EXTENDED GIBONACCI SUMS OF POLYNOMIAL PRODUCTS OF ORDER 3 REVISITED

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Abstract. We investigate two gibonacci sums of polynomial products of order 3, and their Pell and Jacobsthal counterparts.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary complex variable, $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials, and $n \geq 0$.

Fibonacci polynomials $f_n(x)$, *Lucas polynomials* $l_n(x)$, *Pell polynomials* $p_n(x)$, *Pell-Lucas polynomials* $q_n(x)$, *Jacobsthal polynomials* $J_n(x)$, and *Jacobsthal-Lucas polynomials* $j_n(x)$ belong to this family $\{z_n(x)\}$. Their numeric counterparts are $F_n = f_n(1)$, $L_n = l_n(1)$, $P_n = p_n(1)$, $2Q_n = q_n(1)$, $J_n = J_n(x)$, and $j_n = j_n(2)$, respectively [4, 3].

As in [3], in the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is no ambiguity; so z_n will mean $z_n(x)$. We let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, and $c_n = J_n(x)$ or $j_n(x)$. Correspondingly, let $G_n = F_n$ or L_n, $B_n = P_n$ or Q_n, and $C_n = J_n$ or j_n.

An *extended gibonacci polynomial product of order m* is a product of gibonacci polynomials z_{n+k} of the form $\prod\limits_{k\geq 0} z_{n+k}^{s_j}$, where $\sum\limits_{s_j \geq 1} s_j = m$ [2, 3].

We now explore two gibonacci sums of polynomial products of order 3, and their Pell and Jacobsthal counterparts.

2. A Gibonacci Sum of Polynomial Products of Order 3

Our investigation hinges on the identity [5]

$$g_{n+1}^3 + xg_n^3 - g_{n-1}^3 = \begin{cases} xg_{3n}, & \text{if } g_n = f_n; \\ x\Delta^2 g_{3n}, & \text{otherwise,} \end{cases} \tag{2.1}$$

and the gibonacci recurrence

$$g_{n+6} = (x^3 + 3x)g_{n+3} + g_n, \tag{2.2}$$

where $\Delta^2 = x^2 + 4$.

Theorem 2.1. *Let $g_n = f_n$ or l_n. Then,*

$$(x^2+3)\left(x\sum_{k=0}^{n} g_k^3 + g_{n+1}^3 + g_n^3\right) = \begin{cases} g_{3n+3} + g_{3n} + 2, & \text{if } g_n = f_n; \\ \Delta^2(g_{3n+3} + g_{3n} - 2) + 4(x+2)(x^2+3), & \text{otherwise.} \end{cases} \tag{2.3}$$

Proof. By recurrence (2.2), we have

$$(x^3+3x)\sum_{k=0}^{n} g_{3k+3} + \sum_{k=0}^{n} g_{3k} = \sum_{k=0}^{n} g_{3k+6},$$

$$(x^3+3x)\left(\sum_{k=0}^{n} g_{3k} + g_{3n+3} - g_0\right) + \sum_{k=0}^{n} g_{3k} = \sum_{k=0}^{n} g_{3k} + g_{3n+6} + g_{3n+3} - g_3 - g_0,$$

$$(x^3+3x)\sum_{k=0}^{n} g_{3k} = g_{3n+3} + g_{3n} - g_3 + (x^3+3x-1)g_0. \tag{2.4}$$

Case 1. Suppose $g_n = f_n$. Then,

$$(x^3+3x)\sum_{k=0}^{n} f_{3k} = f_{3n+3} + f_{3n} - (x^2+1).$$

Consequently, by identity (2.1), we have

$$\begin{aligned} \sum_{k=0}^{n} f_{k+1}^3 + x\sum_{k=0}^{n} f_k^3 - \sum_{k=0}^{n} f_{k-1}^3 &= x\sum_{k=0}^{n} f_{3k}, \\ (x^2+3)\left(x\sum_{k=0}^{n} f_k^3 + f_{n+1}^3 + f_n^3\right) &= f_{3n+3} + f_{3n} + 2. \qquad (2.5) \end{aligned}$$

