# AN INTEGER SEQUENCE WITH A DIVISIBILITY PROPERTY

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Abstract. Modifying the divisibility condition, we introduce a new integer sequence and establish its representation in terms of Fibonacci numbers and the golden mean.

## 1. Introduction

A permutation $\{a_i\}_{i=1}^{\infty}$ of positive integers is said to be divisible if

$$\sum_{i=1}^{n} a_i \equiv 0 \pmod{n} \text{ for all } n \in \mathbb{N}.$$

A concrete example of a permutation with such a property is the OEIS sequence A019444. Our point of interest in the present note is a case of a modified divisibility condition.

Let the sequence $\{z_n\}_{n=1}^{\infty}$ be defined as follows: $z_1 = 1$ and $z_n$ $(n > 1)$ is the least positive integer distinct from $z_1, z_2, \ldots, z_{n-1}$ such that

$$\sum\nolimits_n = \sum_{i=1}^{n} z_i \equiv 1 \pmod{n+1} \text{ for all } n \in \mathbb{N}.$$

We shall prove that the sequence $\{z_n\}_{n=1}^{\infty}$ possesses the properties given in the next theorem.

**Theorem 1.1.** *The sequence $\{z_n\}_{n=1}^{\infty}$ is a permutation of the set of positive integers and the following formulas hold:*

$$z_n = \begin{cases} 1, & \text{for } n = 1; \\ F_{k+1}, & \text{for } n = F_k \ (k > 2); \\ F_{k-1} - 1, & \text{for } n = F_k - 1 \ (k > 4); \\ \lfloor k\tau \rfloor, & \text{for } n = \lfloor k\tau^2 \rfloor,\ n \neq F_k \ (k > 2) \text{ and } n \neq F_k - 1 \ (k > 4); \\ \lfloor k\tau^2 \rfloor, & \text{for } n = \lfloor k\tau \rfloor,\ n \neq F_k \ (k > 2) \text{ and } n \neq F_k - 1 \ (k > 4). \end{cases}$$

Here, $\{F_k\}_{k=1}^{\infty}$ denotes the Fibonacci sequence and $\tau = \frac{1+\sqrt{5}}{2}$ is the golden mean.

## 2. Preliminaries

To prove Theorem 1.1, we need several facts about the Fibonacci sequence and the golden mean. Some of these results are new or possibly new, as indicated in Remark 2.5 at the end of this section. Let $\tau_1 = \frac{\sqrt{5}-1}{2}$ and note that $\tau^2 = \frac{3+\sqrt{5}}{2}$. The straightforward identities $\tau = 1 + \tau_1$, $\tau^2 = 1 + \tau$, and $\tau\tau_1 = 1$ will be used in the sequel without special mention.

**Lemma 2.1.** *For any $k \in \mathbb{N}$, the following identities are valid:*

(i) $\lfloor \lfloor k\tau \rfloor \tau \rfloor = \lfloor k\tau \rfloor + k - 1.$
(ii) $\lfloor (\lfloor k\tau \rfloor + 1)\tau \rfloor = \lfloor k\tau \rfloor + k + 1.$
(iii) $\lfloor (\lfloor k\tau \rfloor + k)\tau \rfloor = 2\lfloor k\tau \rfloor + k.$
(iv) $\lfloor k\tau \rfloor = \lfloor j\tau \rfloor$ *for $j \in \mathbb{N}$ if and only if $k = j$.*

*Proof.* We check (iii) and (iv). The first two identities can be derived analogously. The proof of (iii) is accomplished through the following chain of equivalent statements, the last one being obviously true:

$$k+2\lfloor k\tau\rfloor<(\lfloor k\tau\rfloor+k)\,\tau<k+2\lfloor k\tau\rfloor+1$$
$$\Longleftrightarrow \lfloor k\tau\rfloor<(\lfloor k\tau\rfloor+k)\,(\tau-1)<\lfloor k\tau\rfloor+1$$
$$\Longleftrightarrow \lfloor k\tau\rfloor<(\lfloor k\tau\rfloor+k)\,\tau_1<\lfloor k\tau\rfloor+1$$
$$\Longleftrightarrow \lfloor k\tau\rfloor\tau<\lfloor k\tau\rfloor+k<(\lfloor k\tau\rfloor+1)\,\tau$$
$$\Longleftrightarrow \lfloor k\tau\rfloor(\tau-1)<k<(\lfloor k\tau\rfloor+1)\,(\tau-1)+1$$
$$\Longleftrightarrow \lfloor k\tau\rfloor\tau_1<k<(\lfloor k\tau\rfloor+1)\,\tau_1+1$$
$$\Longleftrightarrow \lfloor k\tau\rfloor<k\tau<\lfloor k\tau\rfloor+1+\tau.$$

