EXPLICIT FORMULAS FOR SUMS INVOLVING THE SQUARES OF THE FIRST n TRIBONACCI NUMBERS

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Abstract. In this paper, we present explicit formulas for various sums involving the squares of the first n Tribonacci numbers.

1. Introduction

The sequence of Tribonacci numbers $\{T_k\}_{k=0}^{\infty}$ is defined by the initial conditions $T_0 = 0$, $T_1 = T_2 = 1$ and the Tribonacci recurrence relation $T_{n+3} = T_n + T_{n+1} + T_{n+2}$ [1]. In 2008, E. Kilic [5] found a closed form expression for the sum $\sum_{k=0}^{n} T_k$ of the first n Tribonacci numbers. In particular, he proved the identity

$$\sum_{k=0}^{n} T_k = \frac{1}{2}\left(T_{n+2} + T_n - 1\right),$$

for all nonnegative integers n, by means of generating matrix calculations. Because of the above mentioned Tribonacci recurrence relation $T_n = T_{n+3} - T_{n+2} - T_{n+1}$, E. Kilic's formula is equivalent to the formula $\sum_{k=0}^{n} T_k = \frac{1}{2}\left(T_{n+3} - T_{n+1} - 1\right)$ for all nonnegative integers n. More identities for sums of Tribonacci numbers, for example for sums of the form $\sum_{k=0}^{n} T_{mk}$, with m a positive integer and n a nonnegative integer, can be found in [1, 5, 3, 9].

We had the idea to write this paper after we solved the Advanced Fibonacci Quarterly Problem H-828 [2] proposed by Kenneth B. Davenport. The problem wanted a closed form solution for the sum $\sum_{k=0}^{n} kT_k^2$, where again $\{T_k\}_{k=0}^{\infty}$ is the Tribonacci sequence and n is a nonnegative integer. We gave the following closed form solution for this problem:

$$\sum_{k=0}^{n} kT_k^2 = \left(\frac{1}{2}n+1\right)T_nT_{n+2} + (n+2)\,T_{n+1}T_{n+2} - \left(\frac{1}{4}n+1\right)T_n^2 - \left(n+\frac{7}{4}\right)T_{n+1}^2 - \left(\frac{1}{4}n+\frac{3}{4}\right)T_{n+2}^2 - \frac{1}{2}T_nT_{n+1} + \frac{1}{2}.$$

Substituting the relation $T_n = T_{n+3} - T_{n+2} - T_{n+1}$ into this equation, we get for all nonnegative integers n, the equivalent form

$$\sum_{k=0}^{n} kT_k^2 = \frac{1}{2}(n+3)\,T_{n+1}T_{n+3} + (n+3)\,T_{n+2}T_{n+3} - \left(\frac{5}{4}n+\frac{9}{4}\right)T_{n+1}^2 - \left(n+\frac{11}{4}\right)T_{n+2}^2 - \left(\frac{1}{4}n+1\right)T_{n+3}^2 - \frac{1}{2}T_{n+1}T_{n+2} + \frac{1}{2}.$$

These two formulas are different from the formulas for the sums $\sum_{k=0}^{n} T_k$ and $\sum_{k=0}^{n} T_{mk}$. They are the two-dimensional analogues of the formulas for $\sum_{k=0}^{n} T_k$ and belong to the same family of summation formulas for Tribonacci numbers.

In this paper, we give the corresponding explicit formula for the sum $\sum_{k=0}^{n} T_k^2$ of the squares of the first n Tribonacci numbers. Hideyuki Ohtsuka proposed the same problem in *The Fibonacci Quarterly* as H-715 [8], which had been solved by various readers with the highlighted solution given by Zbigniew Jakubczyk [4]. Unlike that of Ohtsuka and Jakubczyk, the summation in our formula starts from zero, but both formulas are equivalent. The derivation given in the next section of this paper varies from that of Jakubczyk. To prove the formula for $\sum_{k=0}^{n} T_k^2$, we will use the method of generating functions [11].

In the second half of this paper, we present and prove the closed form expressions for the three sums $\sum_{k=0}^{n} kT_k$, $\sum_{k=0}^{n} k^2T_k$, and $\sum_{k=0}^{n} k^2T_k^2$ by employing the same method.

We have searched the above three formulas in the literature and on the internet, but we could find none of them. Therefore, we believe that these formulas are original.

