EXACT DIVISIBILITY BY POWERS OF THE BALANCING AND LUCAS-BALANCING NUMBERS

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Abstract. We obtain exact divisibility results for the powers of the balancing and Lucas-balancing numbers. This gives all the results analogous to those of Fibonacci and Lucas numbers from 1970 to 2019. For example, Hoggatt and Bicknell-Johnson (1977) and Benjamin and Rouse (2009) proved that if $F_n^k \mid m$, then $F_n^{k+1} \mid F_{nm}$, which was later generalized by Pongsriiam (2014) to include the exact divisibility such as $F_n^{k+1} \parallel F_{nm}$, provided that $F_n^k \parallel m$, $n \geq 3$, and $n \not\equiv 3 \pmod 6$. Here, F_n is the nth Fibonacci number. For the balancing numbers B_n, we show that $B_n^k \parallel m$ if and only if $B_n^{k+1} \parallel B_{nm}$ for all $k \geq 1$ and $m, n \geq 2$. The corresponding results for the Lucas-balancing numbers are also given.

1. Introduction

For integers $m, n \geq 1$, and $k \geq 0$, we say that m^k exactly divides n and write $m^k \parallel n$ if $m^k \mid n$ and $m^{k+1} \nmid n$. In this paper, we explore the exact divisibility by powers of the balancing and Lucas-balancing numbers. Recall that n is said to be a balancing number if it is a solution to the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r),$$

for some $r \in \mathbb{N}$. The sequence $(B_n)_{n\geq 0}$ of balancing numbers is defined by the recurrence $B_{n+1} = 6B_n - B_{n-1}$ for $n \geq 1$, with $B_0 = 0$ and $B_1 = 1$, and the sequence $(C_n)_{n\geq 0}$ of Lucas-balancing numbers is defined by the same recursive pattern as $(B_n)_{n\geq 0}$, but with the initial values $C_0 = 1$ and $C_1 = 3$ (see [2, 17, 25]). The literature regarding divisibility properties of the Fibonacci numbers F_n and Lucas numbers L_n is well-known, but those of the balancing and Lucas-balancing numbers are, to some extent, not very prevalent. Recall that one of the tools used in Matijasevich's solution to Hilbert's 10th problem [12, 13, 14] is the following divisibility relation:

$$F_n^2 \mid F_{nm} \quad \text{if and only if} \quad F_n \mid m. \tag{1.1}$$

Hoggatt and Bicknell-Johnson [5] gave a generalization of (1.1) by replacing F_n^2 by F_n^3, and for a general k, they proved that

$$\text{if } F_n^k \mid m, \text{ then } F_n^{k+1} \mid F_{nm}. \tag{1.2}$$

Benjamin and Rouse [3] and Seibert and Trojovský [29] also provided a different proof of (1.2). Then the investigation on exact divisibility results for a subsequence of $(F_n)_{n\geq 1}$ began with the work of Tangboonduangjit, et al. [18, 31] and was generalized by Onphaeng and Pongsriiam [15]. The most general results in this direction were obtained by Pongsriiam [20], where (1.2) is extended to include the divisibility and exact divisibility for both the Fibonacci and Lucas numbers. Finally, Onphaeng and Pongsriiam [16] have recently given the converse of the results in [20], which completely answers these questions for the Fibonacci and Lucas numbers.

In this paper, we show divisibility properties analogous to those in [3, 5, 16, 20, 29] for the balancing and Lucas-balancing numbers. For other related and recent results on Fibonacci, Lucas, balancing, and Lucas-balancing numbers, see [4, 6, 22, 24, 26, 27, 30].

2. Preliminaries

We first recall some basic properties of B_n and C_n in the following lemmas.

Lemma 2.1. ([17] and [25]) *Let m and n be positive integers. Then, the following statements hold.*

(i) $B_n \mid B_m$ *if and only if* $n \mid m$. *Furthermore,* $\gcd(B_m, B_n) = B_{\gcd(m,n)}$.
(ii) $\gcd(B_n, C_n) = 1$.
(iii) $B_{2n} = 2B_nC_n$.
(iv) (Binet's formula) $B_m = \frac{\alpha^{2m}-\beta^{2m}}{\alpha^2-\beta^2}$ *and* $C_m = \frac{\alpha^{2m}+\beta^{2m}}{2}$, *where* $\alpha = 1+\sqrt{2}$ *and* $\beta = 1-\sqrt{2}$.

