POWERS OF TWO IN GENERALIZED LUCAS SEQUENCES

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Abstract. For an integer $k \geq 2$, let $(L_n^{(k)})_n$ be the k-generalized Lucas sequence that starts with $0, \ldots, 0, 2, 1$ (k terms) and each term afterwards is the sum of the k preceding terms. In this paper, we find all powers of two that appear in k-generalized Lucas sequences; i.e., we study the Diophantine equation $L_n^{(k)} = 2^m$ in positive integers n, k, m with $k \geq 2$.

1. Introduction

Let $k \geq 2$ be an integer. We consider a generalization of Lucas sequence called the k-generalized Lucas sequence $(L_n^{(k)})_{n \geq -(k-2)}$ defined as

$$L_n^{(k)} = L_{n-1}^{(k)} + L_{n-2}^{(k)} + \cdots + L_{n-k}^{(k)} \quad \text{for all } n \geq 2, \tag{1.1}$$

with the initial conditions $L_{-(k-3)}^{(k)} = \cdots = L_{-1}^{(k)} = 0$, $L_0^{(k)} = 2$, and $L_1^{(k)} = 1$. If $k = 2$, we obtain the classical Lucas sequence

$$L_0 = 2, \quad L_1 = 1, \quad \text{and} \quad L_n = L_{n-1} + L_{n-2} \quad \text{for} \quad n \geq 2.$$

$$(L_n)_{n \geq 0} = \{\underline{2}, \underline{1}, 3, \underline{4}, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, \ldots\}.$$

If $k = 3$, then the 3-Lucas sequence is

$$(L_n^{(3)})_{n \geq -1} = \{0, \underline{2}, \underline{1}, 3, 6, 10, 19, 35, \underline{64}, 118, 217, 399, 734, 1350, 2483, 4567, \ldots\}.$$

If $k = 4$, then the 4-Lucas sequence is

$$(L_n^{(4)})_{n \geq -2} = \{0, 0, \underline{2}, \underline{1}, 3, 6, 12, 22, 43, 83, 160, 308, 594, 1145, 2207, 4254, 8200, \ldots\}.$$

Finding perfect powers in a binary recurrence sequence is an interesting problem in number theory. For example, in [4], Bugeaud, Mignotte, and Siksek proved that 1, 2, and 4 are the only powers of 2 that appear in the Lucas sequence. In [2], Bravo and Luca found all powers of two that are k-generalized Fibonacci numbers. In general, there are several finiteness theorems for perfect powers in any nondegenerate binary recurrence sequences. For example, Pethő [9] and Shorey and Stewart [10] proved independently that there are only finitely many perfect powers with an exponent greater than 1 in any nondegenerate binary recurrence sequence, which are, in principle, effectively computable. But, finding the perfect powers is sometimes a challenge.

In this paper, we investigate the problem of finding powers of 2 in the k-generalized Lucas sequences. Namely, we determine all the solutions of the Diophantine equation

$$L_n^{(k)} = 2^m, \tag{1.2}$$

in positive integers n, k, m with $k \geq 2$. Following the argument from [2], we prove the following result.

Theorem 1.1. *All the solutions of the Diophantine equation* (1.2) *in positive integers* n, k, m *with* $k \geq 2$ *are*

$$(n, k, m) \in \{(0, k, 1), (1, k, 0), (3, 2, 2), (7, 3, 6)\}. \tag{1.3}$$

Namely, we have

$$L_0^{(k)} = 2, \quad L_1^{(k)} = 1 = 2^0, \quad L_3^{(2)} = 4 = 2^2, \text{and} \quad L_7^{(3)} = 64 = 2^6.$$

Our proof of Theorem 1.1 is mainly based on linear forms in logarithms of algebraic numbers and a reduction algorithm originally introduced by Baker and Davenport in [1]. Here, we use a version from Dujella and Pethő in [5, Lemma 5(a)].