2. The Explicit Formula for the Sum of the Squares of the First n Tribonacci Numbers

The method of proof used here is the same as in our solution to the Advanced Problem H-828 and as in [10]. In this section, we prove

Theorem 2.1. *(Explicit Formula for the Sum of the Squares of the First n Tribonacci Numbers)*
We have, for all nonnegative integers n, the closed form expression

$$\sum_{k=0}^{n} T_k^2 = \frac{1}{2}T_nT_{n+2} + T_{n+1}T_{n+2} - \frac{1}{4}T_n^2 - T_{n+1}^2 - \frac{1}{4}T_{n+2}^2 + \frac{1}{4}.$$

This is for all nonnegative integers n equivalent, by substituting the relation $T_n = T_{n+3} - T_{n+2} - T_{n+1}$ into the above expression, to

$$\sum_{k=0}^{n} T_k^2 = \frac{1}{2}T_{n+1}T_{n+3} + T_{n+2}T_{n+3} - \frac{5}{4}T_{n+1}^2 - T_{n+2}^2 - \frac{1}{4}T_{n+3}^2 + \frac{1}{4}.$$

Proof. Let the function $s(x)$ be defined by

$$s(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

For the Tribonacci numbers $\{T_n\}_{n=0}^{\infty}$, we have, for nonnegative integers n, the explicit formula [6, 7]

$$T_n = \frac{\alpha_1^n}{4\alpha_1 - \alpha_1^2 - 1} + \frac{\alpha_2^n}{4\alpha_2 - \alpha_2^2 - 1} + \frac{\alpha_3^n}{4\alpha_3 - \alpha_3^2 - 1} \quad ,$$

where α_1, α_2, and α_3 are the three roots of the polynomial

$$q(x) := x^3 - x^2 - x - 1.$$

Using this explicit formula for the Tribonacci numbers from above and the geometric series identity

$$\sum_{k=0}^{\infty} y^k = \frac{1}{1-y} \quad \text{for all complex numbers } y \text{ with } |y| < 1,$$

we can calculate that

$$\sum_{n=0}^{\infty} T_n^2 x^n = \sum_{n=0}^{\infty}\left(\frac{\alpha_1^n}{4\alpha_1-\alpha_1^2-1}+\frac{\alpha_2^n}{4\alpha_2-\alpha_2^2-1}+\frac{\alpha_3^n}{4\alpha_3-\alpha_3^2-1}\right)^2 x^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{\alpha_1^{2n}}{\left(4\alpha_1-\alpha_1^2-1\right)^2}+\frac{\alpha_2^{2n}}{\left(4\alpha_2-\alpha_2^2-1\right)^2}+\frac{\alpha_3^{2n}}{\left(4\alpha_3-\alpha_3^2-1\right)^2}+\frac{2(\alpha_1\alpha_2)^n}{\left(4\alpha_1-\alpha_1^2-1\right)\left(4\alpha_2-\alpha_2^2-1\right)}\right.$$
$$\left.+\frac{2(\alpha_1\alpha_3)^n}{\left(4\alpha_1-\alpha_1^2-1\right)\left(4\alpha_3-\alpha_3^2-1\right)}+\frac{2(\alpha_2\alpha_3)^n}{\left(4\alpha_2-\alpha_2^2-1\right)\left(4\alpha_3-\alpha_3^2-1\right)}\right)x^n$$

$$= \frac{\sum_{n=0}^{\infty}\left(\alpha_1^2 x\right)^n}{\left(4\alpha_1-\alpha_1^2-1\right)^2}+\frac{\sum_{n=0}^{\infty}\left(\alpha_2^2 x\right)^n}{\left(4\alpha_2-\alpha_2^2-1\right)^2}+\frac{\sum_{n=0}^{\infty}\left(\alpha_3^2 x\right)^n}{\left(4\alpha_3-\alpha_3^2-1\right)^2}+\frac{2\sum_{n=0}^{\infty}(\alpha_1\alpha_2 x)^n}{\left(4\alpha_1-\alpha_1^2-1\right)\left(4\alpha_2-\alpha_2^2-1\right)}$$
$$+\frac{2\sum_{n=0}^{\infty}(\alpha_1\alpha_3 x)^n}{\left(4\alpha_1-\alpha_1^2-1\right)\left(4\alpha_3-\alpha_3^2-1\right)}+\frac{2\sum_{n=0}^{\infty}(\alpha_2\alpha_3 x)^n}{\left(4\alpha_2-\alpha_2^2-1\right)\left(4\alpha_3-\alpha_3^2-1\right)}$$