Case 2. Suppose $g_n = l_n$. Then,

$$\begin{aligned} (x^3+3x)\sum_{k=0}^{n} l_{3k} &= l_{3n+3} + l_{3n} - (x^3+3x) + 2(x^3+3x-1) \\ &= l_{3n+3} + l_{3n} + x^3 + 3x - 2. \end{aligned}$$

By identity (2.1), we have

$$\begin{aligned} \sum_{k=0}^{n} l_{k+1}^3 + x\sum_{k=0}^{n} l_k^3 - \sum_{k=0}^{n} l_{k-1}^3 &= x\Delta^2\sum_{k=0}^{n} l_{3k}, \\ x\sum_{k=0}^{n} l_k^3 + l_{n+1}^3 + l_n^3 + x^3 - 8 &= \frac{x\Delta^2}{x^3+3x}\left(l_{3n+3} + l_{3n} + x^3 + 3x - 2\right), \\ (x^2+3)\left(x\sum_{k=0}^{n} l_k^3 + l_{n+1}^3 + l_n^3\right) &= \Delta^2(l_{3n+3} + l_{3n} - 2) + 4(x+2)(x^2+3). \qquad (2.6) \end{aligned}$$

Combining the two cases yields the desired result. □

This theorem has interesting byproducts. It follows from formula (2.3) that

$$4\left(\sum_{k=0}^{n} G_k^3 + G_{n+1}^3 + G_n^3\right) = \begin{cases} G_{3n+3} + G_{3n} + 2, & \text{if } G_n = F_n; \\ 5(G_{3n+3} + G_{3n} - 2) + 48, & \text{otherwise}; \end{cases} \tag{2.7}$$

$$4\sum_{k=0}^{n} G_k^3 + 4G_{n-1}^3 = \begin{cases} G_{3n+3} - 3G_{3n} + 2, & \text{if } G_n = F_n; \\ 5(G_{3n+3} - 3G_{3n}) + 38, & \text{otherwise}; \end{cases}$$

$$2\sum_{k=0}^{n} G_k^3 + 2G_{n-1}^3 = \begin{cases} G_{3n-1} + 1, & \text{if } G_n = F_n; \\ 5G_{3n-1} + 19, & \text{otherwise}. \end{cases} \tag{2.8}$$

Formula (2.7) implies that $G_{3n+3}+G_{3n} \equiv 2 \pmod 4$. Formula (2.8), with $G_n = F_n$, appears in [8].

Next, we investigate the Pell implications of Theorem 2.1.

3. Pell Implications

Because $b_n(x) = g_n(2x)$, formula (2.3) has a Pell counterpart:

$$(4x^2+3)\left(2x\sum_{k=0}^{n} b_k^3+b_{n+1}^3+b_n^3\right) = \begin{cases} b_{3n+3}+b_{3n}+2, & \text{if } b_n = p_n; \\ 4(x^2+1)(b_{3n+3}+b_{3n}-2)+8(x+1)(4x^2+3), & \text{otherwise.} \end{cases}$$

This yields

$$7\left(2\sum_{k=0}^{n} B_k^3 + B_{n+1}^3 + B_n^3\right) = \begin{cases} B_{3n+3}+B_{3n}+2, & \text{if } B_n = P_n; \\ 2(B_{3n+3}+B_{3n})+12, & \text{otherwise.} \end{cases} \tag{3.1}$$

Because $B_{n+3} = 12B_n + 5B_{n-1}$ and

$$B_{n+1}^3 + 2B_n^3 - B_{n-1}^3 = \begin{cases} 2B_{3n}, & \text{if } B_n = P_n; \\ 4B_{3n}, & \text{otherwise} \end{cases}$$

by formula (2.1) [5], it then follows that

$$7\left(\sum_{k=0}^{n} B_k^3 - B_n^3 + B_{n-1}^3\right) = \begin{cases} 5B_{3n-1}-B_{3n}+1, & \text{if } B_n = P_n; \\ 10B_{3n-1}-2B_{3n}+12, & \text{otherwise.} \end{cases}$$

Next, we explore the Jacobsthal ramifications of Theorem 2.1.

