HOW TO ADD TWO NATURAL NUMBERS IN BASE PHI

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ABSTRACT. In the base phi representation, any natural number is written uniquely as a sum of powers of the golden mean with coefficients 0 and 1, where it is required that the product of two consecutive digits is always 0. In this self-contained paper, we give a new and short proof of the recursive structure of the base phi representations of the natural numbers.

1. Introduction

Base phi representations were introduced by George Bergman in 1957 ([1]). Let the golden mean be given by $\varphi = (1+\sqrt{5})/2$.
Ignoring leading and trailing zeros, any natural number N can be written uniquely as

$$N = \sum_{i=-\infty}^{\infty} d_i \varphi^i,$$

with digits $d_i = 0$ or 1, and where $d_i d_{i+1} = 11$ is not allowed. As usual, we denote the base phi representation of N as $\beta(N)$, and we write these representations with a 'decimal' point as

$$\beta(N) = d_L d_{L-1} \dots d_1 d_0 \cdot d_{-1} d_{-2} \dots d_{R+1} d_R.$$

We give ourselves the freedom to write also nonadmissible representations in this notation. For example, because $4 = 2 \times 2$ and $\beta(2) = 10 \cdot 01$, we will write $\beta(4) \doteq 20 \cdot 02$. Here, the $\doteq$ sign indicates what we consider a nonadmissible representation.

Our concern will be the recursive structure of the set of all numbers in their base phi representation. One can say that this structure was discovered in a series of papers[1] [5], [6], [7], and [8]. A version of the recursive structure theorem is given in Proposition 3.1 and Proposition 3.2 in [8]. Referring to these two propositions, the authors state: "The full result is expressed in the following propositions, and was proved in Lemma 3.8 of [7]." However, Lemma 3.8 in [7] consists of 11 statements, all (except the rather trivial number (11)) about frequencies of occurrences of 1's and 0's. This means that, at least formally, there is no proof of the recursive structure theorem. We will fill this gap in the Section 3.

Finally, we mention that the recursive structure theorem plays a crucial role in the papers [2] and [3].

2. Adding Two Base Phi Numbers

We first mention that the natural number 2 has representation $\beta(2) = 10 \cdot 01$, because $\varphi + \varphi^{-2} = 2$. That this is correct, can be computed, using the equation $\varphi^2 = \varphi + 1$. With some more work, one finds that $\beta(4) = 101 \cdot 01$.

[1]N.B.: these authors write the representations in reverse order.

A more convenient way to find the β-representations is to add $\beta(1) = 1\cdot$ repeatedly. When we add two base phi numbers, then, in general, there is a carry both to the left and (two places) to the right:

$$\beta(5) = \beta(4+1) \doteq \beta(4) + \beta(1) = 101\cdot 01 + 1 \doteq 102\cdot 01 \doteq 110\cdot 02 = 1000\cdot 1001.$$

Here, we used twice that $2\varphi^n = \varphi^{n+1} + \varphi^{n-2}$ for all integers n, a direct consequence of $\beta(2) = 10{\cdot}01$. Note that there is not only a *double carry*, but that we also have to get rid of the 11's, by replacing them with 100's. This is allowed because of the equation $\varphi^{n+2} = \varphi^{n+1} + \varphi^n$. We call this operation a *golden mean shift*.

For the convenience of the reader, we provide a list of the base phi representations of the first 24 natural numbers:

N	$\beta(N)$
1	$1\cdot$
2	$10\cdot 01$
3	$100\cdot 01$
4	$101\cdot 01$
5	$1000\cdot 1001$
6	$1010\cdot 0001$
7	$10000\cdot 0001$
8	$10001\cdot 0001$

N	$\beta(N)$
9	$10010\cdot 0101$
10	$10100\cdot 0101$
11	$10101\cdot 0101$
12	$100000\cdot 101001$
13	$100010\cdot 001001$
14	$100100\cdot 001001$
15	$100101\cdot 001001$
16	$101000\cdot 100001$

N	$\beta(N)$
17	$101010\cdot 000001$
18	$1000000\cdot 000001$
19	$1000001\cdot 000001$
20	$1000010\cdot 010001$
21	$1000100\cdot 010001$
22	$1000101\cdot 010001$
23	$1001000\cdot 100101$
24	$1001010\cdot 000101$