$$= \frac{\frac{1}{\left(4\alpha_1-\alpha_1^2-1\right)^2}}{1-\alpha_1^2 x}+\frac{\frac{1}{\left(4\alpha_2-\alpha_2^2-1\right)^2}}{1-\alpha_2^2 x}+\frac{\frac{1}{\left(4\alpha_3-\alpha_3^2-1\right)^2}}{1-\alpha_3^2 x}+\frac{\frac{2}{\left(4\alpha_1-\alpha_1^2-1\right)\left(4\alpha_2-\alpha_2^2-1\right)}}{1-\alpha_1\alpha_2 x}$$
$$+\frac{\frac{2}{\left(4\alpha_1-\alpha_1^2-1\right)\left(4\alpha_3-\alpha_3^2-1\right)}}{1-\alpha_1\alpha_3 x}+\frac{\frac{2}{\left(4\alpha_2-\alpha_2^2-1\right)\left(4\alpha_3-\alpha_3^2-1\right)}}{1-\alpha_2\alpha_3 x}$$

$$= \frac{p(x)}{\left(1-\alpha_1^2 x\right)\left(1-\alpha_2^2 x\right)\left(1-\alpha_3^2 x\right)(1-\alpha_1\alpha_2 x)(1-\alpha_1\alpha_3 x)(1-\alpha_2\alpha_3 x)}$$

$$= \frac{p(x)}{(1-3x-x^2-x^3)(1+x+x^2-x^3)}$$

$$= \frac{p(x)}{(x^3-x^2-x-1)(x^3+x^2+3x-1)},$$

where $p(x)$ is the unique polynomial of degree less than six, such that the above generating function $\frac{p(x)}{(x^3-x^2-x-1)(x^3+x^2+3x-1)}$ for $\sum_{n=0}^{\infty} T_n^2 x^n$ generates at least the first six terms of the Taylor series $\sum_{n=0}^{\infty} T_n^2 x^n$ correctly. If this is the case, then the function $\frac{p(x)}{(x^3-x^2-x-1)(x^3+x^2+3x-1)}$ agrees with $\sum_{n=0}^{\infty} T_n^2 x^n$ for all nonnegative integers n and generates every Taylor series coefficient of $\sum_{n=0}^{\infty} T_n^2 x^n$ correctly. This polynomial $p(x)$ turns out to be

$$p(x) = -x\left(x^3+x^2+x-1\right).$$

Therefore, we obtain that

$$f_1(x) = \sum_{n=0}^{\infty} T_n^2 x^n = -\frac{x\left(x^3+x^2+x-1\right)}{(x^3-x^2-x-1)(x^3+x^2+3x-1)}.$$

In the same way, we get the following generating function identities

$$f_2(x) = \sum_{n=0}^{\infty} T_{n+1}^2 x^n = -\frac{x^3+x^2+x-1}{(x^3-x^2-x-1)(x^3+x^2+3x-1)},$$

$$f_3(x) = \sum_{n=0}^{\infty} T_{n+2}^2 x^n = -\frac{x^5+x^3-5x^2-2x-1}{(x^3-x^2-x-1)(x^3+x^2+3x-1)},$$

$$f_4(x)=\sum_{n=0}^{\infty}T_nT_{n+1}x^n=\frac{x\left(x^2+1\right)}{\left(x^3-x^2-x-1\right)\left(x^3+x^2+3x-1\right)},$$
$$f_5(x)=\sum_{n=0}^{\infty}T_nT_{n+2}x^n=\frac{2x}{\left(x^3-x^2-x-1\right)\left(x^3+x^2+3x-1\right)},$$
$$f_6(x)=\sum_{n=0}^{\infty}T_{n+1}T_{n+2}x^n=\frac{x^2+1}{\left(x^3-x^2-x-1\right)\left(x^3+x^2+3x-1\right)}.$$

Because of the calculation

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}T_k^2\right)x^n=-\frac{x\left(x^3+x^2+x-1\right)}{(1-x)\left(x^3-x^2-x-1\right)\left(x^3+x^2+3x-1\right)}\\
&=s(x)f_1(x)\\
&=\frac{1}{2}f_5(x)+f_6(x)-\frac{1}{4}f_1(x)-f_2(x)-\frac{1}{4}f_3(x)+\frac{1}{4}s(x)\\
&=\frac{1}{2}\sum_{n=0}^{\infty}T_nT_{n+2}x^n+\sum_{n=0}^{\infty}T_{n+1}T_{n+2}x^n-\frac{1}{4}\sum_{n=0}^{\infty}T_n^2x^n-\sum_{n=0}^{\infty}T_{n+1}^2x^n\\
&\quad-\frac{1}{4}\sum_{n=0}^{\infty}T_{n+2}^2x^n+\frac{1}{4}\sum_{n=0}^{\infty}x^n\\
&=\sum_{n=0}^{\infty}\left[\frac{1}{2}T_nT_{n+2}+T_{n+1}T_{n+2}-\frac{1}{4}T_n^2-T_{n+1}^2-\frac{1}{4}T_{n+2}^2+\frac{1}{4}\right]x^n,
\end{aligned}$$