From this point on, we let $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$. In addition, we sometimes apply Lemma 2.1 without reference. Furthermore, if $x \in \mathbb{R}$, then we write $\lfloor x \rfloor$ to denote the largest integer not exceeding x.

Lemma 2.2. *If m and n are positive integers, then*

$$B_{mn} = \sum_{j=1}^{n} \binom{n}{j} B_m^j B_{m-1}^{n-j} (-1)^{n-j} B_j = \sum_{j=1}^{m} \binom{m}{j} B_n^j B_{n-1}^{m-j} (-1)^{m-j} B_j.$$

Proof. Applying the Binet formula given in Lemma 2.1, we get $\alpha^2 B_m - B_{m-1} = \alpha^{2m}$ and $\beta^2 B_m - B_{m-1} = \beta^{2m}$. Putting the values of α^{2m} and β^{2m} into $B_{mn} = \frac{(\alpha^{2m})^n-(\beta^{2m})^n}{4\sqrt{2}}$ and then applying the binomial expansion, we obtain the first equality. Because $B_{mn} = B_{nm}$, we can interchange the role of m and n to obtain the second equality. □

Lemma 2.3. *Let m and n be positive integers. Then, the following statements hold.*

(i) $B_{mn} = \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{m-2j-1} B_n^{2j+1} C_n^{m-2j-1} 8^j$.
(ii) $C_{mn} = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{m-2j} B_n^{2j} C_n^{m-2j} 8^j$.

Proof. By the Binet formula, we obtain that $\alpha^{2n} = C_n + \sqrt{8}B_n$ and $\beta^{2n} = C_n - \sqrt{8}B_n$. Substituting α^{2n} and β^{2n} in $B_{mn} = \frac{(\alpha^{2n})^m-(\beta^{2n})^m}{4\sqrt{2}}$ and applying the binomial expansion, we get

$$\begin{aligned} B_{mn} &= \frac{1}{4\sqrt{2}} \sum_{i=0}^{m} \binom{m}{i} \sqrt{8}^i B_n^i C_n^{m-i} (1-(-1)^i) \\ &= \frac{1}{4\sqrt{2}} \sum_{i=0}^{m} \binom{m}{m-i} \sqrt{8}^i B_n^i C_n^{m-i} (1-(-1)^i). \qquad (2.1) \end{aligned}$$

If i is even, then $1-(-1)^i = 0$. So, we consider the sum in (2.1) when i is odd, say $i = 2j+1$. Then (2.1) becomes

$$B_{mn} = \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{m-2j-1} B_n^{2j+1} C_n^{m-2j-1} 8^j,$$

which is the same as (i). The proof of (ii) is similar. □

Lemma 2.4. ([15]) *Let k, ℓ, m, and s be positive integers and $s^k \mid m$. Then, $s^{k+\ell} \mid \binom{m}{j} s^j$ for all $1 \le j \le m$ satisfying $2^{j-\ell+1} > j$. In particular, $s^{k+1} \mid \binom{m}{j} s^j$ for all $1 \le j \le m$, and $s^{k+2} \mid \binom{m}{j} s^j$ for all $3 \le j \le m$.*

For each $n \in \mathbb{N}$, the order (or the rank) of appearance of n in the balancing sequence, denoted by $\alpha(n)$, is the smallest positive integer k such that $n \mid B_k$. Similarly, the order of appearance of n in the Fibonacci sequence, denoted by $z(n)$, is the smallest $k \in \mathbb{N}$ such that $n \mid F_k$. It is well-known that $n \mid F_m$ if and only if $z(n) \mid m$, and $n \mid B_m$ if and only if $\alpha(n) \mid m$. In addition, if p is a prime and $n \in \mathbb{N}$, we write $\nu_p(n)$ to denote the p-adic valuation (or p-adic order) of n, which is defined to be the nonnegative integer k such that $p^k \parallel n$.

