \text{and} \quad G(n,k) = \frac{(n-k)!}{k!}, \tag{2.2}$$

so that $e_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} F(n,k)$. Now, check that

$$F(n+2,k) - (n+1)F(n+1,k) + F(n,k) = G(n-1,k) - G(n-1,k-1). \tag{2.3}$$

In the usual application of the WZ-method, the next step is to sum over all integer values of k. In this case, because the sums are finite and cannot be extended in a natural manner to infinite sums, the boundary terms have to be addressed. Summing from $k = 1$ to $\overline{n} = \lfloor \frac{n-1}{2} \rfloor$ gives

$$\sum_{k=1}^{\overline{n}} F(n+2,k) - (n+1)\sum_{k=1}^{\overline{n}} F(n+1,k) + \sum_{k=1}^{\overline{n}} F(n,k) = G(n-1,\overline{n}) - G(n-1,0). \tag{2.4}$$

This is now written in terms of $e_n = \sum_{k=0}^{\overline{n}} F(n,k)$. The formulas are divided according to the parity of n.

Assume first that n is even. Then, $\overline{n} = n/2 - 1$ and $\overline{n+2} = n/2$. Therefore,

$$\begin{aligned}
\sum_{k=1}^{\overline{n}} F(n+2,k) &= \sum_{k=1}^{n/2-1} F(n+2,k) \\
&= -F(n+2,0) + \sum_{k=0}^{\overline{n+2}} F(n+2,k) - F(n+2,n/2) \\
&= e_{n+2} - (n+1)! - n/2 - 1.
\end{aligned}$$

Similarly,

$$\sum_{k=1}^{\overline{n}} F(n+1,k) = e_{n+1} - n! - 1 \text{ and } \sum_{k=1}^{\overline{n}} F(n,k) = e_n - (n-1)! \tag{2.5}$$

and (2.4) gives $e_n - (n-1)e_{n-1} + e_{n-2} = 0$. The case n is odd is treated in the same manner. $\square$

The recurrence (2.1), with $n-2$ instead on n, gives an expression for e_{n-2}. Replacing it in (2.1) gives a second form of this recurrence.

Corollary 2.2. *The sequence e_n satisfies the recurrence*

$$e_n = (n-1)e_{n-1} - (n-3)e_{n-3} + e_{n-4}, \tag{2.6}$$

with $e_1 = e_2 = 1$, $e_3 = 3$, and $e_4 = 8$.

The WZ-method used above also produces recurrences for the subsequences $\{e_{2n}\}$ and $\{e_{2n-1}\}$.

Theorem 2.3. *The sequence $T_{1,n} = e_{2n+1}$ satisfies the recurrence*

$$(n+1)T_{1,n+2} - (2n+3)(2n^2+6n+3)T_{1,n+1} + (n+2)T_{1,n} = 2(2n+3) \tag{2.7}$$

and $T_{2,n} = e_{2n}$ satisfies

$$(2n+1)T_{2,n+2} - 2(n+1)(4n^2+8n+1)T_{2,n+1} + (2n+3)T_{2,n} = 2(2n+1)(2n+3). \tag{2.8}$$

3. The p-adic Valuations of the Sequence $\{e_n\}$

This section begins the discussion on prime factorization of the sequence $\{e_n\}$. These properties are expressed in terms of the p-adic valuation of e_n, the highest power of p that divides e_n.

Using the expressions for e_{2n} and e_{2n+1} in (1.13) and (1.14), the next result gives $\{e_n\}$ modulo m as sums of at most m terms.