the formula

$$\sum_{k=0}^{n}T_k^2=\frac{1}{2}T_nT_{n+2}+T_{n+1}T_{n+2}-\frac{1}{4}T_n^2-T_{n+1}^2-\frac{1}{4}T_{n+2}^2+\frac{1}{4}$$

is true by equating coefficients and using the uniqueness of power series expansions. This expression is the first formula in the above theorem. □

By substituting the relation $T_{n+2}=T_{n-1}+T_n+T_{n+1}$ into the first formula in the above theorem, we get the formula

$$\sum_{k=0}^{n}T_k^2=\frac{4T_nT_{n+1}-\left(T_{n+1}-T_{n-1}\right)^2+1}{4}$$

of Hideyuki Ohtsuka and Zbigniew Jakubczyk [8, 4].

3. Closed Form Expressions for $\sum_{k=0}^{n}kT_k$, $\sum_{k=0}^{n}k^2T_k$, and $\sum_{k=0}^{n}k^2T_k^2$

We have the following theorem.

Theorem 3.1. *(Explicit Formulas for the Sums $\sum_{k=0}^{n}kT_k$, $\sum_{k=0}^{n}k^2T_k$, and $\sum_{k=0}^{n}k^2T_k^2$) We have for all nonnegative integers n, the closed form expression*

$$\sum_{k=0}^{n}kT_k=\frac{1}{2}\left(nT_n-T_{n+1}+(n-1)T_{n+2}\right)+1,$$

as well as, by using $T_n = T_{n+3} - T_{n+2} - T_{n+1}$*, that*

$$\sum_{k=0}^{n} kT_k = \frac{1}{2}\left(nT_{n+3} - T_{n+2} - (n+1)T_{n+1}\right) + 1.$$

Also, for all nonnegative integers n*, we have the identity*

$$\sum_{k=0}^{n} k^2T_k = \left(\frac{1}{2}n^2 + 1\right)T_n - \left(n - \frac{3}{2}\right)T_{n+1} + \left(\frac{1}{2}n^2 - n + \frac{3}{2}\right)T_{n+2} - 3,$$

as well as, again because of $T_n = T_{n+3} - T_{n+2} - T_{n+1}$*, the equivalent form*

$$\sum_{k=0}^{n} k^2T_k = \left(\frac{1}{2}n^2 + 1\right)T_{n+3} - \left(n - \frac{1}{2}\right)T_{n+2} - \left(\frac{1}{2}n^2 + n - \frac{1}{2}\right)T_{n+1} - 3.$$

Moreover, for all nonnegative integers n*, we have that*

$$\begin{aligned}\sum_{k=0}^{n} k^2T_k^2 = &\left(\frac{1}{2}n^2 + 2n + \frac{7}{2}\right)T_nT_{n+2} + \left(n^2 + 4n + \frac{13}{2}\right)T_{n+1}T_{n+2} - (n+2)\,T_nT_{n+1}\\ &- \left(\frac{1}{4}n^2 + 2n + 4\right)T_n^2 - \left(n^2 + \frac{7}{2}n + \frac{21}{4}\right)T_{n+1}^2 - \left(\frac{1}{4}n^2 + \frac{3}{2}n + \frac{9}{4}\right)T_{n+2}^2 + 1\end{aligned}$$

and by using the relation $T_n = T_{n+3} - T_{n+2} - T_{n+1}$ *we also have that*

$$\begin{aligned}\sum_{k=0}^{n} k^2T_k^2 = &\left(\frac{1}{2}n^2 + 3n + 6\right)T_{n+1}T_{n+3} + \left(n^2 + 6n + \frac{23}{2}\right)T_{n+2}T_{n+3} - (n+3)\,T_{n+1}T_{n+2}\\ &- \left(\frac{5}{4}n^2 + \frac{9}{2}n + \frac{29}{4}\right)T_{n+1}^2 - \left(n^2 + \frac{11}{2}n + \frac{39}{4}\right)T_{n+2}^2 - \left(\frac{1}{4}n^2 + 2n + 4\right)T_{n+3}^2 + 1.\end{aligned}$$