Marques [9] obtained a formula for $z(F_n^k)$ and for $z(L_n^k)$ in some special cases, which were later completed by Pongsriiam [21], and were used in the proof of the converse of exact divisibility results in [16]. For other formulas concerning $z(n)$, see, for example, the recent articles by Marques [10], Marques and Trojovský [11], Trojovský [32], Pongsriiam [23], and Khaochim and Pongsriiam [7, 8]. Here, we recall the formula of $\alpha(B_n^k)$ obtained by Patel, Dutta, and Ray [19].

Lemma 2.5. ([19]) *If B_n is the nth balancing number and α is the order of appearance, then $\alpha(B_n^{k+1}) = nB_n^k$, for every $k \ge 0$ and $n \ge 1$.*

Lengyel's theorem on $\nu_p(F_n)$ and $\nu_p(L_n)$ is often used in the proof of exact divisibility results for F_n and L_n. More general results by Ballot [1], Sanna [28], and Young [33] also lead to formulas for $\nu_p(B_n)$ and $\nu_p(C_n)$ as follows.

Lemma 2.6. *For each $n \in \mathbb{N}$, the p-adic valuation of B_n is given by*

$$\nu_2(B_n) = \nu_2(n)$$

and if $p \ne 2$, then

$$\nu_p(B_n) = \begin{cases} \nu_p(n) + \nu_p(B_{\alpha(p)}), & \text{if } \alpha(p) \mid n; \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, $\nu_2(C_n) = 0$, because C_n is always odd.

3. Main Results

In this section, we will provide some exact divisibility properties related to the sequences (B_n) and (C_n). We first apply Lemma 2.5 to obtain the following theorem.

Theorem 3.1. *Let m and n be positive integers. Then, the following statements hold.*

(i) *$B_n^2 \mid B_{mn}$ if and only if $B_n \mid m$.*
(ii) *$B_n^3 \mid B_{mn}$ if and only if $B_n^2 \mid m$.*

Proof. By the second equality in Lemma 2.2, we obtain

$$B_{mn} \equiv (-1)^{m-1} m B_n B_{n-1}^{m-1} \pmod{B_n^2}.$$

From this, we see that

$$B_n^2 \mid B_{mn} \Leftrightarrow B_n^2 \mid (-1)^{m-1} m B_n B_{n-1}^{m-1} \Leftrightarrow B_n \mid (-1)^{m-1} m B_{n-1}^{m-1} \Leftrightarrow B_n \mid m,$$

where the last equivalence is obtained from $\gcd(B_n, (-1)^{m-1} B_{n-1}^{m-1}) = 1$. Similarly, by Lemma 2.2, we have

$$\begin{aligned} B_{mn} &\equiv (-1)^{m-1} m B_n B_{n-1}^{m-1} + 3m(m-1) B_n^2 B_{n-1}^{m-2} (-1)^{m-2} \pmod{B_n^3} \\ &\equiv (-1)^{m-2} m B_n B_{n-1}^{m-2} (-B_{n-1} + 3(m-1) B_n) \pmod{B_n^3}. \end{aligned}$$

Because $\gcd(B_n, (-B_{n-1} + 3(m-1)B_n)) = \gcd(B_n, B_{n-1}) = 1$, we obtain

$$B_n^3 \mid B_{mn} \Leftrightarrow B_n^3 \mid mB_nB_{n-1}^{m-2} \Leftrightarrow B_n^2 \mid m.$$

This completes the proof. □

We can generalize Theorem 3.1 to higher powers of B_n as shown in Theorem 3.2. Observe that our theorem looks simpler than the one for Fibonacci and Lucas numbers [16, 20] because $\nu_2(B_n)$ and $\nu_2(C_n)$ are simpler than $\nu_2(F_n)$ and $\nu_2(L_n)$.

