MARKOFF EQUATION WITH PELL COMPONENTS

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Abstract. In this paper, we find all triples of Pell numbers $(x, y, z) = (P_i, P_j, P_n)$ satisfying the Markoff equation $x^2 + y^2 + z^2 = 3xyz$.

1. Introduction

The Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz \tag{1}$$

was first considered by A. Markoff [6] in 1879, and is now known as the Markoff equation. A triple (a, b, c) of positive integers with $a \leq b \leq c$ is called a Markoff triple if it satisfies equation (1). The first few such triples are

$$(1, 1, 1),\ (1, 1, 2),\ (1, 2, 5),\ (1, 5, 13),\ (2, 5, 29), (1, 13, 34),\ (1, 34, 89),\ (2, 29, 169),$$

$$(5, 13, 194),\ (1, 89, 233),\ (5, 29, 433),\ (1, 233, 610),\ (2, 169, 985),\ (13, 34, 1325),\ \ldots.$$

A Markoff number (sequence OEIS A002559 in [8]) is a positive integer arising as a component of a Markoff triple. The first few Markoff numbers are

$$\mathbf{1},\ \mathbf{2},\ \mathbf{5},\ 13,\ \mathbf{29},\ 34,\ 89,\ \mathbf{169},\ 194,\ 233,\ 433,\ 610,\ \mathbf{985},\ 1325,\ \ldots.$$

If (a, b, c) is a Markoff triple, one can verify that $(a, c, 3ac - b)$ and $(b, c, 3bc - a)$ are Markoff triples as well. So, any such ordered triple generates additional triples, creating a tree of Markoff triples consisting of all solutions of equation (1). Using this procedure, any Markoff triple can be obtained in a finite number of steps from the basic triple $(1, 1, 1)$. (The tree with $b \leq 100{,}000$ can be found in Zagier [10]).

The Markoff conjecture (first stated by Frobenius [4] in 1913) states that any Markoff number c appears uniquely as the maximal element of a Markoff triple. In other words, if (a_1, b_1, c) and (a_2, b_2, c) are two Markoff triples with $a_i \leq b_i \leq c$, $i = 1, 2$, then $a_1 = a_2$ and $b_1 = b_2$. This conjecture has been proved for many special cases, such as when c is a prime power, or two times an odd prime power, or when one of $3c - 2$ and $3c + 2$ is a prime power (see [1, 2, 7, 11]). For more information and references, see [9].

Although the proof of the conjecture still eludes us, many connections between Markoff numbers and other well-known sequences of numbers have been discovered. For example, in [5], F. Luca and A. Srinivasan determined all triples of Fibonacci numbers $(F_i,\ F_j,\ F_n) = (x,\ y,\ z)$ satisfying the Markoff equation. In this paper, we extend these authors' ideas to the sequence of Pell numbers.

The Pell sequence $(P_n)_{n\geq 0}$ is defined by the binary recurrence

$$P_{n+1} = 2P_n + P_{n-1},$$

with the initial terms $P_0 = 0$ and $P_1 = 1$. The first few members of this sequence are

$$0,\ \mathbf{1},\ \mathbf{2},\ \mathbf{5},\ 12,\ \mathbf{29},\ 70,\ \mathbf{169},\ 408,\ \mathbf{985},\ 2378,\ \mathbf{5741},\ 13860\ \ldots.$$

In addition to $P_2 = 2$, the sequence of Markoff numbers contains all the odd-indexed Pell numbers. It may be verified that the identity

$$2^2 + P_{2m-1}^2 + P_{2m+1}^2 = 3 \cdot 2 \cdot P_{2m-1} P_{2m+1} \tag{2}$$

is valid for all positive integers m. Now, we ask whether there are other solutions $(x, y, z) = (P_i, P_j, P_n)$ to the Markoff equation (1), other than the ones arising from equation (2).

Our result is the following.

Theorem 1.1. *If $(x, y, z) = (P_i, P_j, P_n)$ with $i \leq j \leq n$ is a solution in positive integers to the Markoff equation* (1)*, then either $(x, y, z) \in \{(1, 1, 1), (1, 1, 2), (1, 2, 5)\}$, or it is of the form given by* (2) *with $m \geq 2$.*

We organize this paper as follows. We will recall some properties related to Pell numbers and Markoff triples that will be useful for the proof of Theorem 1.1 in Section 2. The last section is devoted to the proof of this theorem.

2. Auxiliary Results

We begin this section by recalling a few important properties of the Pell sequence.

Let α and β be the roots of the characteristic equation $x^2 - 2x - 1 = 0$ of the Pell numbers, where

$$\alpha = 1 + \sqrt{2} \quad \text{and} \quad \beta = -\frac{1}{\alpha} = 1 - \sqrt{2}.$$

The well-known Binet formula for P_n is given by the roots α and β as

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}} \tag{3}$$

for all $n \geq 0$. Equation (3) also implies that the inequalities

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1} \tag{4}$$

hold for all positive integers n. Also, from the binary recurrence formula we have, for all positive integers j and k,

$$2^k P_j \leq P_{j+k}. \tag{5}$$

One can prove the above inequalities by induction on n or j and k.