3. The Recursive Structure Theorem

The Lucas numbers $(L_n) = (2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, \ldots)$ are defined by

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \quad \text{for } n \ge 2.$$

The Lucas numbers have a particularly simple base phi representation.
From the well-known formula $L_{2n} = \varphi^{2n} + \varphi^{-2n}$, and the recursion $L_{2n+1} = L_{2n} + L_{2n-1}$, we have for all $n \ge 1$

$$\beta(L_{2n}) = 10^{2n}\cdot 0^{2n-1}1, \quad \beta(L_{2n+1}) = 1(01)^n\cdot(01)^n.$$

By iterated application of the double carry and the golden mean shift to $\beta(L_{2n+1}) + \beta(1)$, we find that for all $n \ge 1$

$$\beta(L_{2n+1}+1) = 10^{2n+1}\cdot(10)^n 01.$$

As in [2], we partition the natural numbers into Lucas intervals

$$\Lambda_{2n} := [L_{2n},\, L_{2n+1}] \quad \text{and} \quad \Lambda_{2n+1} := [L_{2n+1}+1,\, L_{2n+2}-1].$$

The basic idea behind this partition is that if

$$\beta(N) = d_L d_{L-1} \ldots d_1 d_0 \cdot d_{-1} d_{-2} \ldots d_{R+1} d_R,$$

then the left most index $L = L(N)$ and the right most index $R = R(N)$ satisfy

$$L(N) = |R(N)| = 2n \quad \text{if and only if} \quad N \in \Lambda_{2n},$$
$$L(N) = 2n+1,\ |R(N)| = 2n+2 \quad \text{if and only if} \quad N \in \Lambda_{2n+1}.$$

This is not hard to see from the simple expressions we have for the β-representations of the Lucas numbers; see also Theorem 1 in [4].

In some sense, odd Lucas intervals are not small enough. To obtain recursive relations, the interval $\Lambda_{2n+1} = [L_{2n+1}+1, L_{2n+2}-1]$ has to be divided into three subintervals. These three intervals are

$$\begin{aligned} I_n :=& [L_{2n+1}+1,\ L_{2n+1}+L_{2n-2}-1],\\ J_n :=& [L_{2n+1}+L_{2n-2},\ L_{2n+1}+L_{2n-1}],\\ K_n :=& [L_{2n+1}+L_{2n-1}+1,\ L_{2n+2}-1]. \end{aligned}$$

Note that I_n and K_n have the same length $L_{2n-2}-1$, and that J_n has length $L_{2n-3}+1$.
It will be convenient to use the free group versions of words of 0's and 1's. For example, $(01)^{-1}0001 = 1^{-1}001$.

Theorem. [Recursive structure theorem]
I *For all* $n \geq 1$ *and* $k = 1, \ldots, L_{2n-1}$ *one has* $\beta(L_{2n}+k) = \beta(L_{2n})+\beta(k) = 10\ldots0\,\beta(k)\,0\ldots01$.
II *For all* $n \geq 2$ *and* $k = 1, \ldots, L_{2n-2}-1$

$$\begin{aligned} I_n: &\quad \beta(L_{2n+1}+k) = 1000(10)^{-1}\beta(L_{2n-1}+k)(01)^{-1}1001,\\ K_n: &\quad \beta(L_{2n+1}+L_{2n-1}+k) = 1010(10)^{-1}\beta(L_{2n-1}+k)(01)^{-1}0001. \end{aligned}$$

Moreover, for all $n \geq 2$ *and* $k = 0, \ldots, L_{2n-3}$

$$J_n: \quad \beta(L_{2n+1}+L_{2n-2}+k) = 10010(10)^{-1}\beta(L_{2n-2}+k)(01)^{-1}001001.$$