2. Preliminary Results

2.1. **Linear Forms in Logarithms.** For any nonzero algebraic number γ of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a \prod_{j=1}^{d} \left(X - \gamma^{(j)}\right)$, we denote the usual absolute logarithmic height of γ by

$$h(\gamma) = \frac{1}{d} \left(\log |a| + \sum_{j=1}^{d} \log \max \left(1, |\gamma^{(j)}|\right) \right).$$

With this notation, Matveev proved the following theorem (see [6]).

Theorem 2.1. *Let* $\gamma_1, \ldots, \gamma_s$ *be real algebraic numbers and let* $b_1, \ldots, b_s$ *be nonzero rational integer numbers. Let* D *be the degree of the number field* $\mathbb{Q}(\gamma_1, \ldots, \gamma_s)$ *over* $\mathbb{Q}$ *and let* A_j *be a positive real number satisfying*

$$A_j = \max\{Dh(\gamma), |\log \gamma|, 0.16\} \quad \text{for} \quad j = 1, \ldots, s.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_s|\}.$$

If $\gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1 \neq 0$, *then*

$$|\gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1| \geq \exp(-1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_s).$$

2.2. **Reduction Algorithm.**

Lemma 2.2. *Let* M *be a positive integer,* p/q *be a convergent of the continued fraction of the irrational* γ *such that* $q > 6M$, *and let* A, B, μ *be some real numbers with* $A > 0$ *and* $B > 1$. *Let*

$$\varepsilon = ||\mu q|| - M \cdot ||\gamma q||,$$

where $||\cdot||$ *denotes the distance from the nearest integer. If* $\varepsilon > 0$, *then there is no solution of the inequality*

$$0 < m\gamma - n + \mu < AB^{-k}$$

in positive integers m, n, *and* k *with*

$$m \leq M \quad \text{and} \quad k \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

2.3. **Properties of k-generalized Lucas Sequence.** In this subsection, we recall some facts and properties of these sequences that will be used later.

The characteristic polynomial of the k-generalized Lucas numbers $(L_n^{(k)})_n$,

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1,$$

is irreducible over $\mathbb{Q}[x]$ and has just one root outside the unit circle; the other roots are strictly inside the unit circle (see, for example, [7], [8] and [11]). In this paper, we denote by $\alpha = \alpha(k)$ the single real root larger than 1, which is located between $2(1-2^{-k})$ and 2 (see [11]). We label these roots as $\alpha_1, \ldots, \alpha_k$ with $\alpha = \alpha_1$. To simplify the notation, in general, we omit the dependence on k of α.

We now consider, for an integer $s \geq 2$, the function

$$f_s(x) = \frac{x-1}{2+(s+1)(x-2)}.$$

With this notation, in the following lemma, we recall some properties of the sequence $(L_n^{(k)})_{n\geq -(k-2)}$, which will be used in the proof of Theorem 1.1.

Lemma 2.3. [3, page 144]

(a) *For all $n \geq 1$ and $k \geq 2$, we have*

$$\alpha^{n-1} \leq L_n^{(k)} \leq 2\alpha^n. \tag{2.1}$$

(b) *The following "Binet-like" formula holds for all $n \geq -(k-2)$:*

$$L_n^{(k)} = \sum_{i=1}^{k} (2\alpha_i - 1) f_k(\alpha_i)\alpha_i^{n-1}. \tag{2.2}$$

(c) *For all $n \geq -(k-2)$, we have*

$$|L_n^{(k)} - (2\alpha - 1) f_k(\alpha)\alpha^{n-1}| < \frac{3}{2}. \tag{2.3}$$

(d) *If $2 \leq n \leq k$, then*

$$L_n^{(k)} = 3 \cdot 2^{n-2}. \tag{2.4}$$

Now, we will prove the following lemma, which is a small variation of the upper bound in inequality (2.1) and will be useful to bound m in terms of n.

Lemma 2.4. *For every positive integer $n \geq 2$, we have*

$$L_n^{(k)} \leq 3 \cdot 2^{n-2}. \tag{2.5}$$

Moreover, if $n \geq k+2$, then the above inequality is strict.