Lemma 3.1. *Suppose $n \equiv n' \bmod m$. Then,*

$$\begin{aligned} e_{2n} &\equiv n' \sum_{j=0}^{m} P(n',j) \qquad \bmod m, \qquad (3.1)\\ e_{2n+1} &\equiv 1 + n' \sum_{j=1}^{m} (n'+j)P(n',j-1) \bmod m. \end{aligned}$$

Proof. From $P(n,j) = \prod_{\ell=1}^{j}(n^2-\ell^2)$, it follows that $P(n,j) \equiv P(n',j) \bmod m$, if $n \equiv n' \bmod m$. The result follows from $P(n,j) \equiv 0 \bmod m$ for $j \geq m$. □

Example 3.2. Lemma 3.1 reduces the computation of $e_n \bmod m$ terms. For example, if $n \equiv 4 \bmod 6$, then $n' = 4$ and

$$e_{2n} = e_{2(6k+4)} \equiv 4\sum_{j=0}^{6} P(4,j) = 5824 \equiv 4 \bmod 6. \tag{3.2}$$

These congruences will be used to determine divisibility properties of $\{e_n\}$. In the case above, e_{2n} is congruent to 4 modulo 6, so it is not divisible by 3; that is, $\nu_3(e_{2(6k+4)}) = 0$.

Corollary 3.3. *The sequence $\{e_n \bmod m\}$ is periodic of length $2m$.*

Proof. This follows directly from Lemma 3.1, and (1.13) and (1.14). □

Corollary 3.4. *For a prime p, we have $e_{2p-j} \equiv (-1)^{j+1}e_{2p+j} \bmod p$.*

Proof. Theorem 2.1 implies that $1 = e_1 \equiv e_{2p+1} \equiv 2pe_{2p} - e_{2p-1} + 2 \bmod p$, which demonstrates that $e_{2p-1} \equiv 1 \bmod p$. Similarly, it can be shown that $1 = e_2 \equiv e_{2p+2} \equiv -e_{2p-2} \bmod p$. Theorem 2.1 also can be used to establish

$$\begin{aligned} e_{2p+j+2} &\equiv (j+1)e_{2p+j+1} - e_{2p+j} + 1 - (-1)^{j+2}, \\ e_{2p-j-2} &\equiv -(j+1)e_{2p-j-1} - e_{2p-j} + 1 - (-1)^{j+2}. \end{aligned}$$

The result follows by induction. □

The rest of the section is devoted to producing closed-form expressions for the p-adic valuations of $\{e_n\}$ for small primes p.

3.1. **The 2-adic Valuation.** The next result gives the 2-adic valuation of e_n.

Theorem 3.5. *For $n \in \mathbb{N}$,*

$$\nu_2(e_n) = \begin{cases} 0, & \text{if} \quad n \not\equiv 0 \bmod 4; \\ 3, & \text{if} \quad n \equiv 4 \bmod 8; \\ 6, & \text{if} \quad n \equiv 8 \bmod 16; \\ \nu_2(n) + 4, & \text{if} \quad n \equiv 0 \bmod 16. \end{cases} \tag{3.3}$$

Proof. Because $\{e_n \bmod 2\}$ is periodic of period 4, the table

n	1	2	3	4
$e_n \bmod 2$	1	1	1	0

shows that $\nu_2(e_n) = 0$, if $n \not\equiv 0 \bmod 4$. Now, assume $n \equiv 0 \bmod 4$. Then (1.13), with $n = 2k$, gives

$$\frac{e_{2k}}{k} \equiv \sum_{j=0}^{m} P(k,j) \bmod m. \tag{3.4}$$

For $n \equiv 4 \bmod 8$, that is, $k \equiv 2 \bmod 4$, one finds $k^2 \equiv 4 \bmod 16$. Therefore, $P(k,2) \equiv 0 \bmod 16$. The relation (3.4) now reduces to

$$\frac{e_{2k}}{k} \equiv P(k,0) + P(k,1) = 4 \bmod 16. \tag{3.5}$$

It follows that $\nu_2(e_{2k}) = \nu_2(k) + 2 = 3$, because $k \equiv 2 \bmod 4$. The last case is decided in the same form. □