Proof.
I As noted in [2], Part **I** follows in a simple way, because adding $\beta(k)$ to $\beta(L_{2n})$ does not give any double carries or golden mean shifts, when k is smaller than L_{2n-1}.
II Part I_n.
Fix a number k with $k \in \{1, \ldots, L_{2n-2}-1\}$. Write, with $L = 2n-1$, $R = -2n$,

$$\beta(L_{2n-1}+k) = 10d_{L-2}\ldots d_0 \cdot d_{-1}\ldots d_{R+2}01.$$

Let us write $\beta(L_{2n}) = 10^{2n} \cdot 0^{2n-1}1$ as

$$\beta(L_{2n}) = 10e_{L-1}\ldots e_0 \cdot e_{-1}\ldots e_{R+2}01,$$

where all e_i are equal to 0. From $L_{2n+1}+k = L_{2n-1}+k+L_{2n}$, we have $\beta(L_{2n+1}+k) \doteq \beta(L_{2n-1}+k)+\beta(L_{2n})$, and so, since $d_L+e_L = 1$, $d_{L-1}+e_{L-1} = 0$, $d_{R+1}+e_{R+1} = 0$, and $d_R+e_R = 2$, one obtains

$$\begin{aligned} \beta(L_{2n+1}+k) &\doteq 110\,d_{L-2}\ldots d_0 \cdot d_{-1}\ldots d_{R+2}\,02\\ &= 1000\,d_{L-2}\ldots d_0 \cdot d_{-1}\ldots d_{R+2}1001\\ &= 1000(10)^{-1}\,\beta(L_{2n-1}+k)(01)^{-1}1001. \end{aligned}$$

II Part J_n.
Note first that

$$\beta(2L_{2n}) \doteq 2\,0^{2n} \cdot 0^{2n-1}\,2 = 1001\,0^{2n-2} \cdot 0^{2n-2}\,1001 = 100\,\beta(L_{2n-2})\,1^{-1}0\,1001.$$

We now exploit the equation $L_{2n+1}+L_{2n-2} = 2L_{2n}$. Fix a number k with $k \in \{0, \ldots, L_{2n-3}\}$. Then,

$$\begin{aligned} \beta(L_{2n+1}+L_{2n-2}+k) &= \beta(2L_{2n}+k) \doteq \beta(2L_{2n})+\beta(k)\\ &= 100\,\beta(L_{2n-2})\,1^{-1}0\,1001+\beta(k)\\ &= 100\beta(L_{2n-2}+k)1^{-1}01001, \end{aligned}$$

where we used Part **I** in the second and in the last step (in the second step a version of Part **I**, with L_{2n} replaced by $2L_{2n}$). Part J_n is now proved, because $10010(10)^{-1} = 100$, and $(01)^{-1}001001 = 1^{-1}01001$.

II Part K_n.
This is more involved than the proofs of Part I_n and Part J_n. We first give a one line proof of the following formula, which can also be found in Lemma 3.3 of [7]. For all $n \geq 1$

$$\beta(L_{2n}-1) = (10)^n \cdot 0^{2n-1}1.$$

Take this formula as an Ansatz, and add $\beta(1)$. The outcome is a matter of applying the golden mean shift n times:

$$\beta(L_{2n}-1)+\beta(1) \doteq (10)^{n-1}11\cdot 0^{2n-1}1 \doteq (10)^{n-2}1100\cdot 0^{2n-1}1 = \cdots = 1(00)^n\cdot 0^{2n-1}1 = \beta(L_{2n}).$$

The truth of the Ansatz follows then from the uniqueness of β-representations.
Note that we proved the K_n-formula

$$\beta(L_{2n+1}+L_{2n-1}+k) = 10\beta(L_{2n-1}+k)(01)^{-1}0001$$

for $k = L_{2n-2}-1$, since

$$L_{2n+1}+L_{2n-1}+L_{2n-2}-1 = L_{2n+1}+L_{2n}-1 = L_{2n+2}-1, \quad L_{2n-1}+L_{2n-2}-1 = L_{2n}-1,$$

and

$$\beta(L_{2n+2}-1) = (10)^{n+1}\cdot 0^{2n+1}1 = 10\,(10)^n\cdot 0^{2n-1}\,11^{-1}\,001 = 10\beta(L_{2n}-1)(01)^{-1}0001.$$

Next, note that $k = L_{2n-2}-1$ in the right side of the K_n-formula gives the last element $L_{2n-1}+L_{2n-2}-1 = L_{2n}-1$ of the Lucas interval $\Lambda_{2n-1} = [L_{2n-1}+1, L_{2n}-1]$, and $k=1$ gives the first element of Λ_{2n-1}. This implies that repeatedly adding $\beta(1)$ to this first element gives β-representations all restricted to the same range. But, because we proved the correctness of the K_n-formula for the last number obtained, the formula must then also be correct for all previous numbers, again by uniqueness of the β-representations. □

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