GENERALIZED FIBONACCI SEQUENCES IN PYTHAGOREAN TRIPLE PRESERVING MATRICES

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Abstract. In this paper, we explore new connections between Pythagorean triples and generalized Fibonacci sequences. We show that Fibonacci and generalized Fibonacci sequences appear in powers of known matrices that give rise to trees of primitive Pythagorean triples.

1. Introduction and Preliminaries

There are many connections between the Fibonacci numbers and Pythagorean triples; e.g., see [1, 2, 3, 11, 12]. In this paper, we present novel connections between Pythagorean triples and Fibonacci and generalized Fibonacci sequences. First, we discuss how the Fibonacci numbers appear in powers of known matrices that give rise to a tree containing all primitive Pythagorean triples and transform any Pythagorean triple into another. Next, we discuss how the Jacobsthal numbers appear in powers of matrices from a different tree containing all primitive Pythagorean triples. Finally, we present a collection of matrices that transform a Pythagorean triple generated by consecutive terms of a generalized Fibonacci sequence into another.

We need a few definitions, which are given below with some illustrating examples.

Definition 1.1. *A Pythagorean triple (PT) is an ordered triple of positive integers,* (a, b, c)*, such that* $a^2 + b^2 = c^2$*. The PT is called primitive provided* $\gcd(a, b, c) = 1$.

It is well-known that any primitive PT is of the form $(m^2 - n^2, 2mn, m^2 + n^2)$ for positive integers m and n with $m > n$, $\gcd(m, n) = 1$, and $m - n$ odd.

There are several definitions of "generalized Fibonacci sequence" in the literature [4, 6], so we need to specify our meaning here.

Definition 1.2. *A generalized Fibonacci sequence* $\{w_n\}$ *is given by* $w_0 = c$*,* $w_1 = d$*, and for* $n \geq 2$*,* $w_n = aw_{n-1} + bw_{n-2}$*, where* $a, b, c, d \in \mathbb{Z}$.

Two well-known examples of generalized Fibonacci sequences are the Pell sequence, given by

$$P_0 = 0,\ P_1 = 1,\ \text{and } P_n = 2P_{n-1} + P_{n-2}, n \geq 2,$$

and the Jacobsthal sequence, defined as

$$J_0 = 0,\ J_1 = 1,\ \text{and } J_n = J_{n-1} + 2J_{n-2}, n \geq 2.$$

Definition 1.3. *A Pythagorean triple preserving matrix (PTPM) is a* 3×3 *matrix that transforms any PT into another PT.*

Palmer and colleagues [9, 10] explored PTPMs at length and provided the general form of a PTPM. As an example, consider the following:

$$\begin{bmatrix} 9 & 8 & 12 \\ 8 & 9 & 12 \\ 12 & 12 & 17 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 119 \\ 120 \\ 169 \end{bmatrix}.$$

The given matrix is a PTPM, and not only transforms the PT in the example to another PT, but does so for any PT given as a column vector.

2. Matrices and Trees of PTs

Hall [2] argued that all primitive PTs can be generated using the matrices U, A, and D as given below.

$$U = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}, \; A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}, \; D = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix}.$$

Hall organized primitive PTs into a tree that has a root of $\begin{bmatrix} 3 & 4 & 5 \end{bmatrix}^T$, with subsequent "Up ($U$)," "Across ($A$)," and "Down ($D$)" branches generated by matrix multiplication. One may easily check that U, A, and D are PTPMs, and furthermore, that they transform any given primitive PT to a primitive PT. For example,

$$U \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \\ 13 \end{bmatrix}, \; A \begin{bmatrix} 5 \\ 12 \\ 13 \end{bmatrix} = \begin{bmatrix} 55 \\ 48 \\ 73 \end{bmatrix}, \text{ and } D \begin{bmatrix} 21 \\ 20 \\ 29 \end{bmatrix} = \begin{bmatrix} 77 \\ 36 \\ 85 \end{bmatrix}.$$

