A BIJECTION FOR THE FIBONOMIAL COEFFICIENTS

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ABSTRACT. The combinatorial properties of the Fibonomial coefficients, defined as $\binom{n}{k}_{\mathcal{F}} = \frac{F_n!}{F_k!F_{n-k}!}$ were originally explored by Benjamin and Plott in 2008 and further examined by Sagan and Savage in 2010. Sagan and Savage gave a combinatorial interpretation of these coefficients in terms of tilings of an $(n-k)\times k$ rectangle containing a path. The proof of this combinatorial interpretation was dependent on showing these tilings satisfied a recurrence known to be satisfied by the Fibonomial coefficients and the more general Lucanomials. In this paper, we give a combinatorial proof that $F_n! = F_k!F_{n-k}!|SSP_{\binom{n}{k}}|$, where $SSP_{\binom{n}{k}}$ is the set of Sagan and Savage tilings of an $(n-k)\times k$ rectangle.

1. Background and Definitions

The well-known Fibonacci sequence is defined recursively by $F_n = F_{n-1} + F_{n-2}$, with initial conditions $F_0 = 0$ and $F_1 = 1$. The nth Fibonacci number, F_n, counts the number of tilings of a strip of length $n-1$ with squares of length 1 and dominoes of length 2.

The Lucas polynomials $\{n\}$ are defined in variables s and t as $\{0\} = 0$, $\{1\} = 1$, and for $n \geq 2$, we have $\{n\} = s\{n-1\} + t\{n-2\}$. If s and t are integers, then the sequence of numbers is called a Lucas sequence. When $s = t = 1$, the sequence is the Fibonacci sequence with $\{n\} = F_n$. The Lucanomials, an analogue of the binomial coefficients, are then defined as

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} = \frac{\{n\}!}{\{k\}!\{n-k\}!},$$

where $\{n\}! = \{n\}\{n-1\}\cdots\{2\}\{1\}$. When $s = t = 1$, $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ is known as the Fibonomial coefficient and written

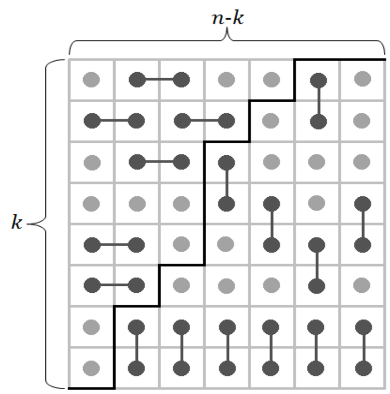
$$\binom{n}{k}_F = \frac{F_n!}{F_k!F_{n-k}!},$$

where $F_n! = F_nF_{n-1}\cdots F_2F_1$.

Fibonomial coefficients were first explored by Benjamin and Plott [1] in 2008 and, in 2010, Sagan and Savage [4] gave a combinatorial interpretation in terms of tilings associated with paths in a $k\times(n-k)$ rectangle. Recently, Bennett, Carrillo, Machacek, and Sagan [3] developed a second combinatorial interpretation in terms of lattice paths. This paper focuses on the Sagan and Savage result and, because $\binom{n}{k}_{\mathcal{F}} = \binom{n}{n-k}_{\mathcal{F}}$, we will interpret the Sagan and Savage result in terms of paths in an $(n-k)\times k$ rectangle, giving a combinatorial proof of an equality using the Sagan and Savage paths. These two existing interpretations of the Fibonomial coefficients have proven to be useful in extensions to different combinatorial objects. The Bennett, Carrillo, Machacek, and Sagan lattices path interpretation has been used by that group to prove several equalities and make extensions to Catalan numbers and finite Coxeter groups.

Sagan and Savage showed that the Fibonomial coefficent $\binom{n}{k}_{\mathcal{F}}$ counts the number of tilings of paths in an $(n-k)\times k$ rectangle with the following properties:

(1) Above the path, rows are tiled with squares and dominoes.
(2) Below the path, each column starts with domino at the bottom, then the rest of the column is tiled with squares and dominoes.



The proof of this combinatorial interpretation relies on the recursive formula for Fibonomial coefficients

$$\binom{n}{k}_{\mathcal{F}} = F_{n-k-1}\binom{n-1}{k-1}_{\mathcal{F}} + F_{k+1}\binom{n-1}{k}_{\mathcal{F}}.$$

If the last step taken is upward, then the top row and the rectangle below can be tiled in F_{k+1} and $\binom{n-1}{k}_{\mathcal{F}}$ ways, respectively, according to rules of Sagan and Savage's interpretation. If the last step is to the right, then the rightmost column and the rectangle to the left can be tiled in F_{n-k-1} and $\binom{n-1}{k-1}_{\mathcal{F}}$ ways, respectively.