3.2. **The 3-adic Valuation.** The 3-adic valuation of $\{e_n\}$ is determined next. The sequence $\{e_n \bmod 3\}$ is periodic with periodic pattern $\{1, 1, 0, 2, 1, 0\}$, so that $\nu_3(e_n)$ is 0, unless $n \equiv 0 \bmod 3$. In the remaining cases, $\nu_3(e_n) \geq 1$. Assume first $n \equiv 3 \bmod 6$, and write $n = 2k+1$ with $k \equiv 1 \bmod 3$. Then $P(k,1) \equiv 0 \bmod 9$, producing $e_{2k+1} \equiv 1 + k \cdot (k+1)P(k,0) \equiv 1 + k(k+1) \bmod 9$. For $k \equiv 1, 4, \text{or } 7 \bmod 9$, this yields $e_{2k+1} \equiv 3 \bmod 9$. It follows that $\nu_3(e_n) = 1$, if $n \equiv 3 \bmod 6$. In the remaining case, $n \equiv 0 \bmod 6$, write $n = 2k$ with $k \equiv 0 \bmod 3$. Then, $k^2 \equiv 0 \bmod 9$, so that $P(k,3) \equiv 0 \bmod 9$. Therefore,

$$\frac{e_{2k}}{k} \equiv P(k,0) + P(k,1) + P(k,2) \equiv 4 \bmod 9.$$

This gives $\nu_3(e_n) = \nu_3(k) = \nu_3(n)$. These results are summarized next.

Theorem 3.6. *For $n \in \mathbb{N}$,*

$$\nu_3(e_n) = \begin{cases} \nu_3(n), & \text{if} \quad n \equiv 0 \bmod 6; \\ 0, & \text{if} \quad n \equiv 1, 2, 4, 5 \mod 6; \\ 1, & \text{if} \quad n \equiv 3 \mod 6. \end{cases} \tag{3.6}$$

3.3. **The 5-adic Valuation.** The 5-adic valuation $\nu_5(e_n)$ is determined as in the previous two cases. The sequence $n \not\equiv 0 \bmod 10$ is periodic of period 10 with periodic pattern $\{1, 1, 3, 3, 1, 2, 3, 4, 1, 0\}$. Thus, e_n is divisible by 5, only when $n \equiv 0 \bmod 10$. In this case, write $n = 2k$ with $k \equiv 0 \mod 5$. Then $k^2 \equiv 0 \mod 25$, so that $P(k, 5) = 0$ and $\dfrac{e_{2k}}{k} \equiv \displaystyle\sum_{j=0}^{4} P(k, j)$. It follows that $e_{2k}/k \equiv 19 \bmod 25$, establishing the next statement.

Theorem 3.7. *For $n \in \mathbb{N}$,*

$$\nu_5(e_n) = \begin{cases} \nu_5(n), & \text{if} \quad n \equiv 0 \mod 10; \\ 0, & \text{if} \quad n \not\equiv 0 \mod 10. \end{cases} \tag{3.7}$$

4. The 7-adic Valuation and the Difference Function Δ

This section describes the 7-adic valuation $\nu_7(e_n)$. For this prime, there is no finite analytic expression for $\nu_7(e_n)$ in terms of $\nu_7(n)$, as in the case of previous primes. The sequence $\{\nu_7(e_n)\}$ is represented here by an infinite tree, with specific branching rules. The discussion begins with some elementary statements.

Lemma 4.1. *Assume $n \not\equiv 0, 6, 8 \bmod 14$. Then, $\nu_7(e_n) = 0$.*

Proof. The value e_n mod 7 is a periodic sequence of period 14, with fundamental period

$$\{1, 1, 3, 1, 3, 0, 6, 0, 3, 6, 3, 6, 1, 0\}. \tag{4.1}$$

Therefore, $e_n \not\equiv 0 \bmod 7$, if $n \not\equiv 0, 6, 8 \bmod 14$ and the statement follows from here. □

Lemma 4.2. *Assume $n \equiv 0 \bmod 14$. Then, $\nu_7(e_n) = \nu_7(n)$.*

Proof. An argument similar to the one used in the case $p = 3$ or $p = 5$ gives this result. The details are omitted. □

The discussion now concentrates on indices of the form $n \equiv 6 \bmod 14$. The analysis for $n \equiv 8 \bmod 14$ is similar.