The matrices U, A, and D transform any given PT into another, and therefore, can be composed together as many times as desired to produce more PTPMs. Here are a few illustrating examples:

$$UA \begin{bmatrix} 21 \\ 20 \\ 29 \end{bmatrix} = \begin{bmatrix} 217 \\ 456 \\ 505 \end{bmatrix}, \; UAD \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 115 \\ 252 \\ 277 \end{bmatrix}, \text{ and } D^3 \begin{bmatrix} 15 \\ 8 \\ 17 \end{bmatrix} = \begin{bmatrix} 99 \\ 20 \\ 101 \end{bmatrix}.$$

Price [11] similarly generated a tree containing all primitive PTs using three matrices:

$$M_1 = \begin{bmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{bmatrix}, \; M_2 = \begin{bmatrix} 2 & 1 & 1 \\ 2 & -2 & 2 \\ 2 & -1 & 3 \end{bmatrix}, \; M_3 = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix}.$$

3. Sequences Appearing in Hall and Price's Matrices

We use F_n to denote the Fibonacci sequence, and P_n to denote the Pell sequence. The powers of U, A, and D have interesting forms, the powers of A in particular.

Theorem 3.1. *For all $n \in \mathbb{N}$,*

(1) $U^n = \begin{bmatrix} 1 & -2n & 2n \\ 2n & 1-2n^2 & 2n^2 \\ 2n & -2n^2 & 2n^2+1 \end{bmatrix},$

$$(2)\ \ A^n = \begin{bmatrix} \frac{P_{2n}+P_{2n-1}+(-1)^n}{2} & \frac{P_{2n}+P_{2n-1}+(-1)^{n-1}}{2} & P_{2n} \\ \frac{P_{2n}+P_{2n-1}+(-1)^{n-1}}{2} & \frac{P_{2n}+P_{2n-1}+(-1)^n}{2} & P_{2n} \\ P_{2n} & P_{2n} & P_{2n}+P_{2n-1} \end{bmatrix},$$

$$(3)\ \ D^n = \begin{bmatrix} 1-2n^2 & 2n & 2n^2 \\ -2n & 1 & 2n \\ -2n^2 & 2n & 2n^2+1 \end{bmatrix}.$$

This theorem is easy to prove by induction, so we omit the proof here. Here is the first example of a generalized Fibonacci sequence appearing in the powers of Hall's matrix A. If we examine products of Hall's matrices, Fibonacci numbers and other generalized Fibonacci numbers appear. In particular, the sequence $\{a_n\}$ that appears in parts (5) and (6) of the following theorem is a generalized Fibonacci sequence with many notable properties [8, A001353].

Theorem 3.2. *For all $n \in \mathbb{N}$,*

$$(1)\ \ (UA)^n = \begin{bmatrix} \frac{1}{2}F_{3n}^2+(-1)^n & F_{3n}^2 & \frac{1}{2}F_{6n} \\ F_{3n}^2 & 2F_{3n}^2+(-1)^n & F_{6n} \\ \frac{1}{2}F_{6n} & F_{6n} & \frac{5}{2}F_{3n}^2+(-1)^n \end{bmatrix}.$$

$$(2)\ \ (AU)^n = \begin{bmatrix} \frac{5}{2}F_{3n}^2+(-1)^n & -F_{6n} & \frac{3}{2}F_{6n} \\ F_{6n} & (-1)^n-2F_{3n}^2 & 3F_{3n}^2 \\ \frac{3}{2}F_{6n} & -3F_{3n}^2 & \frac{9}{2}F_{3n}^2+(-1)^n \end{bmatrix}.$$

$$(3)\ \ (DA)^n = \begin{bmatrix} 2F_{3n}^2+(-1)^n & F_{3n}^2 & F_{6n} \\ F_{3n}^2 & \frac{1}{2}F_{3n}^2+(-1)^n & \frac{1}{2}F_{6n} \\ F_{6n} & \frac{1}{2}F_{6n} & \frac{5}{2}F_{3n}^2+(-1)^n \end{bmatrix}.$$

$$(4)\ \ (AD)^n = \begin{bmatrix} (-1)^n-2F_{3n}^2 & F_{6n} & 3F_{3n}^2 \\ -F_{6n} & \frac{5}{2}F_{3n}^2+(-1)^n & \frac{3}{2}F_{6n} \\ -3F_{3n}^2 & \frac{3}{2}F_{6n} & \frac{9}{2}F_{3n}^2+(-1)^n \end{bmatrix}.$$