Let $SSP_{\binom{n}{k}}$ be the set of Sagan and Savage tilings of an $(n-k)\times k$ rectangle, so $\binom{n}{k}_{\mathcal{F}} = |SSP_{\binom{n}{k}}|$. Then,

$$|SSP_{\binom{n}{k}}| = \frac{F_n!}{F_k!F_{n-k}!}$$

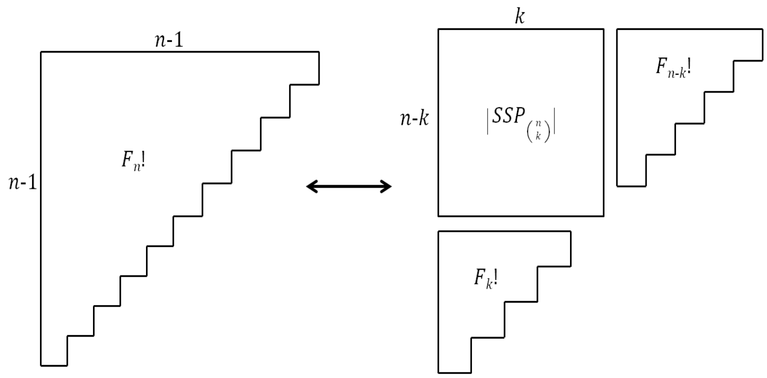
or equivalently

$$F_k!F_{n-k}!|SSP_{\binom{n}{k}}| = F_n!$$

The main result of this paper gives a bijective proof of this equality by providing a bijection between any tiling of the rows of an $(n-1)\times(n-1)$ stairstep shape and a set of three tilings: a tiling of the rows of a $(k-1)\times(k-1)$ stairstep shape, a tiling of the rows of an $(n-k-1)\times(n-k-1)$ stairstep shape, and a Sagan and Savage tiling of an $(n-k)\times k$ rectangle.

2. A Fibonomial Bijection

To begin, we say two elements in positions i and $i+1$ of the same row are *breakable* if they are not connected by a domino, and *unbreakable* if they are connected by a domino. In addition, let A denote the tilings of the rows of an $(n-1)\times(n-1)$ stairstep board corresponding to $F_n!$ and B be the set of tilings corresponding to $|SSP_{\binom{n}{k}}|\cdot F_k!F_{n-k}!$, i.e., a tiling of the rows of a $(k-1)\times(k-1)$ stairstep shape, a tiling of the rows of an $(n-k-1)\times(n-k-1)$ stairstep shape, and a Sagan and Savage tiling of an $(n-k)\times k$ rectangle.

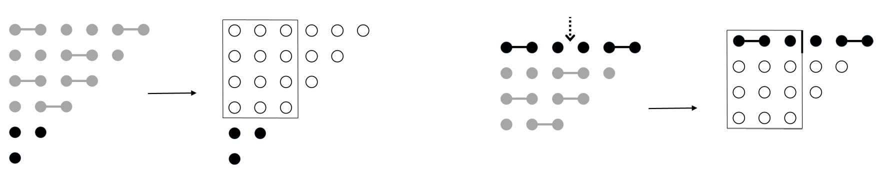


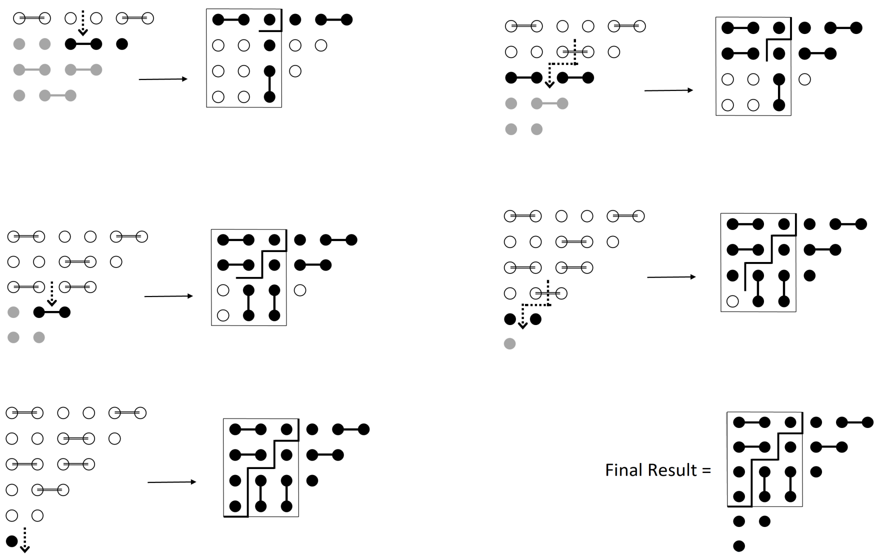
For the bijection, we start with a tiling in A and construct the three tilings in B. Given a tiling in A, rows $n-k+1$ through $n-1$ give a tiling of a $(k-1)\times(k-1)$ stairstep shape, so we remove this portion and use it directly for the tiling in set B counted by $F_k!$.