Proof. In addition to the generating function identities given in the previous section, we have that

$$g_1(x) = \sum_{n=0}^{\infty} T_nx^n = \frac{x}{1 - x - x^2 - x^3},$$

$$g_2(x) = \sum_{n=0}^{\infty} T_{n+1}x^n = \frac{1}{1 - x - x^2 - x^3},$$

$$g_3(x) = \sum_{n=0}^{\infty} T_{n+2}x^n = \frac{1 + x + x^2}{1 - x - x^2 - x^3},$$

$$g_4(x) = \sum_{n=0}^{\infty} nT_nx^n = \frac{x(1+x)\left(1 - x + 2x^2\right)}{\left(1 - x - x^2 - x^3\right)^2},$$

$$g_5(x) = \sum_{n=0}^{\infty} nT_{n+1}x^n = \frac{x\left(1 + 2x + 3x^2\right)}{\left(1 - x - x^2 - x^3\right)^2},$$

$$g_6(x) = \sum_{n=0}^{\infty} nT_{n+2}x^n = \frac{x(1+x)^2\left(2 + x^2\right)}{\left(1 - x - x^2 - x^3\right)^2},$$

$$g_7(x) = \sum_{n=0}^{\infty} n^2T_nx^n = \frac{x\left(1 + x - x^2 + x^3\right)\left(1 + 7x^2 + 4x^3\right)}{\left(1 - x - x^2 - x^3\right)^3},$$

$$g_8(x) = \sum_{n=0}^{\infty} n^2 T_{n+1} x^n = \frac{x\left(1+5x+12x^2+6x^3+11x^4+9x^5\right)}{\left(1-x-x^2-x^3\right)^3},$$
$$g_9(x) = \sum_{n=0}^{\infty} n^2 T_{n+2} x^n = \frac{x\left(1+x\right)\left(2+8x+7x^2+16x^3+4x^4+2x^5+x^6\right)}{\left(1-x-x^2-x^3\right)^3}.$$

Because of the calculation

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} kT_k\right)x^n = \frac{x(1+x)\left(1-x+2x^2\right)}{(1-x)\left(1-x-x^2-x^3\right)^2}\\
&= s(x)g_4(x)\\
&= \frac{1}{2}g_4(x) - \frac{1}{2}g_2(x) + \frac{1}{2}g_6(x) - \frac{1}{2}g_3(x) + s(x)\\
&= \frac{1}{2}\sum_{n=0}^{\infty} nT_n x^n - \frac{1}{2}\sum_{n=0}^{\infty} T_{n+1}x^n + \frac{1}{2}\sum_{n=0}^{\infty} nT_{n+2}x^n - \frac{1}{2}\sum_{n=0}^{\infty} T_{n+2}x^n + \sum_{n=0}^{\infty} x^n\\
&= \sum_{n=0}^{\infty}\left[\frac{1}{2}\left(nT_n - T_{n+1} + (n-1)T_{n+2}\right) + 1\right]x^n,
\end{aligned}$$

the formula

$$\sum_{k=0}^{n} kT_k = \frac{1}{2}\left(nT_n - T_{n+1} + (n-1)T_{n+2}\right) + 1$$

is true by equating coefficients and using the uniqueness of power series expansions. This is the first expression in the above theorem.

Because of the calculation

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} k^2T_k\right)x^n = \frac{x\left(1+x-x^2+x^3\right)\left(1+7x^2+4x^3\right)}{(1-x)\left(1-x-x^2-x^3\right)^3}\\
&= s(x)g_7(x)\\
&= \frac{1}{2}g_7(x) + g_1(x) - g_5(x) + \frac{3}{2}g_2(x) + \frac{1}{2}g_9(x) - g_6(x) + \frac{3}{2}g_3(x) - 3s(x)\\
&= \frac{1}{2}\sum_{n=0}^{\infty} n^2T_n x^n + \sum_{n=0}^{\infty} T_n x^n - \sum_{n=0}^{\infty} nT_{n+1}x^n + \frac{3}{2}\sum_{n=0}^{\infty} T_{n+1}x^n + \frac{1}{2}\sum_{n=0}^{\infty} n^2T_{n+2}x^n\\
&\quad - \sum_{n=0}^{\infty} nT_{n+2}x^n + \frac{3}{2}\sum_{n=0}^{\infty} T_{n+2}x^n - 3\sum_{n=0}^{\infty} x^n\\
&= \sum_{n=0}^{\infty}\left[\left(\frac{1}{2}n^2+1\right)T_n - \left(n-\frac{3}{2}\right)T_{n+1} + \left(\frac{1}{2}n^2 - n + \frac{3}{2}\right)T_{n+2} - 3\right]x^n,
\end{aligned}$$

the formula

$$\sum_{k=0}^{n} k^2T_k = \left(\frac{1}{2}n^2+1\right)T_n - \left(n-\frac{3}{2}\right)T_{n+1} + \left(\frac{1}{2}n^2 - n + \frac{3}{2}\right)T_{n+2} - 3$$

is true by equating coefficients and using the uniqueness of power series expansions. This is the third identity in the above theorem.