$$(5)\ \ (DU)^n = \begin{bmatrix} 6a_n^2+1 & -a_{2n} & 2a_{2n} \\ a_{2n} & 1-2a_n^2 & 4a_n^2 \\ 2a_{2n} & -4a_n^2 & 8a_n^2+1 \end{bmatrix}.$$

$$(6)\ \ (UD)^n = \begin{bmatrix} 1-2a_n^2 & a_{2n} & 4a_n^2 \\ -a_{2n} & 6a_n^2+1 & 2a_{2n} \\ -4a_n^2 & 2a_{2n} & 8a_n^2+1 \end{bmatrix}.$$

where the sequence $\{a_n\}$ is defined as

$$a_0 = 0,\ a_1 = 1,\ a_n = 4a_{n-1} - a_{n-2} \text{ for } n \geq 2.$$

Lemma 3.3. *Let $n \in \mathbb{N}$. Then,*

$$F_{3n+3}^2 = 9F_{3n}^2 + 4F_{6n} + 4(-1)^n.$$

Proof. We make use of the identities $F_{k+1}^2 = 4F_kF_{k-1} + F_{k-2}^2$ [6, p. 59, identity I36] and $F_{6k} = F_{3k}(F_{3k+1} + F_{3k-1})$ [6, p. 59, identity I26], as well as Cassini's identity. Using these identities and the definition of F_n, we obtain the stated identity. □

Lemma 3.4. *Let $n \in \mathbb{N}$. Then,*

$$F_{6n+6} = 20F_{3n}^2 + 9F_{6n} + 8(-1)^n.$$

Proof. We use Lemma 3.3, the identity $F_{6n} = F_{3n}(2F_{3n+1} - F_{3n})$ [6, identity I26] with simple substitution of F_{3n-1} from the definition of the Fibonacci numbers), and we expand F_{3n+3}^2 using the definition of F_n to obtain the stated identity. □

Lemma 3.5. *Let $n \in \mathbb{N}$. Then,*

$$a_{n+1}^2 = 2a_{2n} + 7a_n^2 + 1.$$

Proof. We make use of several identities that follow from the identity of Melham and Shannon [7] for this $\{a_n\}$ sequence:

$$a_na_{n+r+s} - a_{n+r}a_{n+s} = -a_ra_s.$$

In particular, using the identities $a_{2n+1} = a_{n+1}^2 - a_n^2$, $a_{2n} = a_{n+1}a_n - a_na_{n-1}$, $a_{n-1}a_{n+1} = a_n^2 - 1$, as well as the definition of a_n, we obtain the stated identity. □

Lemma 3.6. *Let $n \in \mathbb{N}$. Then,*

$$a_{2n+2} = 24a_n^2 + 7a_{2n} + 4.$$

Proof. Using the identities $a_{2n} = a_{n+1}a_n - a_na_{n-1}$ and $a_{n-1}a_{n+1} = a_n^2 - 1$, as well as the definition of a_n, we obtain the stated identity. □

Proof of Theorem 3.2. We prove parts (1) and (5) of the theorem; the proofs of the other parts have a similar structure. For part (1), we proceed by induction on n. When $n = 1$, the statement is clearly true. Now, suppose it is true for some $n \in \mathbb{N}$. Then,

$$(UA)^{n+1} = (UA)(UA)^n = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 7 & 8 \\ 4 & 8 & 9 \end{bmatrix} \begin{bmatrix} \frac{1}{2}F_{3n}^2 + (-1)^n & F_{3n}^2 & \frac{1}{2}F_{6n} \\ F_{3n}^2 & 2F_{3n}^2 + (-1)^n & F_{6n} \\ \frac{1}{2}F_{6n} & F_{6n} & \frac{5}{2}F_{3n}^2 + (-1)^n \end{bmatrix}.$$