Theorem 3.2. *For all $k \geq 1$ and $m, n \geq 2$, we have $B_n^k \mid m$ if and only if $B_n^{k+1} \mid B_{nm}$.*

Proof. Let $B_n^k \mid m$. Then by Lemma 2.4, $B_n^{k+1} \mid \binom{m}{j} B_n^j$ for all $1 \leq j \leq m$. Then by Lemma 2.2, we obtain $B_n^{k+1} \mid B_{nm}$. Conversely, assume that $B_n^{k+1} \mid B_{nm}$. Then, $\alpha(B_n^{k+1}) \mid nm$. Applying Lemma 2.5 gives us the desired result. Alternatively, to prove that $B_n^k \mid m$, we can show that $\nu_p(B_n^k) \leq \nu_p(m)$ for all primes p dividing B_n. This method will appear again in the proof of Theorem 3.5. If $p = 2$ and $p \mid B_n$, then we obtain, from the assumption $B_n^{k+1} \mid B_{nm}$, that

$$\begin{aligned} 0 \leq \nu_2(B_{nm}) - \nu_2(B_n^{k+1}) &= \nu_2(nm) - (k+1)\nu_2(n) \\ &= \nu_2(n) + \nu_2(m) - (k+1)\nu_2(n) \\ &= \nu_2(m) - k\nu_2(n) = \nu_2(m) - \nu_2(B_n^k), \end{aligned}$$

which implies $\nu_2(B_n^k) \leq \nu_2(m)$. Similarly, if $p \neq 2$ and $p \mid B_n$, then

$$\begin{aligned} 0 \leq \nu_p(B_{nm}) - \nu_p(B_n^{k+1}) &= \nu_p(nm) + \nu_p(B_{\alpha(p)}) - (k+1)(\nu_p(n) + \nu_p(B_{\alpha(p)})) \\ &= \nu_p(m) - k(\nu_p(n) + \nu_p(B_{\alpha(p)})) = \nu_p(m) - \nu_p(B_n^k), \end{aligned}$$

which leads to $\nu_p(B_n^k) \leq \nu_p(m)$. Therefore, $\nu_p(B_n^k) \leq \nu_p(m)$ for all p dividing B_n, as required. This completes the proof. □

Theorem 3.3. *For all $k \geq 1$ and $m, n \geq 2$, we obtain $B_n^k \parallel m$ if and only if $B_n^{k+1} \parallel B_{nm}$.*

Proof. Assume that $B_n^k \parallel m$. Then, $B_n^k \mid m$. By Theorem 3.2, we obtain $B_n^{k+1} \mid B_{nm}$. So, to prove that $B_n^{k+1} \parallel B_{nm}$, it is enough to show that $B_n^{k+2} \nmid B_{nm}$. Applying Theorem 3.2 again, but replacing k by $k+1$, we see that $B_n^{k+2} \mid B_{nm}$ implies $B_n^{k+1} \mid m$, which contradicts the assumption $B_n^k \parallel m$. So $B_n^{k+2} \nmid B_{nm}$, as required. For the converse part, suppose $B_n^{k+1} \parallel B_{nm}$. So $B_n^{k+1} \mid B_{nm}$, and hence, $B_n^k \mid m$ by Theorem 3.2. If $B_n^{k+1} \mid m$, then again, we would have $B_n^{k+2} \mid B_{nm}$, contradicting $B_n^{k+1} \parallel B_{nm}$. Therefore, $B_n^{k+1} \nmid m$. Hence, $B_n^k \parallel m$ and the proof is complete. □

Theorem 3.4. *Let $m, r \geq 1$. If r is even, then $C_m^2 \nmid C_{mr}$. If r is odd, then*

(i) *$C_m^2 \mid C_{mr}$ if and only if $C_m \mid r$.*
(ii) *$C_m^3 \mid C_{mr}$ if and only if $C_m^2 \mid r$.*

Proof. By the Binet formula, we get $\alpha^{2m} = C_m + \sqrt{8}B_m$ and $\beta^{2m} = C_m - \sqrt{8}B_m$. Hence, proceeding in the same way as in Lemma 2.3, we obtain

$$C_{mr} = \frac{(\alpha^{2m})^r + (\beta^{2m})^r}{2} = \frac{1}{2}\sum_{j=0}^{r} \binom{r}{j} C_m^{r-j} B_m^j \sqrt{8}^j (1 + (-1)^j). \tag{3.1}$$