Proof. The proof follows from formula (2.4), $L_n^{(k)} = 3 \cdot 2^{n-2} - 2 < 3 \cdot 2^{n-2}$ for $n = k+1$, and induction for $n \geq k+2$ using the recurrence

$$L_n^{(k)} = L_{n-1}^{(k)} + \cdots + L_{n-k}^{(k)} < 3 \cdot 2^{n-3} + \cdots + 3 \cdot 2^{n-k-2} \leq 3(2^{n-3} + \cdots + 1) < 3 \cdot 2^{n-2}.$$

□

3. The Proof of Theorem 1.1

The proof of Theorem 1.1 will be done in three steps.

3.1. **Setup.** Clearly, $L_0^{(k)} = 2$ and $L_1^{(k)} = 1 = 2^0$ for all $k \geq 2$. We call these types of solutions *trivial* solutions. Now, assume that we have a nontrivial solution (n, k, m) of equation (1.2). By inequality (2.1) and Lemma 2.4, we have

$$\alpha^{n-1} \leq L_n^{(k)} = 2^m < 3 \cdot 2^{n-2}.$$

So, we get

$$n \leq m\left(\frac{\log 2}{\log \alpha}\right) + 1 \quad \text{and} \quad m < n. \tag{3.1}$$

In addition to this, by using $\log 2/\log \alpha < 3/2$, it follows immediately from (3.1) that

$$m < n < \frac{3}{2}m + 1. \tag{3.2}$$

Because the Diophantine equation (1.2) was already solved for $k = 2$, we may assume that $k \geq 3$. Because the solutions to equation (1.2) are nontrivial and $L_n^{(k)} = 3 \cdot 2^{n-2}$ for $2 \leq n \leq k$ (see relation (2.4)), in the remainder of the article we suppose that $n \geq k + 1$. So, we get $n \geq 4$ and $m \geq 3$.

Now, we give an inequality for n and m in terms of k.

Lemma 3.1. *If (n, k, m) is a nontrivial solution in integers of (1.2) with $k \geq 2$ and $n \geq k+1$, then the inequalities*

$$m < n < 2.2 \cdot 10^{14} k^4 \log^3 k \tag{3.3}$$

hold.

Proof. Equation (1.2) and inequality (2.3) imply that

$$|2^m - (2\alpha - 1)f_k(\alpha)\alpha^{n-1}| < \frac{3}{2}. \tag{3.4}$$

Dividing both sides of the above inequality by the positive number $(2\alpha - 1)f_k(\alpha)\alpha^{n-1}$ and using $2 + (k + 1)(\alpha - 2) < 2$ with $1/(2\alpha - 1) < 1/2$, we get

$$|2^m \cdot \alpha^{-(n-1)} \cdot ((2\alpha - 1)f_k(\alpha))^{-1} - 1| < \frac{3}{\alpha^{n-1}}. \tag{3.5}$$

To prove (3.3), we use Theorem 2.1. We take $t = 3$ and

$$\gamma_1 = 2, \quad \gamma_2 = \alpha, \quad \gamma_3 = (2\alpha - 1)f_k(\alpha), \quad b_1 = m, \quad b_2 = -(n - 1), \quad b_3 = -1.$$

Let

$$\Lambda = 2^m \cdot \alpha^{-(n-1)} \cdot ((2\alpha - 1)f_k(\alpha))^{-1} - 1. \tag{3.6}$$

We check that $\Lambda \neq 0$. Assuming $\Lambda = 0$, we are led to

$$2^m = \frac{(2\alpha - 1)(\alpha - 1)}{2 + (k + 1)(\alpha - 2)}\alpha^{n-1}.$$

Conjugating the above relation by the automorphism of Galois $\sigma : \alpha \mapsto \alpha_i$ for some $i > 1$ and then taking absolute values, we have

$$8 < 2^m = \left|\frac{(2\alpha_i - 1)(\alpha_i - 1)}{2 + (k + 1)(\alpha_i - 2)}\alpha_i^{n-1}\right| < \frac{6}{k - 1} < 8.$$

Thus, $\Lambda \neq 0$.