The first column of this product simplifies to

$$\begin{bmatrix} \frac{9}{2}F_{3n}^2 + 2F_{6n} + (-1)^n \\ 9F_{3n}^2 + 4F_{6n} + 4(-1)^n \\ 10F_{3n}^2 + \frac{9}{2}F_{6n} + 4(-1)^n \end{bmatrix}.$$

Now,

$$\frac{9}{2}F_{3n}^2 + 2F_{6n} + (-1)^n = \frac{1}{2}\left(9F_{3n}^2 + 4F_{6n} + 4(-1)^n\right) + (-1)^{n+1} = \frac{1}{2}F_{3(n+1)}^2 + (-1)^{n+1},$$

where the last equality is from Lemma 3.3. Also, from Lemma 3.3, we have that $9F_{3n}^2+4F_{6n}+4(-1)^n = F_{3(n+1)}^2$. Finally, for the first column, by Lemma 3.4, we have

$$10F_{3n}^2 + \frac{9}{2}F_{6n} + 4(-1)^n = \frac{1}{2}(20F_{3n}^2 + 9F_{6n} + 8(-1)^n) = \frac{1}{2}F_{6n+6}.$$

The second column of this product simplifies to

$$\begin{bmatrix} 9F_{3n}^2 + 4F_{6n} + 4(-1)^n \\ 18F_{3n}^2 + 8F_{6n} + 7(-1)^n \\ 20F_{3n}^2 + 9F_{6n} + 8(-1)^n \end{bmatrix}.$$

Again, by Lemma 3.1, $9F_{3n}^2 + 4F_{6n} + 4(-1)^n = F_{3(n+1)}^2$. Also,

$$18F_{3n}^2 + 8F_{6n} + 7(-1)^n = 2F_{3(n+1)}^2 + (-1)^{n+1}.$$

Finally for the second column, by Lemma 3.4,

$$20F_{3n}^2 + 9F_{6n} + 8(-1)^n = F_{6(n+1)}.$$

The third column of the product simplifies to

$$\begin{bmatrix} 10F_{3n}^2 + \frac{9}{2}F_{6n} + 4(-1)^n \\ 20F_{3n}^2 + 9F_{6n} + 8(-1)^n \\ \frac{45}{2}F_{3n}^2 + 10F_{6n} + 9(-1)^n \end{bmatrix}.$$

We have already established the result for the first two entries of this column. Using Lemma 3.3, we have

$$\frac{45}{2}F_{3n}^2 + 10F_{6n} + 9(-1)^n = \frac{5}{2}(9F_{3n}^2 + 4F_{6n} + 4(-1)^n) + (-1)^{n+1}$$

$$= \frac{5}{2}F_{3(n+1)}^2 + (-1)^{n+1}.$$

For part (5), we also proceed by induction on n. When $n = 1$, the statement is clearly true. Now, suppose it is true for some $n \in \mathbb{N}$. Then,

$$(DU)^{n+1} = (DU)(DU)^n = \begin{bmatrix} 7 & -4 & 8 \\ 4 & -1 & 4 \\ 8 & -4 & 9 \end{bmatrix} \begin{bmatrix} 6a_n^2 + 1 & -a_{2n} & 2a_{2n} \\ a_{2n} & 1 - 2a_n^2 & 4a_n^2 \\ 2a_{2n} & -4_n^2 & 8a_n^2 + 1 \end{bmatrix}.$$

The first column of this product simplifies to

$$\begin{bmatrix} 42a_n^2 + 12a_{2n} + 7 \\ 24a_n^2 + 7a_{2n} + 4 \\ 48a_n^2 + 14a_{2n} + 8 \end{bmatrix} = \begin{bmatrix} 6a_{n+1}^2 + 1 \\ a_{2n+2} \\ 2a_{2n+2} \end{bmatrix},$$

where the first entry is a result of Lemma 3.5, and the other two entries are a result of Lemma 3.6. The second column of this product simplifies to

$$\begin{bmatrix} -24a_n^2 - 7a_{2n} - 7 \\ -14a_n^2 - 4a_{2n} - 1 \\ -28a_n^2 - 8a_{2n} - 4 \end{bmatrix} = \begin{bmatrix} -a_{2n+2} \\ 1 - a_{n+1}^2 \\ -4a_{n+1}^2 \end{bmatrix},$$