If j is odd, then $1+(-1)^j=0$. So, we consider (3.1) when j is even, say $j=2i$. Then,

$$C_{mr}=\sum_{i=0}^{\lfloor \frac{r}{2}\rfloor}\binom{r}{2i}C_m^{r-2i}B_m^{2i}8^i. \tag{3.2}$$

Now, it can be said that, except for the last term, all the terms on the right side of (3.2) are divisible by C_m^2. If r is even, then the last term is $8^{r/2}B_m^r$ and so, $C_{mr}\equiv 8^{r/2}B_m^r \pmod{C_m^2}$. Because $\gcd(C_m,2)=1=\gcd(C_m,B_m)$, we obtain $C_m^2\nmid C_{mr}$. Suppose r is odd. Then all terms in (3.2), except the last term, are divisible by C_m^3. Therefore,

$$C_{mr}\equiv rC_mB_m^{r-1}8^{\frac{r-1}{2}} \pmod{C_m^3}.$$

From this, we see that $C_m^3\mid C_{mr}\Leftrightarrow C_m^2\mid r$ and $C_m^2\mid C_{mr}\Leftrightarrow C_m\mid r$. □

By Theorem 3.4, it is natural to consider the divisibility of C_{nm} by C_n^{k+1} only when m is odd, as shown in the next theorem.

Theorem 3.5. *Suppose $m,n\geq 1$ and m is odd. Then $C_n^k\mid m$ if and only if $C_n^{k+1}\mid C_{nm}$.*

Proof. Assume that $C_n^k\mid m$. By virtue of Lemma 2.4, $C_n^{k+1}\mid\binom{m}{j}C_n^j$ for $1\leq j\leq m$. So, if we replace j by $m-2j$, we have $C_n^{k+1}\mid\binom{m}{m-2j}C_n^{m-2j}$ for $1\leq m-2j\leq m$, which simplifies to $0\leq j\leq\frac{m-1}{2}$. Then, by virtue of Lemma 2.3, $C_n^{k+1}\mid C_{nm}$. Conversely, assume that $C_n^{k+1}\mid C_{nm}$. To show that $C_n^k\mid m$, we prove that $\nu_p(C_n^k)\leq\nu_p(m)$ for every prime p dividing C_n. Because C_n is odd, we consider only $p\mid C_n$ and $p\geq 3$. If $\alpha(p)\mid n$, then we obtain, from Lemmas 2.1 and 2.6, that $\nu_p(C_n)=\nu_p\left(\frac{B_{2n}}{2B_n}\right)=\nu_p(B_{2n})-\nu_p(2B_n)=\nu_p(B_{2n})-\nu_p(B_n)=(\nu_p(2n)+\nu_p(B_{\alpha(p)}))-(\nu_p(n)+\nu_p(B_{\alpha(p)}))=0$, which is not the case. Similarly, if $\alpha(p)\nmid 2n$, then $\nu_p(C_n)=\nu_p\left(\frac{B_{2n}}{2B_n}\right)=0$, which is false. Therefore, $\alpha(p)\mid 2n$ and $\alpha(p)\nmid n$. Thus, $\nu_p(C_n)=\nu_p\left(\frac{B_{2n}}{2B_n}\right)=\nu_p(B_{2n})=\nu_p(n)+\nu_p(B_{\alpha(p)})$. Now, because $C_n^{k+1}\mid C_{nm}$, we obtain

$$\begin{aligned}0\geq\nu_p(C_n^{k+1})-\nu_p(C_{nm})&=\nu_p(C_n^k)+\nu_p(C_n)-\nu_p(B_{2nm})+\nu_p(B_{nm})\\&=\nu_p(C_n^k)+\nu_p(B_{nm})+\nu_p(n)+\nu_p(B_{\alpha(p)})-\nu_p(2mn)-\nu_p(B_{\alpha(p)})\\&=\nu_p(C_n^k)+\nu_p(B_{nm})-\nu_p(m)\geq\nu_p(C_n^k)-\nu_p(m),\end{aligned}$$

which implies $\nu_p(C_n^k)\leq\nu_p(m)$, as required. The proof is complete. □

Theorem 3.6. *Suppose $m,n\geq 1$ and m is odd. Then $C_n^k\parallel m$ if and only if $C_n^{k+1}\parallel C_{nm}$.*

