EXTENDED GIBONACCI SUMS OF POLYNOMIAL PRODUCTS OF ORDERS 4 AND 5

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Abstract. We explore two Fibonacci and Jacobsthal sums of polynomial products of orders 4 and 5, and extract their Pell, Vieta, and Chebyshev counterparts. We also confirm the Fibonacci and Jacobsthal sums of polynomial products of orders 4 and 5 using graph-theoretic tools.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary complex variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 4, 7]. *Pell polynomials* $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [4].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial* [2, 7]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$ and $j_n(1) = L_n$.

Let $a(x) = x$ and $b(x) = -1$. When $g_0(x) = 0$ and $g_1(x) = 1$, $g_n(x) = V_n(x)$, the nth *Vieta polynomial*; and when $g_0(x) = 2$ and $g_1(x) = x$, $g_n(x) = v_n(x)$, the nth *Vieta-Lucas polynomial* [3, 7].

Finally, let $a(x) = 2x$ and $b(x) = -1$. When $g_0(x) = 1$ and $g_1(x) = x$, $g_n(x) = T_n(x)$, the nth *Chebyshev polynomial of the first kind*; and when $g_0(x) = 1$ and $g_1(x) = 2x$, $g_n(x) = U_n(x)$, the nth *Chebyshev polynomial of the second kind* [3, 7].

The Jacobsthal, Vieta, and Chebyshev subfamilies are closely related by the relationships in Table 1, where $i = \sqrt{-1}$ [3, 7].

Table 1. Relationships Among the Subfamilies

$$
\begin{aligned}
J_n(x) &= x^{(n-1)/2} f_n(1/\sqrt{x}) & \qquad j_n(x) &= x^{n/2} l_n(1/\sqrt{x}) \\
V_n(x) &= i^{n-1} f_n(-ix) & v_n(x) &= i^n l_n(-ix) \\
V_n(2x) &= U_{n-1}(x) & v_n(2x) &= 2T_n(x).
\end{aligned}
$$

In the interest of clarity, concision, and convenience, we omit the argument in the functional notation, when there is no ambiguity; so z_n will mean $z_n(x)$. We also omit a lot of basic algebra.

A *gibonacci polynomial product of order* m is a product of gibonacci polynomials g_{n+k} of the form $\prod_{k\geq 0} g_{n+k}^{s_j}$, where $\sum_{s_j\geq 1} s_j = m$ [8].

1.1. Fibonacci and Jacobsthal Sums of Polynomial Products of Orders 2 and 3.

It is well-known that Fibonacci and Jacobsthal polynomials satisfy the sums of products of orders 2 and 3 in Table 2 [4, 6, 7].

TABLE 2. Fibonacci and Jacobsthal Sums of Polynomial Products of Orders 2 and 3

$$
\begin{array}{lcl}
f_{m+n} &=& f_{m+1}f_n + f_m f_{n-1} \\
xf_{2n} &=& f_{n+1}^2 - f_{n-1}^2 \\
f_{2n+1} &=& f_{n+1}^2 + f_n^2 \\
xf_{3n} &=& f_{n+1}^3 + xf_n^3 - f_{n-1}^3
\end{array}
\qquad
\begin{array}{lcl}
J_{m+n} &=& J_{m+1}J_n + xJ_m J_{n-1} \\
J_{2n} &=& J_{n+1}^2 - x^2 J_{n-1}^2 \\
J_{2n+1} &=& J_{n+1}^2 - x^2 J_{n-1}^2 \\
J_{3n} &=& J_{n+1}^3 + xJ_n^3 - x^3 J_{n-1}^3
\end{array}
$$

With this background, we begin our discourse with a formula for f_{4n} as a sum of polynomial products of order 4.

2. A Fibonacci Sum of Polynomial Products of Order 4

By the Fibonacci addition formula in Table 2, we have

$$
\begin{aligned}
xf_{4n} &= f_{2n+1}(xf_{2n}) + (xf_{2n})f_{2n-1} \\
&= \left(f_{n+1}^2 + f_n^2\right)\left(f_{n+1}^2 - f_{n-1}^2\right) + \left(f_{n+1}^2 - f_{n-1}^2\right)\left(f_n^2 + f_{n-1}^2\right) \\
&= f_{n+1}^4 + 2f_{n+1}^2 f_n^2 - 2f_n^2 f_{n-1}^2 - f_{n-1}^4 \\
&= f_{n+1}^4 + 2f_n^2(xf_n + f_{n-1})^2 - 2f_n^2 f_{n-1}^2 - f_{n-1}^4 \\
&= f_{n+1}^4 + 2x^2 f_n^4 + 4xf_n^3 f_{n-1} - f_{n-1}^4. \qquad (2.1)
\end{aligned}
$$

Identity (2.1) can also be established in two other ways, namely, using the identities

1) $f_{4n} = f_{2n}l_{2n}$ and $l_{2n} = f_{n+1}^2 + 2f_n^2 + f_{n-1}^2$; and
2) $f_{4n} = f_{3n+1}f_n + f_{3n}f_{n-1}$ and $xf_{3n} = f_{n+1}^3 + xf_n^3 - f_{n-1}^3$.