Also, we have that

$$
\begin{aligned}
f_7(x) &= \sum_{n=0}^{\infty} nT_n^2 x^n = \frac{x\left(2x^9+3x^8+4x^7+2x^6+8x^5+12x^3+2x^2-2x+1\right)}{\left(x^3-x^2-x-1\right)^2\left(x^3+x^2+3x-1\right)^2},\\
f_8(x) &= \sum_{n=0}^{\infty} nT_{n+1}^2 x^n = \frac{x\left(3x^8+4x^7+6x^6-4x^5-12x^3+14x^2+4x+1\right)}{\left(x^3-x^2-x-1\right)^2\left(x^3+x^2+3x-1\right)^2},\\
f_9(x) &= \sum_{n=0}^{\infty} nT_{n+2}^2 x^n = \frac{x\left(x^{10}+2x^8-8x^7-8x^5+22x^4+24x^3+11x^2+16x+4\right)}{\left(x^3-x^2-x-1\right)^2\left(x^3+x^2+3x-1\right)^2},\\
f_{10}(x) &= \sum_{n=0}^{\infty} nT_nT_{n+1} x^n = -\frac{x\left(x-1\right)\left(3x^7+3x^6+9x^5+9x^4+15x^3+7x^2+x+1\right)}{\left(x^3-x^2-x-1\right)^2\left(x^3+x^2+3x-1\right)^2},\\
f_{11}(x) &= \sum_{n=0}^{\infty} nT_nT_{n+2} x^n = -\frac{2x\left(5x^6+3x^4-12x^3-3x^2-1\right)}{\left(x^3-x^2-x-1\right)^2\left(x^3+x^2+3x-1\right)^2},\\
f_{12}(x) &= \sum_{n=0}^{\infty} nT_{n+1}T_{n+2} x^n = -\frac{2x\left(2x^7+4x^5-3x^4+2x^3-8x^2-4x-1\right)}{\left(x^3-x^2-x-1\right)^2\left(x^3+x^2+3x-1\right)^2}
\end{aligned}
$$

and that

$$
\begin{aligned}
h_1(x) &= \sum_{n=0}^{\infty} n^2T_n^2 x^n\\
&= -\frac{x(x+1)\left(4x^{14}+5x^{13}+7x^{12}+48x^{11}+90x^{10}+37x^9+157x^8-100x^7+164x^6-237x^5+141x^4-44x^3-18x^2+3x-1\right)}{\left(x^3-x^2-x-1\right)^3\left(x^3+x^2+3x-1\right)^3},\\
h_2(x) &= \sum_{n=0}^{\infty} n^2T_{n+1}^2 x^n\\
&= -\frac{x\left(9x^{14}+16x^{13}+27x^{12}+30x^{11}+99x^{10}+22x^9+201x^8-4x^7+151x^6-204x^5-179x^4+22x^3-51x^2-10x-1\right)}{\left(x^3-x^2-x-1\right)^3\left(x^3+x^2+3x-1\right)^3},\\
h_3(x) &= \sum_{n=0}^{\infty} n^2T_{n+2}^2 x^n\\
&= -\frac{x\left(x^{16}+3x^{14}-2x^{13}+19x^{12}+6x^{11}+77x^{10}+108x^9+51x^8+140x^7-187x^6+78x^5-403x^4-290x^3-69x^2-40x-4\right)}{\left(x^3-x^2-x-1\right)^3\left(x^3+x^2+3x-1\right)^3},\\
h_4(x) &= \sum_{n=0}^{\infty} n^2T_nT_{n+1} x^n = \frac{x\left(9x^{14}+27x^{12}+54x^{11}+93x^{10}+14x^9-9x^8-8x^7+31x^6+144x^5-51x^4+50x^3+27x^2+2x+1\right)}{\left(x^3-x^2-x-1\right)^3\left(x^3+x^2+3x-1\right)^3},\\
h_5(x) &= \sum_{n=0}^{\infty} n^2T_nT_{n+2} x^n = \frac{2x\left(25x^{12}+26x^{10}-66x^9+27x^8-16x^7+92x^6+72x^5-61x^4+72x^3+18x^2+2x+1\right)}{\left(x^3-x^2-x-1\right)^3\left(x^3+x^2+3x-1\right)^3},\\
h_6(x) &= \sum_{n=0}^{\infty} n^2T_{n+1}T_{n+2} x^n = \frac{2x\left(8x^{13}+24x^{11}+3x^{10}+48x^9-57x^8-6x^7-52x^6+78x^5+136x^4+30x^3+33x^2+10x+1\right)}{\left(x^3-x^2-x-1\right)^3\left(x^3+x^2+3x-1\right)^3}.
\end{aligned}
$$