For (iv), suppose $n=\lfloor k\tau\rfloor=\lfloor j\tau\rfloor$. Adding up relations $n\le j\tau<n+1$ and $-n-1<-k\tau\le -n$, we get $-1<(j-k)\,\tau<1$. Multiplication by $\tau_1$ gives $-\tau_1<j-k<\tau_1$. This yields $j-k=0$ because $0<\tau_1<1$. □

Next, we adopt the notation

$$x\prec y\Longleftrightarrow x+2\le y.$$

**Theorem 2.A.** *(Zeckendorf) Every positive integer $n$ has the unique representation of the form*

$$n=F_{i_1}+F_{i_2}+\cdots+F_{i_r}, \tag{$\Delta$}$$

*where $i_1\prec i_2\prec\cdots\prec i_r$, and $i_1\ge 2$.*

Relation $(\Delta)$ is known as the Fibonacci representation of $n$.

**Definition 2.2.** *With respect to $(\Delta)$, we introduce sets $A_1$, $A_2$, $A_3$, and $A_4$ as follows:*
*$n\in A_1\Leftrightarrow i_1=2$ and $i_2$ is odd or $n=1$.*
*$n\in A_2\Leftrightarrow i_1=2$ and $i_2$ is even.*
*$n\in A_3\Leftrightarrow i_1>2$ and $i_1$ is even.*
*$n\in A_4\Leftrightarrow i_1>2$ and $i_1$ is odd.*

We also make use of the function $e:\mathbb{N}\to\mathbb{N}$ defined by

$$e\,(n)=F_{i_1-1}+F_{i_2-1}+\cdots+F_{i_r-1},$$

where $n$ is represented using $(\Delta)$.
For the sake of completeness of exposition, we recall the following theorem.

**Theorem 2.B.** *(Beatty-Skolem) If $\alpha$ and $\beta$ are positive irrational numbers such that $\frac{1}{\alpha}+\frac{1}{\beta}=1$, then the sets $S_\alpha=\{\lfloor n\alpha\rfloor:n\in\mathbb{N}\}$ and $S_\beta=\{\lfloor n\beta\rfloor:n\in\mathbb{N}\}$ form a disjoint decomposition of the set of positive integers, i.e., $S_\alpha\cup S_\beta=\mathbb{N}$ and $S_\alpha\cap S_\beta=\emptyset$.*

The next lemma provides some useful information about the sets $A_1$, $A_2$, $A_3$, and $A_4$ and the function $e$.

**Lemma 2.3.**
(i) *If $n\in A_1\cup A_2$ and $n=1+F_{i_2}+\cdots+F_{i_r}$, then*
$\lfloor(1+F_{i_2}+\cdots+F_{i_r})\,\tau\rfloor=1+F_{i_2+1}+\cdots+F_{i_r+1}$.
(ii) *If $n\in A_3$ and $n=F_{i_1}+F_{i_2}+\cdots+F_{i_r}$, then $\lfloor(F_{i_1}+\cdots+F_{i_r})\,\tau\rfloor=F_{i_1+1}+\cdots+F_{i_r+1}-1$.*
(iii) *If $n\in A_4$ and $n=F_{i_1}+F_{i_2}+\cdots+F_{i_r}$, then $\lfloor(F_{i_1}+\cdots+F_{i_r})\,\tau\rfloor=F_{i_1+1}+\cdots+F_{i_r+1}$.*
(iv) *If $a\,(n)=\lfloor n\tau\rfloor$ and $b\,(n)=\lfloor n\tau^2\rfloor$, then $e\,(a\,(n))=n$ and $e\,(b\,(n))=a\,(n)$.*

(v) $e(n) = \lfloor (n+1)\tau_1 \rfloor$.

(vi) *If* $j$ *is odd and* $j > 1$, *then* $e(F_j - 1) = F_{j-1}$ *and if* $j$ *is even and* $j > 2$, *then* $e(F_j - 1) = F_{j-1} - 1$.