where the first entry is a result of Lemma 3.6, and the other two entries are a result of Lemma 3.5. The third column of this product simplifies to

$$\begin{bmatrix} 48a_n^2 + 14a_{2n} + 8 \\ 28a_n^2 + 8a_{2n} + 4 \\ 56a_n^2 + 16a_{2n} + 9 \end{bmatrix} = \begin{bmatrix} 2a_{2n+2} \\ 4a_{n+1}^2 \\ 8a_{n+1}^2 + 1 \end{bmatrix},$$

where the first entry is a result of Lemma 3.6, and the other two entries are a result of Lemma 3.5.

□

The powers of Price's matrices also have interesting forms. In particular, Jacobsthal numbers appear in the powers of M_2.

Theorem 3.7. *For all $n \in \mathbb{N}$,*

(1) $$M_1^n = \begin{bmatrix} 2^n & 2^n - 1 & 1 - 2^n \\ 2^n(1-2^n) & 2^n & 2^n(2^n-1) \\ 2^n(1-2^n) & 2^n - 1 & 2^n(2^n-1)+1 \end{bmatrix}.$$

(2) $$M_2^n = \begin{bmatrix} 2^n J_n + 4J_{n-1}^2 & (-1)^{n+1} J_n & 2^n J_n + 4J_{n-1}^2 - 1 \\ 2^n J_n & (-2)^n & 2^n J_n \\ 2^n J_n + 4J_n J_{n-1} & (-1)^n J_n & 2^n J_n + 4J_n J_{n-1} + 1 \end{bmatrix}.$$

(3) $$M_3^n = \begin{bmatrix} 2^n & 1 - 2^n & 2^n - 1 \\ 2^n(2^n-1) & 2^n & 2^n(2^n-1) \\ 2^n(2^n-1) & 2^n - 1 & 2^n(2^n-1)+1 \end{bmatrix}.$$

We need the following identities involving Jacobsthal numbers. (For a discussion of Jacobsthal numbers and their properties, see [5].)

Lemma 3.8. *For $n \geq 0$, $J_{n+2} - J_n = 2^n$ and $J_n + J_{n+1} = 2^n$.*

Lemma 3.9. *[Simson's Lemma] For* $n \geq 1$, $J_{n-1}J_{n+1} - J_n^2 = (-1)^n 2^{n-1}$.

Lemma 3.10. *For* $n \geq 0$, $J_n + (-1)^n = 2J_{n-1}$.

These lemmas can be proven using the closed form of the Jacobsthal numbers.

Proof of Theorem 3.7. The proof is by induction on n. We omit the proofs of parts (1) and (3) because they follow from matrix multiplication and straightforward simplification. Consider part (2) of the theorem. The statement is clearly true when $n = 1$. Now, suppose it is true for some $n \in \mathbb{N}$.

Then,

$$M_2^{n+1} = M_2 M_2^n = \begin{bmatrix} 2 & 1 & 1 \\ 2 & -2 & 2 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2^n J_n + 4J_{n-1}^2 & (-1)^{n+1} J_n & 2^n J_n + 4J_{n-1}^2 - 1 \\ 2^n J_n & (-2)^n & 2^n J_n \\ 2^n J_n + 4J_n J_{n-1} & (-1)^n J_n & 2^n J_n + 4J_n J_{n-1} + 1 \end{bmatrix}.$$

We consider the individual entries of $M_2^{n+1} = (x_{i,j})$. For the entry $x_{1,1}$, we have

$$\begin{aligned} 2(2^n J_n + 4J_{n-1}^2) + 2^n J_n + 2^n J_n + 4J_n J_{n-1} &= 2^{n+2} J_n + 4J_{n-1}J_{n+1} \\ &= 2^{n+2} J_n + 4J_n^2 + 4(-1)^n 2^{n-1} \\ &= 2^{n+1} J_{n+1} + 4J_n^2, \end{aligned}$$

using Lemmas 3.9 and 3.10. The proof for entry $x_{1,3}$ reduces to the proof of entry $x_{1,1}$.