Proof. Let $C_n^k\parallel m$. So, $C_n^k\mid m$ and $C_n^{k+1}\nmid m$. By virtue of Theorem 3.5, $C_n^{k+1}\mid C_{nm}$. The only thing left to prove is $C_n^{k+2}\nmid C_{nm}$. Now, by Lemma 2.4, we have $C_n^{k+2}\mid\binom{m}{j}C_n^j$ for $3\leq j\leq m$, which can be rewritten as $C_n^{k+2}\mid\binom{m}{m-2j}C_n^{m-2j}$ for $0\leq j\leq\frac{m-3}{2}$. Then, by virtue of Lemma 2.3,

$$C_{nm}\equiv mC_nB_n^{m-1}8^{\frac{m-1}{2}} \pmod{C_n^{k+2}}.$$

Now, we have

$$C_n^{k+2}\mid C_{nm}\Leftrightarrow C_n^{k+1}\mid m.$$

Because $C_n^{k+1}\nmid m$, we obtain $C_n^{k+2}\nmid C_{nm}$. Conversely, let $C_n^{k+1}\parallel C_{nm}$. Because $C_n^{k+1}\mid C_{nm}$, we obtain, by Theorem 3.5, that $C_n^k\mid m$. If $C_n^{k+1}\mid m$, then we again apply Theorem 3.5 to get $C_n^{k+2}\mid C_{nm}$, which contradicts the assumption $C_n^{k+1}\parallel C_{nm}$. Hence, $C_n^{k+1}\nmid m$, which gives the desired result that $C_n^k\parallel m$. □

Theorem 3.7. *Let m and r be positive integers. If r is odd, then $C_m \nmid B_{mr}$. Suppose r is even. Then, the following statements hold.*

(i) $C_m \mid B_{mr}$.
(ii) $C_m^2 \mid B_{mr}$ *if and only if* $C_m \mid r$.
(iii) $C_m^3 \mid B_{mr}$ *if and only if* $C_m^2 \mid r$.

Proof. Proceeding in the similar manner as in Theorem 3.4, we obtain

$$B_{mr} = \frac{(\alpha^{2m})^r - (\beta^{2m})^r}{4\sqrt{2}} = \frac{1}{4\sqrt{2}} \sum_{j=0}^{r} \binom{r}{j} C_m^{r-j} B_m^j \sqrt{8}^j (1 - (-1)^j)$$
$$= \sum_{i=0}^{\lfloor \frac{r-1}{2} \rfloor} \binom{r}{2i+1} C_m^{r-2i-1} B_m^{2i+1} 8^i. \tag{3.3}$$

Case 1. r is odd. Then, all terms except the last term in (3.3) are divisible by C_m. Therefore, $B_{mr} \equiv B_m^r 8^{\frac{r-1}{2}} \pmod{C_m}$. Because $C_m > 1$ and $\gcd(C_m, B_m) = 1 = \gcd(C_m, 8)$, we see that $C_m \nmid B_{mr}$.
Case 2. r is even. Then, all terms except the last term in (3.3) are divisible by C_m^3. The last term corresponds to $i = (r-2)/2$ and so

$$B_{mr} \equiv r C_m B_m^{r-1} 8^{\frac{r-2}{2}} \pmod{C_m^3}. \tag{3.4}$$

Because (3.4) also holds when mod C_m^3 is replaced by mod C_m and mod C_m^2, we see that C_m divides B_{mr}, $C_m^2 \mid B_{mr} \Leftrightarrow C_m^2 \mid rC_m \Leftrightarrow C_m \mid r$, and $C_m^3 \mid B_{mr} \Leftrightarrow C_m^3 \mid rC_m \Leftrightarrow C_m^2 \mid r$. This completes the proof. □

Because $C_n \nmid B_{nm}$, if m is odd, it is natural to extend Theorem 3.7 under the assumption that m is even as follows.