4. Jacobsthal Consequences

Identity (2.1) has a Jacobsthal counterpart [6]:

$$c_{n+1}^3 + xc_n^3 - x^3c_{n-1}^3 = \begin{cases} c_{3n}, & \text{if } c_n = J_n(x); \\ (4x+1)c_{3n}, & \text{otherwise.} \end{cases} \tag{4.1}$$

This implies

$$C_{n+1}^3 + 2C_n^3 - 8C_{n-1}^3 = \begin{cases} C_{3n}, & \text{if } C_n = J_n; \\ 9C_{3n}, & \text{otherwise.} \end{cases} \tag{4.2}$$

By the Jacobsthal recurrence, we have $c_{n+6} = (3x+1)c_{n+3} + x^3c_n$. Consequently,

$$\begin{aligned} (3x+1)\sum_{k=0}^{n} c_{3k+3} + x^3\sum_{k=0}^{n} c_{3k} &= \sum_{k=0}^{n} c_{3k+6}, \\ (3x+1)\left(\sum_{k=0}^{n} c_{3k} + c_{3n+3} - c_0\right) + x^3\sum_{k=0}^{n} c_{3k} &= \sum_{k=0}^{n} c_{3k} + c_{3n+6} + c_{3n+3} - c_3 - c_0, \\ (x^3+3x)\sum_{k=0}^{n} c_{3k} &= c_{3n+6} - 3xc_{3n+3} - c_3 + 3xc_0 \\ &= c_{3n+3} + x^3c_{3n} - c_3 + 3xc_0. \end{aligned} \tag{4.3}$$

Case 1. Suppose $c_n = J_n(x)$. Then,

$$(x^3+3x)\sum_{k=0}^{n} J_{3k}(x) = J_{3n+3}(x) + x^3 J_{3n}(x) - x - 1.$$

Because $J_0(x) = 0$ and $J_{-1}(x) = 1/x$, it then follows by identity (4.1) that

$$\begin{aligned}
\sum_{k=0}^{n} J_{k+1}^3(x) + x\sum_{k=0}^{n} J_k^3(x) - x^3\sum_{k=0}^{n} J_{k-1}^3(x) &= \sum_{k=0}^{n} J_{3k}(x),\\
(x^3+3x)\left[(1+x-x^3)\sum_{k=0}^{n} J_k^3(x) + J_{n+1}^3(x) + x^3 J_n^3(x) - 1\right] &= J_{3n+3}(x) + x^3 J_{3n}(x) - x - 1,\\
(x^3+3x)\left[(1+x-x^3)\sum_{k=0}^{n} J_k^3(x) + J_{n+1}^3(x) + x^3 J_n^3(x)\right] &= J_{3n+3}(x) + x^3 J_{3n}(x)\\
&\quad + x^3 + 2x - 1. \qquad (4.4)
\end{aligned}$$

Case 2. Suppose $c_n = j_n(x)$. By equation (4.3), we have

$$(x^3+3x)\sum_{k=0}^{n} j_{3k}(x) = j_{3n+3}(x) + x^3 j_{3n}(x) + 3x - 1.$$

Because $j_0(x) = 2$ and $j_{-1}(x) = -1/x$, it follows by identity (4.1) that

$$\begin{aligned}
&\sum_{k=0}^{n} j_{k+1}^3(x) + x\sum_{k=0}^{n} j_k^3(x) - x^3\sum_{k=0}^{n} j_{k-1}^3(x) = (4x+1)\sum_{k=0}^{n} j_{3k}(x),\\
&(1+x-x^3)\sum_{k=0}^{n} j_k^3(x) + j_{n+1}^3(x) + x^3 j_n^3(x) - 7 = \frac{4x+1}{x^3+3x}\left[j_{3n+3}(x) + x^3 j_{3n}(x) + 3x - 1\right],\\
&(x^3+3x)\left[(1+x-x^3)\sum_{k=0}^{n} j_k^3(x) + j_{n+1}^3(x) + x^3 j_n^3(x)\right] = (4x+1)\left[j_{3n+3}(x) + x^3 j_{3n}(x)\right]\\
&\qquad + 7x^3 + 12x^2 + 20x - 1. \qquad (4.5)
\end{aligned}$$

Combining the two cases, we get the following result.