*Proof.* (i) We have

$$\begin{aligned}
(1 + F_{i_2} + \cdots + F_{i_r})\tau &= \left(\frac{\tau + \tau_1}{\tau + \tau_1} + \frac{\tau^{i_2} + (-\tau_1)^{i_2}}{\tau + \tau_1} + \cdots + \frac{\tau^{i_r} + (-\tau_1)^{i_r}}{\tau + \tau_1}\right)\tau \\
&= \frac{\tau^2 + 1}{\tau + \tau_1} + \frac{\tau^{i_2+1} - (-\tau_1)^{i_2-1}}{\tau + \tau_1} + \cdots + \frac{\tau^{i_r+1} - (-\tau_1)^{i_r-1}}{\tau + \tau_1} \\
&= \frac{\tau^2 - \tau_1^2}{\tau + \tau_1} + \frac{\tau^{i_2+1} + (-\tau_1)^{i_2+1}}{\tau + \tau_1} + \cdots + \frac{\tau^{i_r+1} + (-\tau_1)^{i_r+1}}{\tau + \tau_1} \\
&\quad + \frac{1 + \tau_1^2 - (-\tau_1)^{i_2+1} - (-\tau_1)^{i_2-1} - \cdots - (-\tau_1)^{i_r+1} - (-\tau_1)^{i_r-1}}{\tau + \tau_1} \\
&= 1 + F_{i_2+1} + \cdots + F_{i_r+1} + \frac{1 + \tau_1^2}{\tau + \tau_1}\left(1 - (-\tau_1)^{i_2-1} - \cdots - (-\tau_1)^{i_r-1}\right).
\end{aligned}$$

The above equality implies

$$\begin{aligned}
(1 + F_{i_2} + \cdots + F_{i_r})\tau &> 1 + F_{i_2+1} + \cdots + F_{i_r+1} + \frac{1 + \tau_1^2}{\tau + \tau_1}\left(1 - \tau_1^2 - \tau_1^4 - \cdots\right) \\
&> 1 + F_{i_2+1} + \cdots + F_{i_r+1}
\end{aligned}$$

and

$$\begin{aligned}
(1 + F_{i_2} + \cdots + F_{i_r})\tau &< 1 + F_{i_2+1} + \cdots + F_{i_r+1} + \frac{1 + \tau_1^2}{\tau + \tau_1}\left(1 + \tau_1^2 + \tau_1^4 + \cdots\right) \\
&= 1 + F_{i_2+1} + \cdots + F_{i_r+1} + 1.
\end{aligned}$$

Combining these two estimates, we obtain (i).

(ii) and (iii) can be deduced in a similar way as (i).

(iv) follows easily from (i), (ii), and (iii). For example, if $n \in A_3$ and $n = F_{i_1} + \cdots + F_{i_r}$, then $b(n) = F_{i_1+2} + \cdots + F_{i_r+2} - 1 = F_3 + F_5 + \cdots + F_{i_1+1} + F_{i_2+2} + \cdots + F_{i_r+2}$. So, we get $e(b(n)) = F_2 + F_4 + \cdots + F_{i_1} + F_{i_2+1} + \cdots + F_{i_r+1} = F_{i_1+1} - 1 + F_{i_2+1} + \cdots + F_{i_r+1} = a(n)$.

(v) also follows from (i), (ii), and (iii). To see this, it is sufficient to note that $\lfloor (n+1)\tau_1 \rfloor = \lfloor (n+1)\tau \rfloor - (n+1)$, and then go through the cases $n+1 \in A_1 \cup A_2$, $n+1 \in A_3$, and $n+1 \in A_4$.

(vi) If $j$ is odd and $j > 1$, then $e(F_j - 1) = e(F_2 + F_4 + \cdots + F_{j-1}) = F_1 + F_3 + \cdots + F_{j-2} = F_{j-1}$. If $j$ is even and $j > 2$, then $e(F_j - 1) = e(F_1 + F_3 + \cdots + F_{j-1} - 1) = e(F_3 + \cdots + F_{j-1}) = F_2 + \cdots + F_{j-2} = F_{j-1} - 1$. ☐

**Lemma 2.4.** *Let* $S(\tau) = \{\lfloor n\tau \rfloor : n \in \mathbb{N}\}$ *and* $S(\tau^2) = \{\lfloor n\tau^2 \rfloor : n \in \mathbb{N}\}$. *Then,*

(i) $S(\tau) \cup S(\tau^2) = \mathbb{N}$ *and* $S(\tau) \cap S(\tau^2) = \emptyset$.

(ii) $S(\tau^2) = A_4$.