Because of the calculation

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} k^2T_k^2\right)x^n \\
&= -\frac{x(x+1)\big(4x^{14}+5x^{13}+7x^{12}+48x^{11}+90x^{10}+37x^9+157x^8-100x^7+164x^6-237x^5+141x^4-44x^3-18x^2+3x-1\big)}{(1-x)(x^3-x^2-x-1)^3(x^3+x^2+3x-1)^3} \\
&= s(x)h_1(x) \\
&= \frac{1}{2}h_5(x)+2f_{11}(x)+\frac{7}{2}f_5(x)+h_6(x)+4f_{12}(x)+\frac{13}{2}f_6(x)-f_{10}(x)-2f_4(x)-\frac{1}{4}h_1(x) \\
&\quad -2f_7(x)-4f_1(x)-h_2(x)-\frac{7}{2}f_8(x)-\frac{21}{4}f_2(x)-\frac{1}{4}h_3(x)-\frac{3}{2}f_9(x)-\frac{9}{4}f_3(x)+s(x) \\
&= \frac{1}{2}\sum_{n=0}^{\infty} n^2T_nT_{n+2}x^n+2\sum_{n=0}^{\infty} nT_nT_{n+2}x^n+\frac{7}{2}\sum_{n=0}^{\infty} T_nT_{n+2}x^n+\sum_{n=0}^{\infty} n^2T_{n+1}T_{n+2}x^n \\
&\quad +4\sum_{n=0}^{\infty} nT_{n+1}T_{n+2}x^n+\frac{13}{2}\sum_{n=0}^{\infty} T_{n+1}T_{n+2}x^n-\sum_{n=0}^{\infty} nT_nT_{n+1}x^n-2\sum_{n=0}^{\infty} T_nT_{n+1}x^n \\
&\quad -\frac{1}{4}\sum_{n=0}^{\infty} n^2T_n^2x^n-2\sum_{n=0}^{\infty} nT_n^2x^n-4\sum_{n=0}^{\infty} T_n^2x^n-\sum_{n=0}^{\infty} n^2T_{n+1}^2x^n-\frac{7}{2}\sum_{n=0}^{\infty} nT_{n+1}^2x^n \\
&\quad -\frac{21}{4}\sum_{n=0}^{\infty} T_{n+1}^2-\frac{1}{4}\sum_{n=0}^{\infty} n^2T_{n+2}^2x^n-\frac{3}{2}\sum_{n=0}^{\infty} nT_{n+2}^2x^n-\frac{9}{4}\sum_{n=0}^{\infty} T_{n+2}^2x^n+\sum_{n=0}^{\infty} x^n \\
&= \sum_{n=0}^{\infty}\Bigg[\left(\frac{1}{2}n^2+2n+\frac{7}{2}\right)T_nT_{n+2}+\left(n^2+4n+\frac{13}{2}\right)T_{n+1}T_{n+2}-(n+2)\,T_nT_{n+1} \\
&\quad -\left(\frac{1}{4}n^2+2n+4\right)T_n^2-\left(n^2+\frac{7}{2}n+\frac{21}{4}\right)T_{n+1}^2-\left(\frac{1}{4}n^2+\frac{3}{2}n+\frac{9}{4}\right)T_{n+2}^2+1\Bigg]x^n,
\end{aligned}$$

the formula

$$\begin{aligned}
\sum_{k=0}^{n} k^2T_k^2 &= \left(\frac{1}{2}n^2+2n+\frac{7}{2}\right)T_nT_{n+2}+\left(n^2+4n+\frac{13}{2}\right)T_{n+1}T_{n+2}-(n+2)\,T_nT_{n+1} \\
&\quad -\left(\frac{1}{4}n^2+2n+4\right)T_n^2-\left(n^2+\frac{7}{2}n+\frac{21}{4}\right)T_{n+1}^2-\left(\frac{1}{4}n^2+\frac{3}{2}n+\frac{9}{4}\right)T_{n+2}^2+1
\end{aligned}$$

is again true by equating coefficients and using the uniqueness of power series expansions. This is the fifth formula in the theorem. □

