ON THE SUM OF DIGITS OF THE ZECKENDORF REPRESENTATIONS OF TWO CONSECUTIVE NUMBERS

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Abstract. Let $\mathbb{F}_0$ and $\mathbb{F}_1$ be sets of natural numbers with even and odd sums of digits of their Zeckendorf representation. Suppose $F_{i,j}(X) = \sharp\{n < X : n \in \mathbb{F}_i, n+1 \in \mathbb{F}_j\}$. We prove asymptotic formulas for $F_{i,j}(X)$.

1. Introduction

Let n be a natural number, and

$$n = \sum_{k=0}^{\infty} n_k 2^k,$$

with $n_k \in \{0, 1\}$, be the binary representation of n. Suppose

$$\mathbb{N}_0 = \{n : n \in \mathbb{N}, \sum_{i=0}^{\infty} n_i \equiv 0 \pmod 2\},$$

and

$$\mathbb{N}_1 = \mathbb{N} \setminus \mathbb{N}_0.$$

The sets $\mathbb{N}_i$ were first studied by Gelfond [2], who proved the uniform distribution of numbers from these sets in arithmetic progressions.

Later, many interesting results about $\mathbb{N}_i$ were proved. For example, Eminyan [1] proved the following result.

Theorem 1.1. *Let $N_{i,j}(X)$ be the number of the natural solutions of the equation $n - m = 1$, $n, m \leq X$, $n \in \mathbb{N}_i$, $m \in \mathbb{N}_j$, $i, j = 0, 1$. Then,*

$$N_{i,j}(X) = \frac{X}{6} + O(\log X),$$

if $i = j$, and

$$N_{i,j}(X) = \frac{X}{3} + O(\log X),$$

if $i \neq j$.

Now, consider the sequence of Fibonacci numbers $\{F_k\}$: $F_0 = 0$, $F_1 = 1$, $F_{k+2} = F_{k+1} + F_k$ and recall that any natural n has Zeckendorf representation [6]

$$n = \sum_{k=2}^{\infty} f_k F_k,$$

where $f_k \in \{0, 1\}$, $f_k f_{k+1} = 0$, and $f_k = 0$ for $k \geq k_0(n)$. An analogue of Gelfond's result for the Zeckendorf representation was proved in [3]. Our goal is to prove an analogue of Eminyan's result.

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2. Main Result

Consider the sets

$$\mathbb{F}_0 = \{n : n \in \mathbb{N}, \sum_{k=0}^{\infty} f_k \equiv 0 \pmod 2\} \text{ and } \mathbb{F}_1 = \mathbb{N} \setminus \mathbb{F}_0.$$

Let $F_{i,j}(X)$ be the number of the natural solutions of the equation $n - m = 1$, $n, m \le X$, $n \in \mathbb{F}_i$, $m \in \mathbb{F}_j$, $i, j = 0, 1$. In other words,

$$F_{i,j}(X) = \#\{n < X : n \in \mathbb{F}_i,\ n+1 \in \mathbb{F}_j\}.$$

Our main result is the following theorem.

Theorem 2.1. *For $i = j$, we have*

$$F_{i,j}(X) = \frac{\sqrt{5}}{10} X + O(\log X). \tag{2.1}$$

For $i \ne j$, we have

$$F_{i,j}(X) = \frac{5 - \sqrt{5}}{10} X + O(\log X). \tag{2.2}$$

Assume

$$\varepsilon(n) = \begin{cases} 1, & n \in \mathbb{F}_0; \\ -1, & n \in \mathbb{F}_1. \end{cases}$$

Then, it is easy to see that

$$F_{i,j}(X) = \sum_{n \le X} \frac{(-1)^i \varepsilon(n) + 1}{2} \frac{(-1)^j \varepsilon(n+1) + 1}{2}. \tag{2.3}$$

Define two sums: $S_1(X) = \sum_{0 \le n < X} \varepsilon(n)$, and $S_2(X) = \sum_{0 \le n < X} \varepsilon(n)\varepsilon(n+1)$.

Lemma 2.2. *For $S_1(X)$, the following estimate holds.*

$$S_1(X) = O(\log X). \tag{2.4}$$

Lemma 2.3. *For $S_2(X)$, the following asymptotic formula holds.*

$$S_2(X) = \frac{2\sqrt{5} - 5}{5} X + O(\log X). \tag{2.5}$$

The proof of (2.1) and (2.2) is immediately obtained by substituting (2.4) and (2.5) in (2.3). So, to prove Theorem 2.1, it is sufficient to prove Lemmas 2.2 and 2.3.