(iii) $n - 1 \in S(\tau) \Longleftrightarrow \lfloor n\tau \rfloor = \lfloor (n-1)\tau \rfloor + 2$ *(*$n > 1$*).*

(iv) $n - 1 \in S(\tau^2) \Longleftrightarrow \lfloor n\tau \rfloor = \lfloor (n-1)\tau \rfloor + 1$ *(*$n > 1$*).*

*Proof.* (i) This follows immediately from Theorem 2.B because $\frac{1}{\tau} + \frac{1}{\tau^2} = 1$.

(ii) Let $b(n) = \lfloor n\tau^2 \rfloor$ and $n = F_{i_1} + F_{i_2} + \cdots + F_{i_r}$ be the Fibonacci representation of $n$.

If $n \in A_1 \cup A_2$, then $b(n) = \lfloor n\tau^2 \rfloor = \lfloor n\tau \rfloor + n = 1 + F_{i_2+1} + \cdots + F_{i_r+1} + 1 + F_{i_2} + \cdots + F_{i_r} = F_3 + F_{i_2+2} + \cdots + F_{i_r+2} \in A_4$.

If $n \in A_3$, then $b(n) = F_{i_1+2} + \cdots + F_{i_r+2} - 1 = F_3 + F_5 + \cdots + F_{i_1+1} + F_{i_2+2} + \cdots + F_{i_r+2} \in A_4$.

If $n \in A_4$, then $b(n) = F_{i_1+2} + \cdots + F_{i_r+2} \in A_4$.

The above formulas imply the inclusion $S(\tau^2) \subset A_4$.

Next, we check the opposite inclusion.

Let $n \in A_4$ and $n = F_{i_1} + F_{i_2} + \cdots + F_{i_r}$ be the Fibonacci representation of $n$. If $i_1 > 3$, then $F_{i_1-2} + F_{i_2-2} + \cdots + F_{i_r-2}$ belongs to $A_4$ and $\lfloor (F_{i_1-2} + F_{i_2-2} + \cdots + F_{i_r-2})\tau^2 \rfloor = F_{i_1} + F_{i_2} + \cdots + F_{i_r}$.

If $i_1 = 3$, then $\lfloor (1 + F_{i_2-2} + \cdots + F_{i_r-2})\tau^2 \rfloor = F_3 + F_{i_2} + \cdots + F_{i_r}$. Thus, $A_4 \subset S(\tau^2)$.

(iii) The number of elements of $S(\tau)$ that are not larger than $n-1$ equals $\lfloor \frac{n}{\tau} \rfloor$. Indeed, let $k$ be the largest positive integer such that $\lfloor k\tau \rfloor \le n-1$. Then, $\lfloor k\tau \rfloor \le n-1 < \lfloor (k+1)\tau \rfloor$ gives $k\tau < n < (k+1)\tau$, i.e., $k = \lfloor \frac{n}{\tau} \rfloor$. If $n-1 \in S(\tau)$, then

$\lfloor \frac{n}{\tau} \rfloor = 1 + \lfloor \frac{n-1}{\tau} \rfloor \iff \lfloor n\tau_1 \rfloor = 1 + \lfloor (n-1)\tau_1 \rfloor \iff \lfloor n\tau \rfloor - n = 1 + \lfloor (n-1)\tau \rfloor - (n-1) \iff \lfloor n\tau \rfloor = \lfloor (n-1)\tau \rfloor + 2$.

(iv) can be checked in an analogous way as (iii). □

**Remark 2.5.** *Lemma 2.1 (i) and Lemma 2.1 (ii) appear as a theorem in* [4]. *Lemma 2.1 (iii) is possibly new. A proof of Zeckendorf's theorem can be found in* [5] *(see also* [3]*). Lemma 2.3 (i), (ii), and (iii) are ours. Carlitz studied the function e in* [2]*, where Lemma 2.3 (iv) can be found. Lemma 2.3 (v) is possibly new. Lemma 2.4 (ii), (iii), and (iv) are ours. Regarding the Beatty-Skolem theorem, we refer to* [1, 4] *(see also* [3]*).*

## 3. Proof of Theorem 1.1

Let $M_n = \frac{\sum_n - 1}{n+1}$ for $n \in \mathbb{N}$. We have $z_1 = 1$ and $M_1 = 0$; $z_2 = 3$ and $M_2 = 1$. We start by establishing the following proposition and its corollary.