4. Conclusion

We have presented and proved closed form expressions for the sums $\sum_{k=0}^{n} kT_k$, $\sum_{k=0}^{n} k^2T_k$, and $\sum_{k=0}^{n} k^2T_k^2$. Moreover, the observations in this paper show that there exist closed form expressions for Tribonacci sums like $\sum_{k=0}^{n} kT_k$, $\sum_{k=0}^{n} k^2T_k$, $\sum_{k=0}^{n} T_k^2$, $\sum_{k=0}^{n} kT_k^2$, and $\sum_{k=0}^{n} k^2T_k^2$ using only Tribonacci numbers. All these formulas belong to the same family of Tribonacci summation formulas and have a similar shape.

The same method can be used for other Tribonacci sums. For example,

$$\begin{aligned}\sum_{k=0}^{n} T_k^3 &= \frac{3}{11}T_n^3 - \frac{1}{22}T_{n+1}^3 + \frac{3}{22}T_{n+2}^3 + \frac{7}{11}T_n^2 T_{n+1} + \frac{7}{22}T_n^2 T_{n+2} + \frac{10}{11}T_n T_{n+1}^2 + \frac{3}{22}T_{n+1}^2 T_{n+2}\\ &\quad - \frac{1}{22}T_n T_{n+2}^2 - \frac{5}{22}T_{n+1}T_{n+2}^2 - \frac{10}{11}T_n T_{n+1} T_{n+2} + \frac{3}{22}n\\ &= \frac{3}{22}T_{n-1}^3 + \frac{15}{22}T_n^3 + \frac{4}{11}T_{n-1}^2 T_n + \frac{2}{11}T_{n-1}^2 T_{n+1} + \frac{7}{11}T_{n-1}T_n^2 + \frac{3}{22}T_n^2 T_{n+1} + \frac{1}{22}T_n T_{n+1}^2\\ &\quad + \frac{1}{11}T_{n-1}T_{n+1}^2 - \frac{7}{11}T_{n-1}T_n T_{n+1} + \frac{3}{22}n,\end{aligned}$$

where we have used the recurrence relation $T_{n+2} = T_{n-1} + T_n + T_{n+1}$. This is also equivalent to

$$\sum_{k=0}^{n} T_k^3 = \frac{\left(15T_n^2 + 3T_nT_{n+1} + T_{n+1}^2\right)T_n + 2\left(7T_n^2 - 7T_nT_{n+1} + T_{n+1}^2\right)T_{n-1} + 4(2T_n + T_{n+1})T_{n-1}^2 + 3T_{n-1}^3 + 3n}{22}$$

for all nonnegative integers n. The proof is analogous to the previous one.

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References

[1] E, Choi and J. Jo, *On partial sum of Tribonacci numbers*, International Journal of Mathematics and Mathematical Sciences, Hindawi Publishing Corporation, vol. 2015.

[2] K. Davenport, *Problem H-828*, The Fibonacci Quarterly, **56.3** (2018), 284. https://www.fq.math.ca/Problems/August2018AdvancedREV.pdf.

[3] N. Irmak and M. Alp, *Tribonacci numbers with indices in arithmetic progression and their sums*, Miskolc Mathematical Notes, **14.1** (2013), 125–133.

[4] Z. Jakubczyk, *Solution H-715*, The Fibonacci Quarterly, **51.3** (2013), 285–286. https://www.fq.math.ca/Problems/August2013advanced.pdf.

[5] E. Kilic, *Tribonacci sequences with certain indices and their sums*, Ars Combinatoria, **86** (2008), 13–22.

[6] T. Noe, T. Piezas III, and E. W. Weisstein, *Tribonacci number*, From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/TribonacciNumber.html.

[7] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, http://oeis.org/A000073.

[8] H. Ohtsuka, *Problem H-715*, The Fibonacci Quarterly, **50.1** (2012), 90. https://www.fq.math.ca/Problems/February2012advanced.pdf.

[9] S. Pethe, *Some identities for Tribonacci sequences*, The Fibonacci Quarterly, **26.2** (1988), 144–151.

[10] H. Prodinger and S. J. Selkirk, *Sums of squares of Tetranacci numbers: A generating function approach*, preprint, June 2019. https://arxiv.org/pdf/1906.08336.pdf.

[11] H. S. Wilf, *Generatingfunctionology*, Academic Press, Boston, 1990.

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