For the entry $x_{1,2}$, we have

$$\begin{aligned} 2(-1)^{n+1} J_n + (-2)^n + (-1)^n J_n &= (-1)^{n+1} J_n + (-2)^n \\ &= (-1)^n (2^n - J_n) \\ &= (-1)^n J_{n+1} = (-1)^{n+2} J_{n+1}, \end{aligned}$$

using Lemma 3.8.

For the entry $x_{2,1}$, we have

$$\begin{aligned} 2(2^n J_n + 4J_{n-1}^2) - 2(2^n J_n) + 2(2^n J_n + 4J_n J_{n-1}) &= 4J_n(2^{n-1} + J_{n-1}) + 4J_{n-1}(J_n + 2J_{n-1}) \\ &= 4J_{n+1}(J_{n-1} + J_n) \\ &= 4J_{n+1}(J_{n+1} - J_{n-1}) \\ &= 4J_{n+1}(2^{n-1}) = 2^{n+1} J_{n+1}, \end{aligned}$$

using Lemma 3.8 and the recursion formula for Jacobsthal numbers. The proof of entry $x_{2,3}$ is identical after initial cancellation. The proof for the entry $x_{2,2}$ is a trivial simplification.

For the entry $x_{3,1}$, we have

$$\begin{aligned} 2^{n+1} J_n + 8J_{n-1}^2 - 2^n J_n + 3(2^n J_n) + 12 J_n J_{n-1} &= 2^{n+2} J_n + 4J_{n-1}J_{n+2} \\ &= 2^{n+2} J_n + 2J_{n+2}(J_{n+1} - J_n) \\ &= 2J_n(2^{n+1} - J_{n+2}) + 2J_{n+1}J_{n+2} \\ &= 2J_n J_{n+1} + 2J_{n+1}J_{n+2} \\ &= 2^{n+1} J_{n+1} + 4J_n J_{n+1}, \end{aligned}$$

using Lemmas 3.8 and the recursion formula for Jacobsthal numbers. The proof of entry $x_{3,3}$ is similar.

For the entry $x_{3,2}$, we have

$$\begin{aligned}2(-1)^{n+1}J_n-(-2)^n+3(-1)^nJ_n &= (-1)^nJ_n-(-2)^n\\ &= (-1)^n(J_n-2^n)\\ &= (-1)^n(-J_{n+1})=(-1)^{n+1}J_{n+1},\end{aligned}$$

using Lemma 3.8.

□

4. Generalized Fibonacci Sequences in Other PTPMs

Palmer and colleagues [10] found the general form of a PTPM, and the following result is from them.

Theorem 4.1. *Let r, s, t, u, m, and n be any real or complex numbers. Then,*

$$\begin{bmatrix}\frac{(r^2-t^2)-(s^2-u^2)}{2} & rt-su & \frac{(r^2+t^2)-(s^2+u^2)}{2}\\ rs-tu & ru+st & rs+tu\\ \frac{(r^2-t^2)+(s^2-u^2)}{2} & rt+su & \frac{(r^2+t^2)+(s^2+u^2)}{2}\end{bmatrix}\begin{bmatrix}m^2-n^2\\ 2mn\\ m^2+n^2\end{bmatrix}=\begin{bmatrix}M^2-N^2\\ 2MN\\ M^2+N^2\end{bmatrix},$$

where $M=mr+nt$ and $N=ms+nu$.

We can use this result to construct a PTPM that maps a PT generated by consecutive terms of a generalized Fibonacci sequence to the next such PT. That is, suppose $w_k=aw_{k-1}+bw_{k-2}$. If we choose $r=a$, $t=b$, $s=1$, and $u=0$, then for $m=w_{k-1}$ and $n=w_{k-2}$, we obtain $M=w_k$ and $N=w_{k-1}$. The corresponding PTPM is of the form

$$M_{a,b}=\begin{bmatrix}\dfrac{a^2-b^2-1}{2} & ab & \dfrac{a^2+b^2-1}{2}\\ a & b & a\\ \dfrac{a^2-b^2+1}{2} & ab & \dfrac{a^2+b^2+1}{2}\end{bmatrix}.$$