**Proposition 3.1.** *For any $n > 2$, we have $z_n = M_{n-1}$ and $M_n = M_{n-1}$, if $M_{n-1} \neq z_k$ $(k = 1, \ldots, n-1)$, and $z_n = M_{n-1} + n + 1$ and $M_n = M_{n-1} + 1$, otherwise.*

*Proof.* We proceed by induction. The claim is obviously true for $n = 3$. Suppose that it holds for all $3 \le k \le n-1$. Then, $\{M_k\}_{k=1}^{n-1}$ is nondecreasing. Indeed, from $\sum_{k-1} = kM_{k-1} + 1$ and $\sum_k = (k+1)M_k + 1$, we get $z_k = (k+1)M_k - kM_{k-1}$. Therefore, $M_k = M_{k-1}$ if $z_k = M_{k-1}$, and $M_k = M_{k-1} + 1$ if $z_k = M_{k-1} + k + 1$. Hence, $M_k \ge M_{k-1}$ for $k \le n-1$.

The next observation we need is $M_k \le M_{k-1} + 1 \le M_{k-2} + 2 \le \cdots \le M_1 + k - 1 = k - 1 < k$.

Also, note that $z_k \le M_{k-1} + k + 1$ for $k \le n-1$.

We now derive the claim for $z_n$. If $M_{n-1}$ does not appear among $z_1, \ldots, z_{n-1}$, then we can take $z_n = M_{n-1}$ because $\sum_n = \sum_{n-1} + M_{n-1} = nM_{n-1} + 1 + M_{n-1} \equiv 1 \pmod{n+1}$ and $M_{n-1} < n-1$. Note that $M_n = M_{n-1} < n-1$ in this case.

If $M_{n-1}$ is equal to $z_k$ for some $k \le n-1$, then we let $z_n = M_{n-1} + n + 1$ because

$$\max_{k \le n-1} z_k \le \max\{M_1 + 3, \ldots, M_{n-2} + n\} = M_{n-2} + n \le M_{n-1} + n < M_{n-1} + n + 1,$$

and

$$\sum\nolimits_n = \sum\nolimits_{n-1} + M_{n-1} + n + 1 = nM_{n-1} + 1 + M_{n-1} + n + 1 \equiv 1 \pmod{n+1}.$$

Then,

$$M_n = \frac{\sum_n - 1}{n+1} = M_{n-1} + 1 < n \text{ and } z_n = M_n + n.$$

Finally, we also note that $\{M_k\}_{k=1}^{\infty} = \mathbb{N}_0$. □

**Corollary 3.2.**

(i) *The sequence* $\{M_n\}_{n=1}^{\infty}$ *is nondecreasing.*
(ii) $M_n < n$ *for* $n \in \mathbb{N}$.
(iii) $z_n \leq M_{n-1} + n + 1$ *for* $n \in \mathbb{N}$.
(iv) $\{z_n\}_{n=1}^{\infty}$ *is a permutation of* $\mathbb{N}$.

*Proof.* Assertions (i), (ii), and (iii) follow easily from the proof of Proposition 3.1. We will prove (iv). The mapping $n \mapsto z_n$ is one-to-one because each $z_n$ differs from all $z_l$, $l < n$. Furthermore, $\{z_n\}_{n=1}^{\infty} = \mathbb{N}$. Indeed, for $j \in \mathbb{N}$, there exists $k \geq 2$ such that $j = M_k$. Then, either $j = z_l$ for some $l \leq k$ or $j = M_k = z_{k+1}$ by Proposition 3.1. Hence, $n \mapsto z_n$ is a permutation of $\mathbb{N}$. □

We shall prove our theorem in the following more general form.

**Theorem 3.3.**

(i) $z_1 = 1$ *and* $M_1 = 0$.
(ii) *If* $n = F_k$ $(k > 2)$, *then* $z_n = F_{k+1}$ *and* $M_n = F_{k-1}$.
(iii) *If* $n = \lfloor k\tau^2 \rfloor$, $n \neq F_j$ $(j > 2)$, *and* $n \neq F_j - 1$ $(j > 4)$, *then* $z_n = \lfloor k\tau \rfloor = M_n$.
(iv) *If* $n = \lfloor k\tau \rfloor$, $n \neq F_j$ $(j > 2)$, *and* $n \neq F_j - 1$ $(j > 4)$, *then* $z_n = \lfloor k\tau^2 \rfloor$, $M_n = k$.
(v) *If* $n = F_k - 1$ $(k > 4)$, *then* $z_n = F_{k-1} - 1 = M_n$.