Theorem 4.1. *Let $c_n = J_n(x)$ or $j_n(x)$. Then,*

$$(x^3+3x)\left[(1+x-x^3)\sum_{k=0}^{n} c_k^3(x) + c_{n+1}^3(x) + x^3 c_n^3(x)\right] = A\left[c_{3n+3}(x) + x^3 c_{3n}(x)\right] + B, \quad (4.6)$$

where

$$A = \begin{cases} 1, & \text{if } c_n = J_n(x);\\ 4x+1, & \text{otherwise} \end{cases} \quad \text{and } B = \begin{cases} x^3+2x-1, & \text{if } c_n = J_n(x);\\ 7x^3+12x^2+20x-1, & \text{otherwise.} \end{cases}$$

Clearly, equation (2.7) follows from formula (4.6). When $x = 2$, using equation (4.2), it yields

$$\begin{aligned} 14\left(J_{n+1}^3 + 8J_n^3 - 5\sum_{k=0}^{n} J_k^3\right) &= A^* J_{3n+3} + 8J_{3n} + B^*; & (4.7)\\ 14\left(6J_n^3 + 8J_{n-1}^3 - 5\sum_{k=0}^{n} J_k^3\right) &= A^* J_{3n+3} - 6J_{3n} + B^*, & (4.8)\end{aligned}$$

where $A^* = A(2)$ and $B^* = B(2)$.

It follows, from identities (4.7) and (4.8), that $J_{3n+3}+8J_{3n} \equiv 3 \pmod{14}$ and $j_{3n+3}+8j_{3n} \equiv 9 \pmod{14}$, respectively.

Next, we explore an extended gibonacci sum of polynomial products of order 3.

5. Second Gibonacci Sum of Polynomial Products of Order 3

Using the identity [5]

$$(x^2+1)g_n^3 + 3g_{n+1}g_ng_{n-1} = \begin{cases} g_{3n}, & \text{if } g_n = f_n;\\ \Delta^2 g_{3n}, & \text{otherwise,}\end{cases}$$

we have

$$\begin{aligned} 3\sum_{k=0}^{n} g_{k+1}g_kg_{k-1} &= E\sum_{k=0}^{n} g_{3k} - (x^2+1)\sum_{k=0}^{n} g_k^3,\\ 3(x^3+3x)\sum_{k=0}^{n} g_{k+1}g_kg_{k-1} &= E(x^3+3x)\sum_{k=0}^{n} g_{3k} - (x^2+1)(x^3+3x)\sum_{k=0}^{n} g_k^3, & (5.1)\end{aligned}$$

where

$$E = \begin{cases} 1, & \text{if } g_n = f_n;\\ \Delta^2, & \text{otherwise.}\end{cases}$$

Case 1. Suppose $g_n = f_n$. Then, by formula (2.4), this yields

$$\begin{aligned} 3(x^3+3x)\sum_{k=0}^{n} f_{k+1}f_kf_{k-1} &= (x^3+3x)\sum_{k=0}^{n} f_{3k} - (x^2+1)(x^3+3x)\sum_{k=0}^{n} f_k^3\\ &= \big(f_{3n+3} + f_{3n} - x^2 - 1\big)\\ &\quad - (x^2+1)\left[(f_{3n+3} + f_{3n} + 2) - (x^2+3)(f_{n+1}^3 + f_n^3)\right]\\ &= (x^2+1)(x^2+3)(f_{n+1}^3 + f_n^3) - x^2(f_{3n+3} + f_{3n}) - 3(x^2+1).\end{aligned}$$

Case 2. Let $g_n = l_n$. Again by formulas (2.4) and (5.1), we have

$$\begin{aligned} 3(x^3+3x)\sum_{k=0}^{n} l_{k+1}l_kl_{k-1} &= \Delta^2(x^3+3x)\sum_{k=0}^{n} l_{3k} - (x^2+1)(x^3+3x)\sum_{k=0}^{n} l_k^3\\ &= \Delta^2(l_{3n+3} + l_{3n} + x^3 + 3x - 2) - (x^2+1)\left[\Delta^2(l_{3n+3} + l_{3n} - 2)\right]\\ &\quad + (x^2+1)\left[(x^2+3)(l_{n+1}^3 + l_n^3) - 4(x+2)(x^2+3)\right]\\ &= (x^2+1)(x^2+3)(l_{n+1}^3 + l_n^3) - x^2\Delta^2(l_{3n+3} + l_{3n})\\ &\quad - 3(x^5 + 2x^4 + 4x^3 + 7x^2 + 4x + 4).\end{aligned}$$

Combining the two cases yields the next result.