Experimental results. The statements below come from a *Mathematica* experiment:

$$\nu_7(e_{6+14n}) = \begin{cases} \geq 2, & \text{if } n \equiv 0 \bmod 7; \\ 1, & \text{if } n \not\equiv 0 \bmod 7. \end{cases} \tag{4.2}$$

Now, consider indices of the form $6+2\cdot7^2n$, coming from the indices where the valuation has not been determined yet. The experimental data described above show that $\nu_7(e_{6+2\cdot7^2n}) \geq 2$. The index n is split according to its residue modulo 7. More symbolic experiments give

$$\nu_7(e_{6+2\cdot7^2n}) = \begin{cases} \geq 3, & \text{if } n \equiv 3 \bmod 7; \\ 2, & \text{if } n \not\equiv 3 \bmod 7; \end{cases} \tag{4.3}$$

that may be written as

$$\begin{aligned} \nu_7(e_{6+2\cdot 3\cdot 7^2+2\cdot 7^3 n}) &\geq 3, \\ \nu_7(e_{6+2\cdot a\cdot 7^2+2\cdot 7^3 n}) &= 2 \quad \text{if } a \in \{0,\,1,\,2,\,4,\,5,\,6\}. \end{aligned} \tag{4.4}$$

The next step is to consider indices of the form $6 + 2 \cdot 3 \cdot 7^2 + 2 \cdot 7^3 n$ and again split n in residue classes modulo 7. The data show that there is a single exceptional modular class for which the valuation is at least 4. All the other classes have valuation 3. This process continues indefinitely; a proof is presented below. At each level, there is a unique class whose valuation is not determined, creating an infinite branching tree. The sequence $\{\nu_7(e_n)\}$ has similar characteristics to the 2-adic valuation of the Stirling numbers of the second kind discussed in [3].

The proof of the statement above requires some preliminary results on the relation

$$\frac{e_{2k}}{k} \equiv \sum_{j=0}^{m} P(k,j) \bmod m \tag{4.5}$$

given in Lemma 3.1. Because we are interested in indices of the form $14n + 6$, define

$$\alpha_{1,n} = \frac{e_{2(3+7n)}}{3+7n} \bmod 7^2, \tag{4.6}$$

and use (4.5) to produce the table

n	0	1	2	3	4	5	6
$\alpha_{1,n}$	$0 \cdot 7$	$2 \cdot 7$	$4 \cdot 7$	$6 \cdot 7$	$1 \cdot 7$	$3 \cdot 7$	$5 \cdot 7$

Because $e_{2(3+7\cdot 0)}$ is the only element divisible by 7^2, split the indices of the form $2(3 + 7 \cdot 0 + 7^2 n)$ according to the value n mod 7 and consider the expression

$$\alpha_{2,n} = \frac{e_{2(3+7\cdot 0+7^2 n)}}{3+7\cdot 0+7^2 n} \bmod 7^3. \tag{4.7}$$

Again, (4.5) gives the table

n	0	1	2	3	4	5	6
$\alpha_{2,n}$	$1 \cdot 49$	$3 \cdot 49$	$5 \cdot 49$	$0 \cdot 49$	$2 \cdot 49$	$4 \cdot 49$	$6 \cdot 49$

Note that in each table, the values are increasing by $2 \cdot 7^r$ for an appropriate value of r. We will show that the number 2 is an invariant for $p = 7$ and $n = 6$, and that this pattern continues at every level of the tree for $p = 7$. At this point it is useful to consider first the more general situation where 7 is replaced by a prime $p \geq 7$. From (1.13), it follows that

$$\frac{e_{2n+2c}}{n+c} - \frac{e_{2n}}{n} = \sum_{j=0}^{n+c-1} \left[P(n+c,j) - P(n,j)\right], \tag{4.8}$$

recalling that $P(m,j) = 0$ for $j \geq m$. For a fixed prime p, introduce the notation

$$\Delta(n,j,r) = P(n+p^r,j) - P(n,j) \tag{4.9}$$

to write (4.8) as

$$\frac{e_{2n+2p^r}}{n+p^r} - \frac{e_{2n}}{n} = \sum_{j=0}^{n+c-1} \Delta(n,j,r). \tag{4.10}$$

In what follows, Lemmas 4.3 to 4.7 lead to the definition of a function $\delta_p(n)$, which is independent of both r and j. This function allows us to fully describe the tree structure and explains the patterns seen in the experimental data above for $p = 7$.