Note that when $a=1$ and $b=1$, we obtain the PTPM that maps a PT generated by consecutive Fibonacci numbers to the next such PT, and the matrix is

$$B=\begin{bmatrix}-\dfrac{1}{2} & 1 & \dfrac{1}{2}\\ 1 & 1 & 1\\ \dfrac{1}{2} & 1 & \dfrac{3}{2}\end{bmatrix}.$$

When we say a PT generated by consecutive Fibonacci numbers, we mean a PT $(m^2-n^2, 2mn, m^2+n^2)$, where m and n are consecutive Fibonacci numbers. (Such a PT may contain entries that are not Fibonacci numbers.) We have outlined several such PTs in Table 1. For readability, we have listed them as ordered triples rather than vectors.

m	n	PT
2	1	(3, 4, 5)
3	2	(5, 12, 13)
5	3	(16, 30, 34)
8	5	(39, 80, 89)
13	8	(105, 208, 233)

TABLE 1. PTs Generated by Fibonacci Numbers

The matrix B maps any PT in the table to the PT in the following row, and so on for any such pair of PTs. Fibonacci numbers appear in powers of the matrix B as well. It is easy to verify that for $n \in \mathbb{N}$,

$$B^n = \begin{bmatrix} \frac{1}{2}F_n^2 + (-1)^n & F_n^2 & \frac{1}{2}F_{2n} \\ F_n^2 & 2F_n^2 + (-1)^n & F_{2n} \\ \frac{1}{2}F_{2n} & F_{2n} & \frac{5}{2}F_n^2 + (-1)^n \end{bmatrix}.$$

The PTPMs for the Pell and Jacobsthal sequences are formed by choosing $a = 2$ and $b = 1$, $a = 1$ and $b = 2$, respectively, giving:

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} -2 & 2 & 2 \\ 1 & 2 & 1 \\ -1 & 2 & 3 \end{bmatrix}.$$

We note that the PTPM for the Pell sequence is exactly Hall's matrix A. Recall that all of these matrices are PTPMs, so they additionally map any given PT to a PT. It also appears that the entries of the powers of special cases of such matrices, when $a = 1$ or $b = 1$, can be expressed in terms of the corresponding generalized Fibonacci sequences. We present the case when $b = 1$. The case when $a = 1$ has a similar form with generalized Fibonacci numbers in each entry that also simplifies to the matrix B when $b = 1$.

Theorem 4.2. *Let $\{t_n\}$ be the sequence defined by $t_0 = 0$, $t_1 = 1$, and $t_n = at_{n-1} + t_{n-2}$ for $n \geq 2$. Then, for $n \geq 1$,*

$$M_{a,1}^n = \begin{bmatrix} \frac{a^2}{2}t_n^2 + (-1)^n & at_n^2 & \frac{a}{2}t_{2n} \\ at_n^2 & 2t_n^2 + (-1)^n & t_{2n} \\ \frac{a}{2}t_{2n} & t_{2n} & \left(\frac{a^2}{2} + 2\right)t_n^2 + (-1)^n \end{bmatrix}.$$

The proof of Theorem 4.2 relies on induction and the identities in the following lemmas. We first prove these identities and then discuss when they are needed in the proof of Theorem 4.2. For reference, we state the main result of Melham and Shannon in [7] for the sequence $\{t_n\}$. To obtain the following lemma, take $W_n = Y_n = W_n(0, 1, a, -1)$ to be the fundamental sequence of Lucas with $p = a$ and $q = -1$.