3. Proof of Lemma 2.2

First, consider a natural n with the Zeckendorf representation

$$n = \sum_{i=2}^{k} f_i F_i,$$

where $f_k \ne 0$. Suppose $n' = n - F_k$ and note that

$$\varepsilon(n') = -\varepsilon(n), \tag{3.1}$$

because the Zeckendorf representation of n' can be written as

$$n' = \sum_{i=2}^{k-2} f_i F_i.$$

Further, consider a sum

$$S_1^*(k) = S_1(F_k).$$

Using (3.1), we have

$$S_1^*(k+1) = S_1^*(k) + \sum_{n=F_k}^{F_{k+1}-1} \varepsilon(n) = S_1^*(k) + \sum_{n'=0}^{F_{k-1}-1} \varepsilon(n'+F_k)$$

$$= S_1^*(k) + \sum_{n'=0}^{F_{k-1}-1} (-\varepsilon(n')) = S_1^*(k) - S_1^*(k-1).$$

So, we get

$$S_1^*(k+1) = S_1^*(k) - S_1^*(k-1).$$

The characteristic polynomial $\lambda^2 - \lambda + 1$ of this recurrence relation has roots $\lambda_{1,2} = \frac{1\pm\sqrt{-3}}{2}$. If we note that $|\lambda_{1,2}| = 1$, we obtain the following result.

Lemma 3.1. *There exists a constant C such that*

$$|S_1^*(k)| \le C. \tag{3.2}$$

By more precise calculations, we can prove that the sequence $\{S_1^*(k)\}$ is periodic with the period $\{2, 1, 0, 0, 1, 2\}$, but we do not need this result to prove Lemma 2.2.

Now, suppose that

$$X = F_{k_1} + F_{k_2} + \cdots + F_{k_l},$$

where $k_i \ge k_{i+1} + 2$. Consider the numbers $n^{(t)} = \sum_{i=1}^{t} F_{k_i}$, and represent $S_1(X)$ as

$$S_1(X) = \sum_{n<X} \varepsilon(n) = \sum_{t=1}^{l} \sum_{n=n^{(t-1)}}^{n^{(t)}-1} \varepsilon(n) = \sum_{t=1}^{l} \sum_{n'=0}^{F_{k_t}-1} \varepsilon(n' + n^{(t-1)}).$$

Similarly, as in (3.1), for $n^{(t-1)} \le n < n^{(t)} - 1$, we have

$$\varepsilon(n) = \varepsilon(n - n^{(t-1)})\varepsilon(n^{t-1}) = (-1)^{t-1}\varepsilon(n - n^{(t-1)}).$$

Hence,

$$S_1(X) = \sum_{t=1}^{l} (-1)^{t-1} \sum_{n \le F_{k_t}-1} \varepsilon(n) = \sum_{t=1}^{l} (-1)^{t-1} S_1^*(k_t).$$

Using (3.2), we obtain

$$S_1(X) \le \sum_{t=1}^{l} |S_1^*(k_t)| \le Cl.$$

Further, note that $k_1 \ge l$, and therefore,

$$F_l \le n.$$

Using the Binet formula, we have

$$\frac{\phi^l - (-\phi)^{-l}}{\sqrt{5}} < X,$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden mean. By elementary calculations, we obtain that there exists a constant C_1 such that

$$l < \log_\phi X + C_1.$$

Hence, we have

$$l = O(\log X) \tag{3.3}$$

and Lemma 2.2 is proved.

4. Proof of Lemma 2.3

Similar to the previous section, consider the sum

$$S_2^*(k) = S_2(F_k).$$

We have

$$S_2^*(k+1) = S_2^*(k) + \sum_{n=F_k}^{F_{k+1}-1} \varepsilon(n)\varepsilon(n+1) = S_2^*(k) + \sum_{n'=0}^{F_{k-1}-1} \varepsilon(n'+F_k)\varepsilon(n'+F_k+1).$$