*Proof.* For $n \leq 10$, each of the cases (ii)–(v) appears at least once, as seen in Table 1.

| $n$ | 1 $F_2$ | 2 $F_3$ | 3 $F_4$ | 4 $F_5-1$ | 5 $F_5$ | 6 $\lfloor 4\tau \rfloor$ | 7 $F_6-1$ | 8 $F_6$ | 9 $\lfloor 6\tau \rfloor$ | 10 $\lfloor 4\tau^2 \rfloor$ | 11 $\lfloor 7\tau \rfloor$ | 12 $F_7-1$ | 13 $F_7$ | 14 $\lfloor 9\tau \rfloor$ | 15 $\lfloor 6\tau^2 \rfloor$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $z_n$ | 1 | 3 | 5 | 2 | 8 | 10 | 4 | 13 | 15 | 6 | 18 | 7 | 21 | 23 | 9 |
| $M_n$ | 0 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 | 6 | 7 | 7 | 8 | 9 | 9 |

TABLE 1. The first 15 pairs $(z_n, M_n)$

Thus, the assertion of the theorem is valid for $n \leq 10$. We assume the assertion holds for all $10 \leq m < n$ and prove it is true for $n$.
(ii) If $n = F_k$, then $n - 1 = F_k - 1$. So, by the induction hypothesis, $z_{n-1} = F_{k-1} - 1 = M_{n-1}$. Then, $M_n = M_{n-1} + 1 = F_{k-1}$ and $z_n = M_n + n = F_{k-1} + F_k = F_{k+1}$ according to Proposition 3.1.
(iii) If $n = \lfloor k\tau^2 \rfloor$, $n \neq F_j$, and $n \neq F_j - 1$, then because $\lfloor k\tau^2 \rfloor = k + \lfloor k\tau \rfloor$, Lemma 2.1 (i) gives $n - 1 = k - 1 + \lfloor k\tau \rfloor = \lfloor \lfloor k\tau \rfloor \tau \rfloor$. Obviously, $n - 1 \neq F_j - 1$ because $n \neq F_j$. If $n - 1 = \lfloor \lfloor k\tau \rfloor \tau \rfloor = F_s$, then Lemma 2.3 (iv) yields $\lfloor k\tau \rfloor = e\left(\lfloor \lfloor k\tau \rfloor \tau \rfloor\right) = e\left(F_s\right) = F_{s-1}$. By the same reasoning, $\lfloor k\tau \rfloor = F_{s-1}$ implies $k = F_{s-2}$. Then, we have $n = \lfloor k\tau^2 \rfloor = k + \lfloor k\tau \rfloor = F_{s-2} + F_{s-1} = F_s$, which is not the case. Thus, $M_{n-1} = \lfloor k\tau \rfloor$ and $z_{n-1} = \lfloor \lfloor k\tau \rfloor \tau^2 \rfloor$. Next, we check whether $M_{n-1}$ appears among $z_1, \ldots, z_{n-2}$. (We already have $M_{n-1} \neq z_{n-1}$.)

(1) $\lfloor k\tau \rfloor$ cannot take the form $\lfloor j\tau^2 \rfloor$.
(2) $\lfloor k\tau \rfloor \neq \lfloor j\tau \rfloor$ for $j \neq k$ by Lemma 2.1 (iv).
(3) $\lfloor k\tau \rfloor = F_j$ yields $k = e\left(\lfloor k\tau \rfloor\right) = e\left(F_j\right) = F_{j-1}$ by Lemma 2.3 (iv). Then, we have $n = k + \lfloor k\tau \rfloor = F_{j-1} + F_j = F_{j+1}$, which contradicts the assumption $n \neq F_s$.
(4) Suppose $\lfloor k\tau \rfloor = F_j - 1$. For even $j$, one has $F_j - 1 = F_{j-1} + \cdots + F_3 \in A_4 = S\left(\tau^2\right)$. However, $\lfloor k\tau \rfloor \in S\left(\tau^2\right)$ is impossible. If $j$ is odd, then $k = e\left(\lfloor k\tau \rfloor\right) = e\left(F_j - 1\right) = F_{j-1}$ by Lemma 2.3 (vi). This yields $n = k + \lfloor k\tau \rfloor = F_j - 1 + F_{j-1} = F_{j+1} - 1$, contrary to the assumption $n \neq F_l - 1$.

Hence, $M_{n-1} \notin \{z_1, \dots, z_{n-1}\}$ and $z_n = M_{n-1} = \lfloor k\tau \rfloor = M_n$.
(iv) If $n = \lfloor k\tau \rfloor$, $n \neq F_j$, and $n \neq F_j - 1$, then $n - 1 \neq F_j - 1$. We check the remaining three options.