Theorem 5.1. *Let $g_n = f_n$ or l_n. Then,*

$$3(x^3+3x)\sum_{k=0}^{n} g_{k+1}g_kg_{k-1} = (x^2+1)(x^2+3)(g_{n+1}^3+g_n^3) - Ex^2(g_{3n+3}+g_{3n}) - 3F, \qquad (5.2)$$

where

$$E = \begin{cases} 1, & \text{if } g_n = f_n; \\ \Delta^2, & \text{otherwise} \end{cases} \quad \text{and } F = \begin{cases} x^2+1, & \text{if } g_n = f_n; \\ x^5+2x^4+4x^3+7x^2+4x+4, & \text{otherwise.} \end{cases}$$

It follows from Theorem 5.1 that

$$12\sum_{k=0}^{n} G_{k+1}G_kG_{k-1} = 8(G_{n+1}^3+G_n^3) - E^*(G_{3n+3}+G_{3n}) - 6F^*, \qquad (5.3)$$

where $E^* = E(1)$ and $F^* = F(1)$.

When $G_n = F_n$, this yields

$$\begin{aligned} 12\sum_{k=0}^{n} F_{k+1}F_kF_{k-1} &= 8(F_{n+1}^3+F_n^3) - F_{3n+3} - F_{3n} - 6 \\ &= 8(F_{n+1}^3+F_n^3) - (3F_{3n}+2F_{3n-1}) - F_{3n} - 6, \\ 6\sum_{k=0}^{n} F_{k+1}F_kF_{k-1} &= 2F_{3n} - F_{3n-1} + 4F_{n-1}^3 - 3, \end{aligned}$$

using the Lucas identity $F_{n+1}^3 + F_n^3 - F_{n-1}^3 = F_{3n}$ [5].

On the other hand, with $G_n = L_n$ and the Long identity $L_{n+1}^3 + L_n^3 - L_{n-1}^3 = 5L_{3n}$ [5], we get

$$\begin{aligned} 12\sum_{k=0}^{n} L_{k+1}L_kL_{k-1} &= 8(L_{n+1}^3+L_n^3) - 5(L_{3n+3}+L_{3n}) - 66 \\ &= 8(5L_{3n}+L_{n-1}^3) - 5(3L_{3n}+2L_{3n-1}) - 5L_{3n} - 66, \\ 6\sum_{k=0}^{n} L_{k+1}L_kL_{k-1} &= 10L_{3n} - 5L_{3n-1} + 4L_{n-1}^3 - 33. \end{aligned}$$

Thus,

$$6\sum_{k=0}^{n} G_{k+1}G_kG_{k-1} = E^*(2G_{3n} - G_{3n-1}) + 4G_{n-1}^3 - 3F^*. \qquad (5.4)$$

Using the recurrence $G_{n+3} = 3G_n + 2G_{n-1}$ and identity (2.1), we can rewrite equation (5.4) in a different way:

$$6\sum_{k=0}^{n} G_{k+1}G_kG_{k-1} = 2(G_{n+1}^3+G_n^3+G_{n-1}^3) - E^*G_{3n-1}^3 - 3F^*. \qquad (5.5)$$

Formula (5.5), with $G_n = F_n$, appears in [1] in a slightly different form.

5.1. **Pell Implications.** By virtue of the relationship $b_n(x) = g_n(2x)$, Theorem 5.1 has a Pell byproduct:

$$6(2x^3+3x)\sum_{k=0}^{n} b_{k+1}b_kb_{k-1} = (4x^2+1)(4x^2+3)(b_{n+1}^3+b_n^3) - 4Gx^2(b_{3n+3}+b_{3n}) - 3H, \quad (5.6)$$

where

$$G = \begin{cases} 1, & \text{if } b_n = p_n; \\ 4(x^2+1), & \text{otherwise} \end{cases} \text{ and } H = \begin{cases} 4x^2+1, & \text{if } b_n = p_n; \\ 4(8x^5+8x^4+8x^3+7x^2+2x+1, & \text{otherwise.} \end{cases}$$

It then follows that

$$30\sum_{k=0}^{n} B_{k+1}B_kB_{k-1} = 35(B_{n+1}^3+B_n^3) - 4G^*(B_{3n+3}+B_{3n}) - 3H^*,$$

where $G^* = G(1)$ and $H^* = H(1)$.