From (3.1), we obtain that $\varepsilon(n'+F_k) = -\varepsilon(n')$ for $n' < F_{k-1}$, and $\varepsilon(n'+F_k+1) = -\varepsilon(n'+1)$ for $n' < F_{k-1}-1$. However, it is easy to see that $\varepsilon(n'+F_k+1) = \varepsilon(n')$ for $n' = F_{k-1}-1$. So, we get $\varepsilon(n'+F_k)\varepsilon(n'+F_k+1) = \varepsilon(n')\varepsilon(n'+1)$ for $n' < F_{k-1}-1$, and $\varepsilon(n'+F_k)\varepsilon(n'+F_k+1) = -\varepsilon(n')\varepsilon(n'+1)$ for $n' = F_{k-1}-1$. Therefore,

$$S_2^*(k+1) = S_2^*(k) + \sum_{n'=0}^{F_{k-1}-1} \varepsilon(n')\varepsilon(n'+1) - 2\varepsilon(F_{k-1}-1)\varepsilon(F_{k-1}).$$

Because $\varepsilon(F_{k-1}) = -1$, we have

$$S_2^*(k+1) = S_2^*(k) + S_2^*(k-1) + 2\varepsilon(F_{k-1}-1).$$

Note that the Zeckendorf representation of $F_{k-1}-1$ has the form

$$F_{k-1} - 1 = F_{k-2} + F_{k-4} + F_{k-6} + \cdots$$

So, we have

$$S_2^*(k+1) = S_2^*(k) + S_2^*(k-1) + 2\chi_4(k), \tag{4.1}$$

where

$$\chi_4(k) = \begin{cases} 1, & k \equiv 2,3 \pmod 4; \\ -1, & k \equiv 0,1 \pmod 4. \end{cases}$$

Applying (4.1) four times, we obtain

$$S_2^*(k+1) = S_2^*(k-1) + S_2^*(k-2) + 2S_2^*(k-3) + S_2^*(k-4). \tag{4.2}$$

The characteristic polynomial of (4.2) has roots $\lambda_{1,2} = \frac{1\pm\sqrt{5}}{2}$, $\lambda_{3,4} = \pm\sqrt{-1}$, and $\lambda_5 = -1$. Initial conditions for (4.2) can be found by direct calculations. So, standard techniques from the theory of the recurrent relations leads to

$$S_2^*(k) = \frac{3-\sqrt{-1}}{5}(-\sqrt{-1})^k + \frac{3+\sqrt{-1}}{5}(\sqrt{-1})^k + \frac{2-\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^k + \frac{2+\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^k. \tag{4.3}$$

Equation (4.3) implies the asymptotic formula

$$S_2^*(k) = \frac{2-\sqrt{5}}{5}\phi^k + O(1).$$

Using the Binet formula, we obtain

$$S_2^*(k) = \frac{2\sqrt{5}-5}{5}F_k + O(1). \tag{4.4}$$

Now, to prove Lemma 2.3, we again assume

$$X = F_{k_1} + F_{k_2} + \cdots + F_{k_l},$$

and $n^{(t)} = \sum_{i=1}^{t} F_{k_i}$. So, we can write $S_2(X)$ as

$$S_2(X) = \sum_{n<X} \varepsilon(n)\varepsilon(n+1) = \sum_{t=1}^{l} \sum_{n=n^{(t-1)}}^{n^{(t)}-1} \varepsilon(n)\varepsilon(n+1) = \sum_{t=1}^{l} \sum_{n'=0}^{F_{k_t}-1} \varepsilon(n'+n^{(t-1)})\varepsilon(n'+n^{(t-1)}+1).$$

By the arguments discussed above, we have

$$S_2(X) = \sum_{t=1}^{l} \sum_{n'=0}^{F_{k_t}-1} \varepsilon(n')\varepsilon(n'+1) + O(l) = \sum_{t=1}^{l} S_2^*(k_t) + O(l).$$

Combining this with (3.3) and (4.4), we obtain the required result.

5. Concluding Remarks

There are two interesting ways to generalize Theorem 2.1.

First, we can replace the equation $n - m = 1$ in the definition of the sets $\mathbb{F}_i$ by $n - km = h$. For the binary representations, in [5], it was proved that for odd $k > 1$, we have $N_{i,j}(X) \sim \frac{1}{4}X$. What we can say about the functions $F_{i,j}(X)$ in this case?

The second interesting problem is to generalize Theorem 2.1 to other recurrent sequences.

Also in [4], it was proved that there are infinitely many prime numbers in each set $\mathbb{N}_i$. What can we say about the prime numbers in the sets $\mathbb{F}_i$?

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