Lemma 4.3. *The function* $\Delta(n,j,r)$ *satisfies the recurrence*

$$\Delta(n,j,r) = (n^2-j^2)\Delta(n,j-1,r) + p^r(2n+p^r)P(n+p^r,j-1), \tag{4.11}$$

with initial condition $\Delta(n,0,r)=0$.

Proof. The relation $P(n,j) = (n^2-j^2)P(n,j-1)$ gives

$$\begin{aligned}\Delta(n,j,r) &= P(n+p^r,j) - P(n,j)\\ &= ((n+p^r)^2 - j^2)P(n+p^r,j-1) - (n^2-j^2)P(n,j-1)\\ &= (n^2-j^2)\left(P(n+p^r,j-1) - P(n,j-1)\right) + (2np^r + p^{2r})P(n+p^r,j-1)\\ &= (n^2-j^2)\Delta(n,j-1,r) + p^r(2n+p^r)P(n+p^r,j-1),\end{aligned}$$

as claimed. □

Lemma 4.4. *For* $n,\ j,\ r \in \mathbb{N}$*, the congruence* $\Delta(n,j,r) \equiv 0 \bmod p^r$ *holds. This implies*

$$P(n+p^r,j) \equiv P(n,j) \bmod p^r.$$

Proof. Use induction on j and the statement of Lemma 4.3. □

Because $\Delta(n,j,r)$ is divisible by p^r, it is convenient to define

$$\Delta_p(n,j) = \frac{\Delta(n,j,r)}{p^r} \bmod p. \tag{4.12}$$

Although this function appears to depend on r, the next result demonstrates that the value of $\Delta_p(n,j)$ is independent of r.

Lemma 4.5. *The function* $\Delta_p(n,j)$ *satisfies the recurrence*

$$\Delta_p(n,j) \equiv (n^2-j^2)\Delta_p(n,j-1) + 2nP(n,j-1) \bmod p, \tag{4.13}$$

with $\Delta_p(n,0)=0$.

Proof. This follows directly from Lemma 4.3. □

Lemma 4.6. *Let* p *be prime and* $n<p$*. Then for* $j \geq \max(n,p-n)$*, the equation* $\Delta_p(n,j) \equiv 0 \bmod p$ *holds. In particular,* $\Delta_p(n,j)=0$ *when* $j \geq p$.

Proof. First consider $n \leq \frac{p-1}{2}$, so that $n < p-n < p$. The value $P(n,n)=0$ in recurrence (4.13) yields

$$\Delta_p(n,n+1) \equiv (n^2-(n+1)^2)\Delta_p(n,n) \bmod p. \tag{4.14}$$

Iterating this relation for $j \geq n+1$ gives

$$\Delta_p(n,j) \equiv \left(n^2-(n+1)^2\right)\left(n^2-(n+2)^2\right)\cdots\left(n^2-j^2\right)\Delta_p(n,n) \bmod p. \tag{4.15}$$

The last factor is 0 when $j=p-n$. This shows that $\Delta_p(n,j)=0$ for $j \geq p-n$.