(1) $n-1 = F_j$: Then, $\lfloor k\tau \rfloor = 1+F_j$ and $k = e(\lfloor k\tau \rfloor) = 1+F_{j-1}$. The induction hypothesis gives us $z_{n-1} = F_{j+1}$ and $M_{n-1} = F_{j-1}$. Now, $F_{j-1}$ appears as $z_m$ for $m = F_{j-2} < n-1$. Hence, $M_n = M_{n-1} + 1 = F_{j-1} + 1 = k$ and $z_n = M_n + n = k + \lfloor k\tau \rfloor = \lfloor k\tau^2 \rfloor$.

(2) $n-1 = \lfloor j\tau^2 \rfloor$ and $n-1 \neq F_i$: In this case, $z_{n-1} = M_{n-1} = \lfloor j\tau \rfloor$. Thus, $M_n = M_{n-1} + 1 = \lfloor j\tau \rfloor + 1$ and $z_n = M_n + n$. Then, $\lfloor k\tau \rfloor = \lfloor j\tau^2 \rfloor + 1 = j + \lfloor j\tau \rfloor + 1 = \lfloor (\lfloor j\tau \rfloor + 1)\tau \rfloor$ by Lemma 2.1 (ii). So, $k = \lfloor j\tau \rfloor + 1$. We end up with $M_n = k$ and $z_n = k + \lfloor k\tau \rfloor = \lfloor k\tau^2 \rfloor$.

(3) $n-1 = \lfloor j\tau \rfloor$ and $n-1 \neq F_i$: Then, $\lfloor k\tau \rfloor + k - 1 = \lfloor \lfloor k\tau \rfloor \tau \rfloor = \lfloor (\lfloor j\tau \rfloor + 1)\tau \rfloor = \lfloor j\tau \rfloor + j + 1 = \lfloor k\tau \rfloor + j$. Hence, $k - 1 = j$. The induction hypothesis says that $z_{n-1} = \lfloor j\tau^2 \rfloor$ and $M_{n-1} = j$. Because $\lfloor k\tau \rfloor = \lfloor (k-1)\tau \rfloor + 1$, Lemma 2.4 (iv) implies $k - 1 = j \in S(\tau^2)$. Thus, $j = \lfloor l\tau^2 \rfloor = \lfloor l\tau \rfloor + l$.
Here, $\lfloor l\tau \rfloor = F_s$ is not possible because we would have $l = e(\lfloor l\tau \rfloor) = F_{s-1}$. Then, Lemma 2.1 (iii) yields $n - 1 = \lfloor (\lfloor l\tau \rfloor + l)\tau \rfloor = 2\lfloor l\tau \rfloor + l = 2F_s + F_{s-1} = F_{s+2}$, contradicting $n - 1 \neq F_i$.
It is not the case that $\lfloor l\tau \rfloor = F_s - 1$. Namely, for $s$ odd, we have $l = e(F_s - 1) = F_{s-1}$ by Lemma 2.3 (vi). Then, $n-1 = 2\lfloor l\tau \rfloor + l = 2F_s - 2 + F_{s-1} = F_{s+2} - 2$. This implies $n = F_{s+2} - 1$, which is not the case. Analogously, for $s$ even, $l = e(F_s - 1) = F_{s-1} - 1$. Then, $n - 1 = 2F_s - 2 + F_{s-1} - 1 = F_{s+2} - 3 = F_5 + \cdots + F_{s+1} \in A_4 = S(\tau^2)$ by Lemma 2.4 (ii). This is impossible because $n - 1 \in S(\tau)$.
Thus, the induction hypothesis yields $z_{\lfloor l\tau \rfloor} = \lfloor l\tau^2 \rfloor = j = M_{n-1}$. Note that $z_{\lfloor l\tau \rfloor} \in \{z_1, \dots, z_{n-1}\}$ because $\lfloor l\tau \rfloor < j = k - 1 < \lfloor k\tau \rfloor - 1 = n - 1$. Therefore, $M_n = M_{n-1} + 1 = j + 1 = k$ and $z_n = M_n + n = k + \lfloor k\tau \rfloor = \lfloor k\tau^2 \rfloor$.

(v) $n = F_k - 1$: Then, $n - 1 = F_k - 2$. So, $n - 1 \neq F_j$ and $n - 1 \neq F_j - 1$ for $k > 5$. We consider the remaining two cases.