Next, we construct the path and associated tiling in $SSP_{\binom{n}{k}}$ and a tiling of the stairstep shape $(n-k-1)\times(n-k-1)$. Start with the top row of A and determine whether the elements in positions k and $k+1$ are breakable.

(1) If the elements are unbreakable, a leftward step is created in the upper right corner of the Sagan and Savage path in B; the segment consisting of the domino between elements k and $k+1$ and all the elements in the top row of A to the right of this domino are rotated 90 degrees counter-clockwise and placed under the leftward step of the path just created in B. Place the first $k-1$ elements in the first available row at the bottom of the shape of A, in this case row $n-k+1$.
(2) If the elements in positions k and $k+1$ are breakable, then create a downward step in the upper right corner of the Sagan and Savage path in B. Place the elements in positions 1 through k of the first row in A to the left of the downward step in the path in B just created, and the elements $k+1$ to $n-1$ in the same row of B, but to the right of the rectangle (this portion will be the first row of the tiling of the $(n-k-1)\times(n-k-1)$ stairstep shape).
(3) Assume we just considered elements in positions j and $j+1$ of row i. If these elements were breakable (resp. unbreakable), we next consider elements j and $j+1$ (resp. $j-1$ and j) of row $i+1$ and repeat the process above.
(4) Continue until all of the elements of A have been placed in B and the corresponding stairstep tiling. The last step of the path made should then be connected to the bottom left corner of the rectangle with either leftward steps or downward steps, whichever is appropriate.

Example 1. *Let $n=7$ and $k=3$.*





Theorem 1. *The above procedure gives a bijection between tilings counted by $F_n!$ and the triple of tilings counted by $|SSP_{\binom{n}{k}}| \cdot F_k!F_{n-k}!$.*

Proof. We prove the bijection inductively. If positions k and $k+1$ are breakable, then we place the first k elements of the first row in the top row of the Sagan and Savage tiling of size $(n-k)\times k$ and the elements in positions $k+1$ through $n-k-1$ in the top row of the stairstep shape of size $(n-k-1)\times(n-k-1)$. Now, inductively apply the above procedure on the remaining rows of the given tiling, starting with row 2, to produce a Sagan and Savage tiling of size $(n-k-1)\times k$ and a tiling of a stairstep shape of size $(n-k-2)\times(n-k-2)$, which fit below the single row tilings produced by the first step to produce a Sagan and Savage tiling of size $(n-k)\times k$ and a tiling of a stairstep shape of size $(n-k-1)\times(n-k-1)$.

If positions k and $k+1$ are unbreakable, then we place the elements in positions k through $n-k-1$ in the rightmost column of the Sagan and Savage tiling of size $(n-k)\times k$. We then move the first $k-1$ elements to row $n-k+1$ and continue the procedure by comparing the elements in positions $k-1$ and k in row two. Now, inductively apply the above procedure on the remaining rows of the given tiling, including the newly created row of length $k-1$ in row $n-k+1$, to produce a Sagan and Savage tiling of size $(n-k)\times(k-1)$ and a tiling of a stairstep shape of size $(n-k-1)\times(n-k-1)$. When we combine the Sagan and Savage tiling of size $(n-k)\times(k-1)$ with the rightmost column produced at the first step, we have a Sagan and Savage tiling of size $(n-k)\times k$ and a tiling of a stairstep shape of size $(n-k-1)\times(n-k-1)$. □

References

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[3] C. Bennett, J. Carillo, J. Machacek, and B. Sagan, *Combinatorial interpretations of Lucas analogues of binomial coefficients and Catalan numbers*, **arXiv: 1809.09036**, 26 pp.

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