We will use the following result proved in [5] (p. 2, F. Luca and A. Srinivasan, 2018) in the proof of Theorem 1.1.

Lemma 2.1. *If $(a, b, c) \neq (1, 1, 1)$ satisfies equation* (1) *with $a \leq b \leq c$, then $3ab < b + c$.*

3. The Proof of Theorem 1.1

Assume that $x = P_i$, $y = P_j$, and $z = P_n$ with $i \leq j \leq n$. We rewrite equation (1) as

$$z - 3xy = -\frac{x^2 + y^2}{z}. \tag{6}$$

Now, by inserting the values of x, y, and z and using (3) in (6), we get that

$$\frac{\alpha^n}{2\sqrt{2}} - \frac{3\alpha^{i+j}}{8} = -\frac{P_i^2 + P_j^2}{P_n} + \frac{\beta^n}{2\sqrt{2}} - \frac{3\left(\alpha^i \beta^j + \alpha^j \beta^i - \beta^{i+j}\right)}{8}. \tag{7}$$

We have

$$\frac{P_i^2 + P_j^2}{P_n} \leq \frac{2P_j^2}{P_n} \leq \frac{2\alpha^{2j}}{\alpha^n} \leq 2\alpha^j.$$

Since $\beta = -\alpha^{-1}$, it follows that

$$\left|\frac{\beta^n}{2\sqrt{2}}\right| \le \frac{\alpha^{-j}}{2\sqrt{2}} \le \frac{\alpha^j}{2\sqrt{2}},$$

and taking the absolute value of the last term on the right side of (7), we get

$$\left|\frac{3\left(\alpha^i\beta^j + \alpha^j\beta^i - \beta^{i+j}\right)}{8}\right| \le \frac{3}{8}\left(2\alpha^j + 1\right) \le \frac{9\alpha^j}{8}.$$

Combining the last three inequalities with equation (7), we get that

$$\left|\frac{\alpha^n}{2\sqrt{2}} - \frac{3\alpha^{i+j}}{8}\right| \le \left(2 + \frac{1}{2\sqrt{2}} + \frac{9}{8}\right)\alpha^j < 3.5\alpha^j. \tag{8}$$

Dividing equation (8) by $\alpha^{i+j}/2\sqrt{2}$, we get

$$\left|\alpha^{n-i-j} - \frac{3}{2\sqrt{2}}\right| \le \frac{7\sqrt{2}}{\alpha^i}. \tag{9}$$

We have

$$\min_{n\in Z}\left|\alpha^n - \frac{3}{2\sqrt{2}}\right| = \left|1 - \frac{3}{2\sqrt{2}}\right| > 0.06.$$

From (9), we get that $0.06 < 7\sqrt{2}/\alpha^i$, giving us $i \le 6$. Below, we record what we have proved in the following lemma.

Lemma 3.1. *If $(x, y, z) = (P_i, P_j, P_n)$ satisfies equation (1) with $i \le j \le n$, then $i \in \{0, 1, 2, 3, 4, 5, 6\}$. Out of these, only $P_1 = 1$, $P_2 = 2$, $P_3 = 5$, and $P_5 = 29$ are Markoff numbers.*

Note that $j \ge i \ge 1$. Let $n = j + k$. If $k = 0$, then we have $P_n = P_j$. Because all three components of a Markoff triple are mutually coprime ([3, page 28]), we must have $P_n = P_j = 1$, which forces $P_i = 1$ and we have the triple $(1, 1, 1)$. So we may assume that $k \ge 1$. If $j = 1$, then $i = 1$ and we have the triple $(1, 1, 2)$. If $j = 2$, that is $P_j = 2$, again by the coprimality condition, we must have $P_i = 1$, giving the triple $(1, 2, 5)$. Henceforth, we assume that $j \ge 3$ and $k \ge 1$.

Assume that $P_i = 1$. We have

$$1 + P_j^2 + P_n^2 = 3P_jP_n,$$

which gives us

$$1 + P_j^2 = P_n(3P_j - P_n),$$

and hence, $P_n < 3P_j$. Combining this with inequality (5), we get

$$2^kP_j \le P_n < 3P_j.$$

So $k = 1$, that is, $n = j + 1$. In this case, the Markoff equation (1) becomes

$$1 + P_j^2 = P_{j+1}(3P_j - P_{j+1}) = P_{j+1}(P_j - P_{j-1}). \tag{10}$$

By the recursive formula for P_j, we have $P_j/2 > P_{j-1}$ (as $j \ge 3$), which also gives us $P_j - P_{j-1} > P_j/2$. Similarly, we have $P_{j+1} > 2P_j + 1$. Working on the right side of equation (10) with the last two inequalities,

$$1 + P_j^2 > (2P_j + 1)P_j/2 = P_j^2 + P_j/2 > P_j^2 + 1,$$

which is not true. Hence, $P_i = 1$ is not possible.