(1) $n - 1 = F_k - 2$ cannot have the form $\lfloor j\tau^2 \rfloor$. Namely, for $k$ even, we have $F_k - 2 = F_2 + F_5 + \cdots + F_{k-1} \in A_1$. An odd $k$ leads to $F_k - 2 = F_4 + F_6 + \cdots + F_{k-1} \in A_3$. However, $(A_1 \cup A_3) \cap S(\tau^2) = \emptyset$.

(2) If $n - 1 = \lfloor j\tau \rfloor$, then $z_{n-1} = \lfloor j\tau^2 \rfloor$ and $M_{n-1} = j$. In this case, $j = F_{k-1} - 1$. Indeed, we see, by induction, that $\lfloor (F_{k-1} - 1)\tau \rfloor = F_k - 2$ $(= n - 1 = \lfloor j\tau \rfloor)$. We check $j = F_{k-1} - 1 \notin \{z_1, \dots, z_{n-1}\}$:

2.1. $F_{k-1} - 1$ differs from $F_l - 1$ (for $l \neq k-1$) and from $F_l$.

2.2. Suppose $F_{k-1} - 1 = \lfloor l\tau^2 \rfloor = z_{\lfloor l\tau \rfloor}$, where $\lfloor l\tau \rfloor \neq F_s, F_s - 1$. The case $k-1$ being even yields $\lfloor l\tau \rfloor = e(\lfloor l\tau^2 \rfloor) = e(F_{k-1} - 1) = F_{k-2} - 1$, and this contradicts the assumption that $\lfloor l\tau \rfloor \neq F_s - 1$. The case $k-1$ being odd implies $\lfloor l\tau \rfloor = e(F_{k-1} - 1) = F_{k-2}$, contradicting $\lfloor l\tau \rfloor \neq F_s$.

2.3. Finally, suppose $F_{k-1} - 1 = \lfloor l\tau \rfloor = z_{\lfloor l\tau^2 \rfloor}$, where $\lfloor l\tau^2 \rfloor \neq F_s, F_s - 1$. If $k-1$ is even, then $F_{k-1} - 1 \in S(\tau^2)$, which in our case is impossible. For $k-1$ odd, we get $l = e(\lfloor l\tau \rfloor) = e(F_{k-1} - 1) = F_{k-2}$. Then, $\lfloor l\tau^2 \rfloor = l + \lfloor l\tau \rfloor = F_{k-2} + F_{k-1} - 1 = F_k - 1$. This contradicts the assumption $\lfloor l\tau^2 \rfloor \neq F_s - 1$.

Taking into account that $M_{n-1} = j = F_{k-1} - 1 \notin \{z_1, \dots, z_{n-1}\}$, we get $M_n = M_{n-1} = F_{k-1} - 1$ and $z_n = F_{k-1} - 1$.
This completes the proof of the theorem. □

**Remark 3.4.** *The sequence* $\{z_n\}_{n=1}^{\infty}$ *is initially considered in* [6].

**Remark 3.5.** *The first* 100, 000 *members of the sequence* $\{z_n\}_{n=1}^{\infty}$, *can be obtained by programming our Proposition 3.1 in Python.*

```
z_list = [-1,1,3]
m_list = [-1,0,1]
n = 2
for n in range(2,100000):
if m_list[n] in z_list:
m_list.append(m_list[n]+1)
z_list.append(m_list[n+1]+n+1)
else:
m_list.append(m_list[n])
z_list.append(m_list[n+1])
with open('results,txt', 'w') as f:
f.write('index, M, Z\n')
for i in range(1,100000):
f.write(str(i) + ','+ str(m_list[i]) +','+str(z_list[i])+'\n')
```

The first one hundred members of $\{z_n\}_{n=1}^{\infty}$ are: 1, 3, 5, 2, 8, 10, 4, 13, 15, 6, 18, 7, 21, 23, 9, 26, 28, 11, 31, 12, 34, 36, 14, 39, 41, 16, 44, 17, 47, 49, 19, 52, 20, 55, 57, 22, 60, 62, 24, 65, 25, 68, 70, 27, 73, 75, 29, 78, 30, 81, 83, 32, 86, 33, 89, 91, 35, 94, 96, 37, 99, 38, 102, 104, 40, 107, 109, 42, 112, 43, 115, 117, 45, 120, 46, 123, 125, 48, 128, 130, 50, 133, 51, 136, 138, 53, 141, 54, 144, 146, 56, 149, 151, 58, 154, 59, 157, 159, 61, 162.

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