The argument is similar for $n \geq \frac{p+1}{2}$, beginning with $P(n,p-n)=0$, so that equation 4.15 holds for $j \geq p-n$. Then $\Delta_p(n,j)=0$ for $j \geq n$. □

The identity

$$\frac{e_{2(n+p^r)}}{n+p^r} - \frac{e_{2n}}{n} = \sum_{j=0}^{n-1+p^r} \Delta(n,j,r), \tag{4.16}$$

is now considered modulo p^{r+1}. The periodicity of $\{e_n \bmod p^r\}$ shows that

$$\sum_{j=0}^{n-1+p^r} \Delta(n,j,r) \equiv 0 \bmod p^r. \tag{4.17}$$

Therefore, (4.16) produces

$$\frac{1}{p^r}\left(\frac{e_{2(n+p^r)}}{n+p^r} - \frac{e_{2n}}{n}\right) \equiv \sum_{j=0}^{n-1+p^r} \Delta_p(n,j) \bmod p. \tag{4.18}$$

Lemma 4.6 shows that the sum could be stopped at $j = p$; that is Lemma 4.7.

Lemma 4.7. *For $n \in \mathbb{N}$, p prime, and $r \in \mathbb{N}$,*

$$\frac{1}{p^r}\left(\frac{e_{2(n+p^r)}}{n+p^r} - \frac{e_{2n}}{n}\right) \equiv \sum_{j=0}^{p} \Delta_p(n,j) \bmod p. \tag{4.19}$$

Note: The expression on the right of (4.19) depends only on n and p. Introduce the notation

$$\delta_p(n) = \sum_{j=0}^{p} \Delta_p(n,j) \bmod p. \tag{4.20}$$

Naturally, $0 \leq \delta_p(n) < p$, and this value is independent of r.

Lemma 4.8. *For n, j, $r \in \mathbb{N}$ and p prime,*

$$\frac{e_{2n+2jp^r}}{n+jp^r} - \frac{e_{2n}}{n} \equiv \delta_p(n) j p^r \bmod p^{r+1}. \tag{4.21}$$

Proof. The result follows from (4.19) and a telescoping argument. The details are omitted. □

The previous results are illustrated in the case $p = 7$.

Step 1. The sequence $\{e_n \bmod 7\}$ is periodic with fundamental period

$$\{1, 1, 3, 1, 3, 0, 6, 0, 3, 6, 3, 6, 1, 0\}.$$

Therefore $\nu_7(n) = 0$, if $n \not\equiv 0, 6, 8 \bmod 14$. This is Lemma 4.1.

Step 2. The algorithm requires the value of $\delta_7(3)$. A direct computation of the values for $0 \leq j \leq 3$ (noting that $\Delta_7(n,j) = 0$ for $j \geq 4$) gives

$$\delta_7(3) = \sum_{j=0}^{3} \Delta_7(3,j) = 0 + 6 + 1 + 2 \equiv 2 \bmod 7. \tag{4.22}$$

Step 3. Consider the numbers of the form e_{6+14j}. Because $e_{6+14j} \equiv 0 \bmod 7$, it follows that $\nu_7(e_{2(3+7j)}) \geq 1$. The identity (4.21) becomes

$$\frac{e_{2(3+j7^r)}}{3+j7^r} \equiv \frac{e_6}{3} + 2j \cdot 7^r \bmod 7^{r+1}. \tag{4.23}$$

Now, take $r = 1$. Then,

$$\frac{e_{2(3+7j)}}{3+7j} \equiv 49 + 14j \equiv 14j \bmod 7^2. \tag{4.24}$$

Because $\nu_7(e_{2(3+2j)}) \geq 1$, dividing (4.24) by 7 produces

$$\frac{e_{2(3+7j)}}{7(3+7j)} \equiv 2j \bmod 7. \tag{4.25}$$

Because $\gcd(2,7) = 1$, there is a unique index $j \in \{0, 1, \ldots, 6\}$ such that $2j \equiv 0 \bmod 7$; namely $j = 0$. Then, if $j \neq 0$,

$$\frac{e_{2(3+7j)}}{7(3+7j)} \not\equiv 0 \bmod 7, \tag{4.26}$$

and this implies $\nu_7(e_{2(3+7j)}) \leq 1$. This proves (4.2).