Next, we assume that $P_i = 5$. Then, we have

$$25 + P_j^2 + P_n^2 = 15P_jP_n,$$

or

$$25 + P_j^2 = P_n(15P_j - P_n),$$

giving us $P_n < 15P_j$. By Lemma 2.1, we get $15P_j < P_j + P_n$, or $14P_j < P_n$. Hence,

$$14P_j < P_n < 15P_j.$$

By comparing the last inequalities with (5), we have $2^kP_j < 15P_j$ as $n = j+k$. So, $k \leq 3$, that is $n \leq j+3$. As P_{j+1} and P_{j+2} are both less than $14P_j$, we ignore the possibilities $k = 1, 2$. Now, suppose $k = 3$, so $n = j+3$. By (1), we have

$$25 + P_j^2 = P_{j+3}(15P_j - P_{j+3}) = (5P_{j+1} + 2P_j)(3P_j - 5P_{j-1}). \tag{11}$$

As $P_{j+1} > 2P_j + 1$ ($j \geq 3$), the first factor in the right side of (11) satisfies

$$5P_{j+1} + 2P_j > 12P_j + 5. \tag{12}$$

Similarly, using $P_j > 2P_{j-1}$, we have $(5/2)P_j > 5P_{j-1}$. The second factor in the right side of equation (11) gives us

$$3P_j - 5P_{j-1} > 3P_j - (5/2)P_j = P_j/2. \tag{13}$$

Now, combining (11), (12), and (13), we get

$$25 + P_j^2 > (12P_j + 5)P_j/2 = 6P_j^2 + 5P_j/2,$$

or $25 > 5P_j^2 + 5P_j/2$, which is not true for $j \geq 3$. Therefore, $P_i = 5$ is not possible.

In the case of $P_i = 29$, similar to the above, we get

$$86P_j < P_n < 87P_j.$$

However, the last inequality does not contain a Pell number as $P_{j+5} < 86P_j$, and $P_{j+6} > 87P_j$. Therefore, we are left with the case of $P_i = 2$, leading us to the following lemma.

Lemma 3.2. *If $(x, y, z) = (P_i, P_j, P_n)$ satisfies equation (1) with $i \leq j \leq n$, then either $(x, y, z) \in \{(1, 1, 1), (1, 1, 2), (1, 2, 5)\}$, or $P_i = 2$.*

In the case of $P_i = 2$, we have

$$4 + P_j^2 + P_n^2 = 6P_jP_n,$$

giving us $P_n < 6P_j$. In addition, by Lemma 2.1, we have $6P_j < P_j + P_n$. Hence, we obtain

$$5P_j < P_n < 6P_j. \tag{14}$$

Recall (as explained in the paragraph below Lemma 3.1), that we may assume that $j \geq 3$ and $k \geq 1$. From the inequalities (5) and (14), we also obtain $k \leq 2$, that is, $n - j \leq 2$. So, we have two possible cases: $n = j+1$ or $n = j+2$. But, if $n = j+1$, then $P_{j+1} = 2P_j + P_{j-1} < 5P_j$. This means, P_{j+1} does not lie in the above interval $(5P_j, 6P_j)$. Hence, we must have $n = j+2$. In this case, we have

$$4 + P_{n-2}^2 + P_n^2 = 6P_{n-2}P_n.$$

To complete the proof of Theorem 1.1, it remains to show that n is odd. This follows because if n were even, then P_n is even, which contradicts that the components of any Markoff triple are mutually coprime ([3, page 28]).

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References

[1] A. Baragar, *On the unicity conjecture for Markoff numbers*, Canad. Math. Bull., **39.1**, (1996), 3–9.
[2] J. O. Button, *The uniqueness of the prime Markoff numbers*, J. London Math. Soc., **58** (1998), 9–17.
[3] J. W. S. Cassels, *An Introduction to Diophantine Appoximation*, Cambridge University Press, 1957.
[4] G. Frobenius, *Über die Markoffschen Zahlen*, Akad. Wiss. Sitzungaber, (1913), 458–493.
[5] F. Luca and A. Srinivasan, *Markov equation with Fibonacci components*, The Fibonacci Quarterly, **56.2** (2018), 126–129.
[6] A. A. Markoff, *Sur les formes binaires indéfinies*, Math. Ann., **15** (1879), 381–409.
[7] P. Schmutz, *Systoles of arithmetic surfaces and the Markoff spectrum*, **305** (1996), 191–203.
[8] N. J. A. Sloane, *The Online Encyclopedia of Integer Sequences*, published electronically at https://oeis.org.
[9] A. Srinivasan, *Markoff numbers and ambiguous classes*, J. de Théorie des Nombres de Bordeaux, **21** (2009), 757–770.
[10] D. Zagier, *On the number of Markoff numbers below a given bound*, Math. Comp., **39.160** (1982), 709–723.
[11] Y. Zhang, *Congruence and uniqueness of certain Markoff numbers*, Acta Arith., **128.3** (2007), 295–301.

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