Lemma 4.3 (Special Case of Theorem in Section 2 of [7])**.** *Let* $\{t_n\}$ *be the sequence defined by* $t_0 = 0$*,* $t_1 = 1$*, and* $t_n = at_{n-1} + t_{n-2}$ *for* $n \geq 2$*. Then, for* $n, r, s \geq 0$*,*

$$t_n t_{n+r+s} - t_{n+r} t_{n+s} = (-1)^{n+1} t_r t_s.$$

In the following lemmas, we use different values of n, r, s to obtain identities. For example, we obtain the following two identities from Lemma 4.3:

$$t_{2n-2} - t_n t_{n-1} = t_{n-1} t_{n-2} \text{ and } t_{n-1} t_{n+1} - t_n^2 = (-1)^n.$$

Lemma 4.4. *For* $n \geq 2$*,*

$$t_n^2 = (a^2+1)t_{n-1}^2 + (-1)^{n-1} + \frac{a}{2} t_{2n-2}.$$

Proof. Using the definition of t_n and an identity resulting from Lemma 4.3, we have

$$\begin{aligned}(a^2+1)t_{n-1}^2 + (-1)^{n-1} + \frac{a}{2} t_{2n-2} &= (a^2+1)t_{n-1}^2 + (-1)^{n-1} + \frac{a}{2}(t_{n-1}(t_n + t_{n-2})) \\ &= \frac{a}{2} t_{n-1}(2t_n) + t_{n-1}^2 + (-1)^{n-1} \\ &= at_{n-1}t_n + t_{n-2}t_n = t_n^2.\end{aligned}$$

□

Lemma 4.5. *For* $n \geq 2$*,*

$$t_{2n} = \left(\frac{a^2}{2} + 1\right) t_{2n-2} + \left(\frac{a^3}{2} + 2a\right) t_{n-1}^2 + a(-1)^{n-1}.$$

Proof. We have

$$\begin{aligned}\left(\frac{a^2}{2} + 1\right) t_{2n-2} + \left(\frac{a^3}{2} + 2a\right) t_{n-1}^2 + a(-1)^{n-1} &= \left(\frac{a^2}{2} + 1\right) t_{2n-2} + \left(\frac{a^3}{2} + a\right) t_{n-1}^2 + at_{n-2}t_n \\ &= \left(\frac{a^2}{2} + 1\right) t_{2n-2} + a\left(\frac{a^2}{2} + 1\right)(t_{2n-1} - t_n^2) + at_{n-2}t_n \\ &= t_{2n} + \frac{a^2}{2}(t_{2n} - at_n^2) + a(t_{n-2}t_n - t_n^2) \\ &= t_{2n} + \frac{a^2}{2} t_{2n} - \frac{a^3}{2} t_n^2 - a^2 t_{n-1} t_n = t_{2n}.\end{aligned}$$

□

Lemma 4.6. *For* $n \geq 2$*,*

$$2t_n^2 + (-1)^n = (a^2+2)t_{n-1}^2 + at_{2n-2} + (-1)^{n-1}.$$

Proof. We have

$$\begin{aligned}(a^2+2)t_{n-1}^2 + at_{2n-2} + (-1)^{n-1} &= (a^2+2)t_{n-1}^2 + a(t_{n-1}t_{n-2} + t_{n-1}t_n) + (-1)^{n-1} \\ &= at_{n_1}(at_{n-1} + t_{n-2}) + 2t_{n-1}^2 + at_{n-1}t_n + (-1)^{n-1} \\ &= 2at_{n-1}t_n + 2t_{n-1}^2 + (-1)^{n-1} \\ &= 2t_{n-1}t_{n+1} + (-1)^{n-1} \\ &= t_n^2 + t_{n-1}t_{n+1} = 2t_n^2 + (-1)^n.\end{aligned}$$

□

Proof of Theorem 4.2. When $n = 1$, the formula holds. Consider $M = M_{a,1}M_{a,1}^{n-1}$. Performing matrix multiplication, we get the desired entries in the resultant matrix $M = (m_{i,j})$ by using the identities from Lemmas 4.4, 4.5, and 4.6. The results for entries $m_{1,1}$, $m_{1,2}$, and $m_{2,1}$ follow from Lemma 4.4. The results for entries $m_{1,3}$, $m_{2,3}$, $m_{3,1}$, and $m_{3,2}$ follow from Lemma 4.5. The result for entry $m_{2,2}$ follows from Lemma 4.6. The result for entry $m_{3,3}$ follows from Lemmas 4.4 and 4.5. □

Acknowledgements

We thank the anonymous referee for the careful, prompt review. The referee's comments and suggestions helped improve the quality of this paper.

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MSC2010: 11B39, 11C20, 11D09

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