Next, we investigate the Jacobsthal counterpart of Theorem 5.1.

6. Jacobsthal Companion

We have from [6] that

$$(x+1)c_n^3 + 3xc_{n+1}c_nc_{n-1} = \begin{cases} c_{3n}, & \text{if } c_n = J_n(x); \\ (4x+1)c_{3n}, & \text{otherwise;} \end{cases} \quad (6.1)$$

$$c_{n+1}^3 - c_n^3 - x^3c_{n-1}^3 = 3xc_{n+1}c_nc_{n-1}. \quad (6.2)$$

With A defined as in Theorem 4.1, it follows by identity (6.1) that

$$3x\sum_{k=0}^{n} c_{k+1}c_kc_{k-1} = A\sum_{k=0}^{n} c_{3k} - (x+1)\sum_{k=0}^{n} c_k^3.$$

Case 1. Suppose $c_n = J_n(x)$. Then, by formulas (4.3) and (4.4), we have

$$\begin{aligned} 3x\sum_{k=0}^{n} J_{k+1}J_kJ_{k-1} &= \sum_{k=0}^{n} J_{3k} - (x+1)\sum_{k=0}^{n} J_k^3, \\ 3x(x^3+3x)(1+x-x^3)\sum_{k=0}^{n} J_{k+1}J_kJ_{k-1} &= (1+x-x^3)(J_{3n+3}+x^3J_{3n}-x-1) \\ &= -(x+1)(J_{3n+3}+x^3J_{3n}-x-1) \\ &\quad + (x+1)(x^3+3x)(J_{n+1}^3+x^3J_n^3-1) \\ &= -x^3(J_{3n+3}+x^3J_{3n}) \\ &\quad + (x+1)(x^3+3x)(J_{n+1}^3+x^3J_n^3) - 3x(x+1). \end{aligned}$$

Case 2. Suppose $c_n = j_n(x)$. With $S = \sum_{k=0}^{n} j_{k+1}j_kj_{k-1}$ and $j_n = j_n(x)$, again by formulas (4.3) and (4.4), we get

$$\begin{aligned} 3xS &= (4x+1)\sum_{k=0}^{n} j_{3k} - (x+1)\sum_{k=0}^{n} j_k^3, \\ 3x(x^3+3x)(1+x-x^3)S &= (4x+1)(1+x-x^3)(j_{3n+3}+x^3j_{3n}+3x-1) \\ &\quad - (x+1)\left[(4x+1)(j_{3n+3}+x^3j_{3n})+7x^3+12x^2+20x-1\right] \\ &\quad + (x+1)(x^3+3x)(j_{n+1}^3+x^3j_n^3) \\ &= -(4x+1)x^3(j_{3n+3}+x^3j_{3n}) \\ &\quad + (x+1)(x^3+3x)(j_{n+1}^3+x^3j_n^3) \\ &\quad - 3x(4x^4+2x^3+2x^2+7x+7). \end{aligned}$$

Combining the two cases, we get the next result.

Theorem 6.1. *Let $c_n = J_n(x)$ or $j_n(x)$, and A be as in Theorem 4.1. Then,*

$$3x(x^3+3x)(1+x-x^3)\sum_{k=0}^{n} c_{k+1}c_kc_{k-1} = (x+1)(x^3+3x)(c_{n+1}^3+x^3c_n^3) - Ax^3(c_{3n+3}+x^3c_{3n}) - 3xK,$$

where

$$K = \begin{cases} x+1, & \text{if } c_n = J_n(x); \\ 4x^4+2x^3+2x^2+7x+7, & \text{otherwise.} \end{cases}$$

Clearly, this yields formula (5.3). It also implies

$$210\sum_{k=0}^{n} C_{k+1}C_kC_{k-1} = 4A^*(C_{3n+3}+8C_{3n}) - 21(C_{n+1}^3+8C_n^3) - 3K^*,$$

where $A^* = A(2)$ and $K^* = K(2)$.

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