Theorem 3.8. *Suppose k, m, and n are positive integers and m is even. Then $C_n^k \mid m$ if and only if $C_n^{k+1} \mid B_{nm}$.*

Proof. Let $C_n^k \mid m$. Then, by Lemma 2.4, we obtain $C_n^{k+1} \mid \binom{m}{j} C_n^j$ for all $1 \le j \le m$ and hence, $C_n^{k+1} \mid \binom{m}{m-2j-1} C_n^{m-2j-1}$ for every $0 \le j \le \frac{m-2}{2}$. So, by Lemma 2.3, $C_n^{k+1} \mid B_{mn}$. Conversely, assume that $C_n^{k+1} \mid B_{nm}$. To show that $C_n^k \mid m$, we follow the proof of Theorem 3.5. So, let p be an odd prime dividing C_n. As already shown in the proof of Theorem 3.5, we can assume that $\alpha(p) \mid 2n$ and $\alpha(p) \nmid n$. Then $\alpha(p) \mid nm$, because m is even. Because $C_n^{k+1} \mid B_{nm}$, we have

$$0 \le \nu_p(B_{nm}) - \nu_p(C_n^{k+1}) = \nu_p(n) + \nu_p(m) + \nu_p(B_{\alpha(p)}) - \nu_p(C_n) - \nu_p(C_n^k)$$
$$= \nu_p(m) - \nu_p(C_n^k),$$

which implies $\nu_p(C_n^k) \le \nu_p(m)$. Therefore, $C_n^k \mid m$, as required. □

Theorem 3.9. *Suppose k, m, and n are positive integers and m is even. Then $C_n^k \parallel m$ if and only if $C_n^{k+1} \parallel B_{nm}$.*

Proof. Let $C_n^k \parallel m$. By Theorem 3.8, $C_n^{k+1} \mid B_{nm}$. If $C_n^{k+2} \mid B_{nm}$, we apply Theorem 3.8 again to obtain $C_n^{k+1} \mid m$, which contradicts the assumption $C_n^k \parallel m$. So, $C_n^{k+1} \parallel B_{nm}$. The converse can be proved similarly. If $C_n^{k+1} \parallel B_{nm}$, we apply Theorem 3.8 twice to conclude that $C_n^k \parallel m$. □

Acknowledgment

Asim Patra and Tammatada Khemaratchatakumthorn thank Prapanpong Pongsriiam for the support of this submission. Tammatada Khemaratchatakumthorn is the corresponding author.

References

[1] C. Ballot, *The p-adic valuation of Lucas sequences when p is a special prime*, The Fibonacci Quarterly, **57.3** (2019), 265–275.

[2] A. Behera and G. K. Panda, *On the square roots of triangular numbers*, The Fibonacci Quarterly, **37.2** (1999), 98–105.

[3] A. Benjamin and J. Rouse, *When does F_m^L divide F_n? A combinatorial solution*, in Proceedings of the Eleventh International Conference on Fibonacci Numbers and Their Applications, vol. 194, Congressus Numerantium, 2009, 53–58.

[4] P. Cubre and J. Rouse, *Divisibility properties of the Fibonacci entry point*, Proc. Amer. Math. Soc., **142** (2014), 3771–3785.

[5] V. E. Hoggatt, Jr. and M. Bicknell-Johnson, *Divisibility by Fibonacci and Lucas squares*, The Fibonacci Quarterly, **15.1** (1977), 3–8.

[6] M. Jaidee and P. Pongsriiam, *Arithmetic functions of Fibonacci and Lucas numbers*, The Fibonacci Quarterly, **57.3** (2019), 246–254.

[7] N. Khaochim and P. Pongsriiam, *The general case on the order of appearance of product of consecutive Lucas numbers*, Acta Math. Univ. Comenian., **87** (2018), 277–289.

[8] N. Khaochim and P. Pongsriiam, *On the order of appearance of products of Fibonacci numbers*, Contrib. Discrete Math., **13** (2018), 45–62.

[9] D. Marques, *The order of appearance of powers of Fibonacci and Lucas numbers*, The Fibonacci Quarterly, **50.3** (2012), 239–245.