We have $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{K} = \mathbb{Q}(\alpha)$, so we can take $D = k$. Because $h(\gamma_1) = \log 2$ and $h(\gamma_2) = (\log \alpha)/k < (\log 2)/k = (0.693147\ldots)/k$, it follows that we can take $A_1 = k \log 2$ and $A_2 = 0.7$. Furthermore, because $h(\gamma_3) \leq 6 \log k$ for all $k \geq 3$ (see [3] page 147), we can take

$A_3 = 6k \log k$. By recalling that $m \le n-1$ from (3.2), we can take $B = n-1$. Thus, applying Theorem 2.1 and taking into account inequality (3.5), we obtain

$$n - 1 < 3.5 \cdot 10^{12} k^4 \log^2 k \log(n-1),$$

where we used $1 + \log k < 2\log k$, $1 + \log(n-1) < 2\log(n-1)$, and $1/\log\alpha < 2$, which hold for $k \ge 3$ and $n \ge 4$.

Thus,

$$\frac{n-1}{\log(n-1)} < 3.5 \cdot 10^{12} k^4 \log^2 k. \tag{3.7}$$

Because the function $x \mapsto x/\log x$ is increasing for all $x > e$, it is easy to check that the inequality

$$\frac{x}{\log x} < A \text{ implies } x < 2A\log A, \text{ whenever } A \ge 3.$$

Thus, the desired inequality follows after taking $A = 3.5 \cdot 10^{12} k^4 \log^2 k$ and using inequality (3.7) and $29 + 4\log k + 2\log\log k < 31\log k$, which holds for all $k \ge 3$. □

3.2. **The Case $3 \le k \le 168$.** In this step, we use Lemma 2.2 several times to reduce the upper bound on n.

To apply Lemma 2.2, we let

$$z = m\log 2 - (n-1)\log\alpha - \log\widehat{\mu}, \tag{3.8}$$

where $\widehat{\mu} = (2\alpha - 1)f_k(\alpha)$. Then (3.6) and (2.1) imply that $z \ne 0$ and

$$|e^z - 1| < \frac{3}{\alpha^{n-1}}. \tag{3.9}$$

If $z > 0$ and after dividing both sides by $\log\alpha$ and using $1/\log\alpha < 2$ for all $k \ge 3$, we obtain

$$0 < m\gamma - n + \mu < AB^{-(n-1)}, \tag{3.10}$$

where

$$\gamma = \frac{\log 2}{\log\alpha}, \quad \mu = 1 - \frac{\log\widehat{\mu}}{\log\alpha}, \quad A = 6, \quad \text{and} \quad B = \alpha,$$

Because $\alpha > 1$ is a unit in $\mathcal{O}_{\mathbb{K}}$, α and 2 are multiplicatively independent, so $\gamma \notin \mathbb{Q}$.

For each $k \in [3, 168]$, we find a good approximation of α and a convergent p_ℓ/q_ℓ of the continued fraction of γ such that $q_\ell > 6M$, where $M = \lfloor 2.2 \cdot 10^{14} k^4 \log^3 k \rfloor$, which is an upper bound on m by Lemma 3.3. After doing this, we use Lemma 2.2 on inequality (3.10). A computer search with Pari-gp revealed that the maximum value of $\lfloor \log(Aq/\varepsilon)/\log B \rfloor$ over all $k \in [3, 168]$ is 172, which according to Lemma 2.2, is an upper bound on $n-1$. Hence, we deduce that the possible solutions (n, k, m) of the equation (1.2) for which $k \in [3, 168]$ and $z > 0$ have $n \le 173$; therefore $m \le 172$, since $m < n$.