Step 4. Relation (4.21), with $r = 2$ gives

$$\frac{e_{2(3+7^2j)}}{3 + j \cdot 7^2} = \frac{e_6}{3} + \delta_7(3)j \cdot 7^2 \bmod 7^3, \tag{4.27}$$

which reduces to

$$\frac{e_{2(3+j\cdot 7^2)}}{7^2(3 + j \cdot 7^2)} \equiv 1 + 2j \bmod 7. \tag{4.28}$$

Because $\gcd(2,7) = 1$, there is a unique value of j such that $1 + 2j \equiv 0 \bmod 7$. Solving this congruence gives $j \equiv 3 \bmod 7$. This proves (4.3).

This process may be continued indefinitely, because the condition for finding the unique choice of index j, depends on the value $\delta_7(3) = 2 \not\equiv 0 \bmod 7$.

The algorithm presented in the previous example extends to other primes. The statement of Theorem 4.12 requires some nomenclature.

Definition 4.9. Let $p \geq 3$ be prime and $\{x_n\}$ a sequence of positive integers. The sequence of valuations $\nu_p(x_n)$ is said to have a *tree structure* if there is a finite subset $R = \{r_1, r_2, \ldots, r_m\} \subset \mathbb{N}$ and each element $r \in R$ has a tree associated with it, called *tree associated to the vertex a* and denoted by $\mathcal{T}_p(r)$. This tree has a *root*, vertices arranged according to *levels*, and each vertex $v \in \mathcal{T}_p(r)$ is assigned a collection of natural numbers, called the *index set* of the vertex and denoted by $\mathcal{I}_p(v)$. A vertex $v \in \mathcal{T}_p(r)$ is called *terminal* if the p-adic valuation of x_n has the same value for each $n \in \mathcal{I}_n(v)$; otherwise, the vertex is called *nonterminal*.

The construction of the tree proceeds as follows:

1. For $r \in R$, the root of $\mathcal{T}_p(r)$ is r. This single vertex forms the 0th level of $\mathcal{T}_p(r)$. The index set associated with the root is $\mathcal{I}_p(r) = \{n \in \mathbb{N} : \ n \equiv r \bmod 2p\}$.

2. If the root r is a nonterminal vertex, then level 1 has p vertices, formed by the p congruence classes of r modulo $2p^2$. The vertices at this level are labeled $\{v_{1,0}, v_{1,1,}, \ldots, v_{1,p-1}\}$. For $j_1 \in \{0, 1, \ldots, p-1\}$, the index set of v_{1,j_1} is the collection of numbers congruent to $r + 2pj_1$ modulo $2p^2$. Note that such numbers are also, of necessity, congruent to r modulo $2p$.

3. The branch of the tree corresponding to terminal vertices of level 1 stops at this level. Any branch corresponding to a nonterminal vertex, say v_{1,j_1}, is split into p new vertices. The index set of these new vertices is the collection of numbers congruent to $r + 2j_1p + 2j_2p^2$ modulo $2p^3$, where j_2 varies over $\{0, 1, \ldots, p-1\}$. These numbers are also congruent to r modulo $2p$ and to $r + 2j_1p$ modulo $2p^2$. All these new vertices form the second level of the tree.

4. The next levels of the tree are constructed following the rules described in 3., increasing the power of the prime p in the modularity condition.

For every p, $\nu_p(e_{2p}) > 0$. The trees $\mathcal{T}_p(r)$ with $r \neq 2p$ are the *internal branches* in the tree structure for $\{\nu_p(e_n)\}$.

Definition 4.10. Let p be a prime and $\{x_n\}$ a sequence whose valuations $\{\nu_p(x_n)\}$ have a tree structure. This structure is called *simple* if every internal branch $\mathcal{T}_p(r)$ has the following property: *every level has a single nonterminal vertex.*

Example 4.11. For $p = 7$, there are three classes modulo 14 that serve as roots: 6, 8, and 14. The first level of the tree $\mathcal{T}_7(6)$ has six terminal vertices, corresponding to index sets $\{6 + 14n : n \not\equiv 0 \bmod 7\}$. The valuation of these vertices is always 1. The second level also has six terminal vertices, with valuations 2, corresponding to $\{6 + 2 \cdot 7^2 n : n \not\equiv 3 \bmod 7\}$. This pattern continues indefinitely.