[10] D. Marques, *On the order of appearance of integers at most one away from Fibonacci numbers*, The Fibonacci Quarterly, **50.1** (2012), 36–43.

[11] D. Marques and P. Trojovský, *The order of appearance of the product of five consecutive Lucas numbers*, Tatra Mountains Mathematical Publications, **59** (2014), 65–77.

[12] Y. Matijasevich, *Enumerable sets are Diophantine*, Soviet Math., **11** (1970), 354–358.

[13] Y. Matijasevich, *My collaboration with Julia Robison*, Math. Intelligencer, **14** (1992), 38–45.

[14] Y. Matijasevich, *Hilbert's Tenth Problem*, MIT Press, 1996

[15] K. Onphaeng and P. Pongsriiam, *Subsequences and divisibility by powers of the Fibonacci numbers*, The Fibonacci Quarterly, **52.2** (2014), 163–171.

[16] K. Onphaeng and P. Pongsriiam, *The converse of exact divisibility by powers of the Fibonacci and Lucas numbers*, The Fibonacci Quarterly, **56.4** (2018), 296–302.

[17] G. K. Panda, *Some fascinating properties of balancing numbers*, Congr. Numerantium, **194** (2009), 185–189.

[18] C. Panraksa, A. Tangboonduangjit, and K. Wiboonton, *Exact divisibility properties of some subsequences of Fibonacci numbers*, The Fibonacci Quarterly, **51.4** (2013), 307–318.

[19] B. K. Patel, U. K. Dutta, and P. K. Ray, *Period of balancing sequence modulo powers of balancing and Pell numbers*, Annales Mathematicae et Informaticae, **47** (2017), 177–183.

[20] P. Pongsriiam, *Exact divisibility by powers of the Fibonacci and Lucas numbers*, J. Integer Seq., **17** (2014), Article 14.11.2.

[21] P. Pongsriiam, *A complete formula for the order of appearance of the powers of Lucas numbers*, Commun. Korean Math. Soc., **31** (2016), 447–450.

[22] P. Pongsriiam, *Fibonacci and Lucas numbers associated with Brocard-Ramanujan equation*, Commun. Korean Math. Soc., **32** (2017), 511–522.

[23] P. Pongsriiam, *The order of appearance of factorials in the Fibonacci sequence and certain Diophantine equations*, Period. Math. Hungar., **79** (2019), 141–156.

[24] P. Pongsriiam, *Fibonacci and Lucas numbers which have exactly three prime factors and some unique properties of F_{18} and L_{18}*, The Fibonacci Quarterly, **57.5** (2019), 130–144.

[25] P. K. Ray, *Balancing and Cobalancing Numbers*, Ph.D. thesis, Department of Mathematics, National Institute of Technology, Rourkela, India, 2009.

[26] M. K. Sahukar and G. K. Panda, *Arithmetic functions of balancing numbers*, The Fibonacci Quarterly, **56.3** (2018), 246–251.
[27] M. K. Sahukar and G. K. Panda, *Diophantine equations with balancing-like sequences associated to Brocard-Ramanujan-type problem*, Glas Mat., **54** (2019), 255–270.
[28] C. Sanna, *The p-adic valuation of Lucas sequences*, The Fibonacci Quarterly, **54.2** (2016), 118–124.
[29] J. Seibert and P. Trojovský, *On divisibility of a relation of the Fibonacci numbers*, Int. J. Pure Appl. Math., **46** (2008), 443–448.
[30] C. L. Stewart, *On divisors of Lucas and Lehmer numbers*, Acta Math., **211** (2013), 291–314.
[31] A. Tangboonduangjit and K. Wiboontton, *Divisibility properties of some subsequences of Fibonacci numbers*, East-West J. Math. Spec., vol. 2012, 331–336.
[32] P. Trojovský, *The order of appearance of the sum and difference between two Fibonacci numbers*, Asian-Eur. J. Math., **12** (2019), Article 1950046.
[33] P. T. Young, *p-adic congruences for generalized Fibonacci sequences*, The Fibonacci Quarterly, **32.1** (1994), 2–10.

MSC2010: 11B37, 11B39, 11D09

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