Next, we treat the case $z < 0$. It is easy to see that $2/\alpha^{n-1} < 1/2$ holds for all $k \ge 3$ and $n \ge 4$. Thus, from (3.9), we have that $|e^z - 1| < 1/2$ and therefore, $e^{|z|} < 2$. Since $z < 0$, we have

$$0 < |z| \le e^{|z|} - 1 = e^{|z|}|e^z - 1| < \frac{6}{\alpha^{n-1}}.$$

In a similar way, as was done in the case when $z > 0$, we obtain

$$0 < (n-1)\gamma - m + \mu < AB^{-(n-1)}, \tag{3.11}$$

where

$$\gamma = \frac{\log\alpha}{\log 2}, \quad \mu = \frac{\log\widehat{\mu}}{\log 2}, \quad A = 9, \quad B = \alpha.$$

In this case, we also took $M = \lfloor 2.2 \cdot 10^{14} k^4 \log^3 k \rfloor$, which is an upper bound on $n - 1$ by Lemma 3.1, and we applied Lemma 2.2 to inequality (3.11). In this case, with the help of Pari-gp, we found that the maximum value of $\lfloor \log(Aq/\varepsilon)/\log B \rfloor$ is 174. Thus, the possible solutions (n, k, m) of equation (1.2) in the range $k \in [3, 168]$ and $z < 0$ have $n \leq 175$, so $m \leq 174$.

Finally, we used Mathematica to compare $L_n^{(k)}$ and 2^m for the range $4 \leq n \leq 175$ and $3 \leq m \leq 174$, with $m < n < \dfrac{3m}{2} + 1$ and checked that the only solution of equation (1.2) in this range is that given by Theorem 1.1. Therefore, we have dealt with the case $k \in [3, 168]$.

3.3. **The Case $k > 168$.** Now, we assume that $k > 168$. Thus, we have

$$n < 2.2 \cdot 10^{14} k^4 \log^3 k < 2^{k/2}.$$

In [3, page 150], it was proved that

$$(2\alpha - 1) f_k(\alpha) \alpha^{n-1} = 3 \cdot 2^{n-2} + 3 \cdot 2^{n-1} \eta + \frac{\delta}{2} + \eta\delta,$$

where

$$|\eta| < \frac{2k}{2^k} \quad \text{and} \quad |\delta| < \frac{2^{n+2}}{2^{k/2}}.$$

So, from (3.4) and the above equality, we get

$$\begin{array}{rcl} |2^m - 3 \cdot 2^{n-2}| & < & |(2^m - (2\alpha - 1) f_k(\alpha) \alpha^{n-1} + 3 \cdot 2^{n-1} \eta + \dfrac{\delta}{2} + \eta\delta| \\ & < & \dfrac{3}{2} + \dfrac{3k \cdot 2^n}{2^k} + \dfrac{2^{n+1}}{2^{k/2}} + \dfrac{2^{n+3} k}{2^{3k/2}}. \end{array}$$

We factor $3 \cdot 2^{n-2}$ on the right side of the above inequality and since $1/2^{n-1} < 1/2^{k/2}$ (because $n \geq k + 1$), we obtain

$$\frac{4k}{2^k} < \frac{1}{2^{k/2}}, \quad \frac{8}{3 \cdot 2^{k/2}} < \frac{3}{2^{k/2}}, \quad \text{and} \quad \frac{32k}{3 \cdot 2^{3k/2}} < \frac{1}{2^{k/2}}$$

for $k > 169$. Thus, we get

$$|\frac{2^{m-n+2}}{3} - 1| < \frac{6}{2^{k/2}}. \tag{3.12}$$

As $m < n$ (see (3.2)), we have $m - n + 2 \leq 1$. Then, it follows from (3.12) that

$$\frac{1}{3} < 1 - \frac{2^{m-n+2}}{3} < \frac{6}{2^{k/2}}.$$

So, $2^{k/2} < 18$, which is impossible since $k > 168$.

Therefore, we have no solutions (n, k, m) to equation (1.2) with $k > 168$. This completes the proof of Theorem 1.1.

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