Theorem 4.12. *Let p be an odd prime. For $r \in \mathbb{N}$, define*

$$\Delta_p(n, j; r) = \frac{P(n + p^r, j) - P(n, j)}{p^r} \bmod p, \tag{4.29}$$

and

$$\delta_p(n) = \sum_{j=0}^{p} \Delta_p(n, j; r). \tag{4.30}$$

Then $\delta_p(n) \bmod p$ is independent of r. Assume the condition $\delta_p(n) \not\equiv 0 \bmod p$. Then, the sequence of valuations $\{\nu_p(e_{2a})\}$ has a simple tree structure with roots in the set

$$\mathcal{P}(a) = \{a \in \{1, 2, \ldots, p-1\} : \; e_{2a} \equiv 0 \bmod p\}. \tag{4.31}$$

Example 4.13. For $p = 7$, there are two internal branches to consider, with roots at $r = 2 \cdot 3$ and $r = 2 \cdot 4$. Because $\delta_7(3) = 2$ and $\delta_7(4) = 5$ are nonzero modulo 7, the sequence $\{\nu_p(e_{2n})\}$ has a simple tree structure.

Example 4.14. A similar result holds for the sequence $\nu_p(e_{2a+1})$. It can be shown that

$$\frac{1}{p^r}\left(e_{2(n+p^r)+1} - e_{2n+1}\right) = \sum_{j=1}^{p} n(n+j)\Delta_p(n,j) + \sum_{j=1}^{p}(2n+j)P(n, j-1) \bmod p.$$

The right side does not depend on r, so we may name this quantity $\delta'_p(n)$. Then,

$$e_{2(n+j\cdot p^r)+1} \equiv e_{2n+1} + j \cdot \delta'_p(n) \cdot p^r \bmod p.$$

If $\delta'_p(n) \neq 0 \bmod p$, then the sequence of valuations $\{\nu_p(e_{2a+1})\}$ has a simple tree structure. For example, e_{13} is congruent to $0 \bmod 23$ and congruent to $23 \cdot 14 \bmod 23^2$, meaning that $r = 13 = 2 \cdot 6 + 1$ is a root for $p = 23$. It can be shown that $\delta'_{23}(6) = 21$, so that $e_{13+46j} \equiv e_{13} + 23 \cdot 21j \equiv 23(14 + 21j) \bmod 23^2$. Because 21 and 23 are relatively prime, there is exactly one value of j between 0 and 22 for which $e_{13+46j} \equiv 0 \bmod 23^2$. As it turns out, when $j = 7$, we get $e_{335} \equiv 0 \bmod 23^2$. Because $\delta'_p(n)$ is independent of r, the same pattern continues at each level of the tree, where exactly one branch does not terminate.

Recall that for a prime p, the roots of the branches are in the set $\{1, 2, \ldots, 2p-1\}$. The primes $p < 50$ and their corresponding branches are indicated below. The condition $\delta_p(n) \not\equiv 0 \bmod p$ is satisfied in all these cases. Therefore, there is a simple tree structure in each of these cases.

p	Roots of Branches
7	6, 8
11	10, 12
13	8, 12, 14, 18
23	13, 33
29	22, 36
31	5, 28, 29, 30, 32, 33, 34, 57
37	16, 58
41	11, 71
43	26, 60
47	10, 27, 67, 84

Note that if j is the root of a branch, then so is $2p-j$.

Problem. Is it true that if the valuation $\{\nu_p(e_{2n})\}$ has internal branches, then its tree structure must be simple? In other words, if $\nu_p(e_n) > 0$ for some $n \in \{1, 2, \ldots, 2p-1\}$, must it be true that $\delta_p(n) \not\equiv 0 \mod p$?

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