In the interest of brevity, we omit their proofs.

Identity (2.1) implies that

$$F_{4n} = F_{n+1}^4 + 2F_n^4 + 4F_n^3 F_{n-1} - F_{n-1}^4.$$

Notice that identity (2.1) can be rewritten as

$$
\begin{aligned}
xf_{4n} &= f_{n+1}^4 - 2x^2 f_n^4 - f_{n-1}^4 + 4xf_{n+1}f_n^3 \\
&= f_{n+1}^4 + 4f_{n+1}^2 f_n^2 - 4f_{n+1}f_n^2 f_{n-1} - 2x^2 f_n^4 - f_{n-1}^4.
\end{aligned}
$$

2.1. Pell Consequences.

It also follows from identity (2.1) that

$$
\begin{aligned}
2xp_{4n} &= p_{n+1}^4 + 8x^2 p_n^4 + 8xp_n^3 p_{n-1} - p_{n-1}^4; \\
2P_{4n} &= P_{n+1}^4 + 8P_n^4 + 8P_n^3 P_{n-1} - P_{n-1}^4.
\end{aligned}
$$

Next, we explore a formula for f_{5n} as a sum of polynomial products of order 5.

3. A Fibonacci Sum of Polynomial Products of Order 5

To begin, notice that

$$\begin{aligned} 2x^2 f_{n+1} f_n^4 &= 2x^3 f_n^5 + 2x^2 f_n^4 f_{n-1}; \\ f_{n+1} f_{n-1}^4 &= x f_n f_{n-1}^4 + f_{n-1}^5. \end{aligned}$$

By the Fibonacci addition formula and identity (2.1), we then have

$$\begin{aligned} f_{5n} &= f_{4n+1} f_n + f_{4n} f_{n-1} \\ &= (x f_{4n} + f_{4n-1}) f_n + f_{4n} f_{n-1} \\ &= f_{4n} f_{n+1} + f_{4n-1} f_n; \\ x f_{5n} &= \left(f_{n+1}^4 + 2x^2 f_n^4 + 4x f_n^3 f_{n-1} - f_{n-1}^4\right) f_{n+1} + x f_{4n-1} f_n \\ &= A + B, \end{aligned}$$

where

$$\begin{aligned} A &= f_{n+1}^5 + 2x^2 f_{n+1} f_n^4 + 4x f_{n+1} f_n^3 f_{n-1} - f_{n+1} f_{n-1}^4 \\ &= f_{n+1}^5 + \left(2x^3 f_n^5 + 2x^2 f_n^4 f_{n-1}\right) + 4x f_{n+1} f_n^3 f_{n-1} - \left(x f_n f_{n-1}^4 + f_{n-1}^5\right) \\ &= f_{n+1}^5 + 4x f_{n+1} f_n^3 f_{n-1} + 2x^3 f_n^5 + 2x^2 f_n^4 f_{n-1} - x f_n f_{n-1}^4 - f_{n-1}^5; \\ B &= x f_{4n-1} f_n \\ &= \left(f_{2n}^2 + f_{2n-1}^2\right) x f_n; \\ xB &= \left[\left(f_{n+1}^2 - f_{n-1}^2\right)^2 + x^2\left(f_n^2 + f_{n-1}^2\right)^2\right] f_n \\ &= \left(f_{n+1}^4 + f_{n-1}^4 - 2f_{n+1}^2 f_{n-1}^2\right) f_n + x^2\left(f_n^4 + f_{n-1}^4 + 2f_n^2 f_{n-1}^2\right) f_n \\ &= f_{n+1}^4 f_n + x^2 f_n^5 + (x^2+1) f_n f_{n-1}^4 - 2f_{n+1}^2 f_n f_{n-1}^2 + 2x^2 f_n^3 f_{n-1}(f_{n+1} - x f_n) \\ &= f_{n+1}^4 f_n - 2f_{n+1}^2 f_n f_{n-1}^2 + 2x^2 f_{n+1} f_n^3 f_{n-1} + x^2 f_n^5 - 2x^3 f_n^4 f_{n-1} + (x^2+1) f_n f_{n-1}^4. \end{aligned}$$

Because

$$\begin{aligned} 2x f_{n+1}^2 f_n^2 f_{n-1} &= 2x^2 f_{n+1} f_n^3 f_{n-1} + 2x f_{n+1} f_n^2 f_{n-1}^2 \\ &= 2f_{n+1}^2 f_n f_{n-1}^2 + 2x^2 f_{n+1} f_n^3 f_{n-1} - 2f_{n+1} f_n f_{n-1}^3, \end{aligned}$$

we have

$$\begin{aligned} f_{n+1}^4 f_n &= f_{n+1}^2 f_n (x f_n + f_{n-1})^2 \\ &= x^2 f_{n+1}^2 f_n^3 + f_{n+1}^2 f_n f_{n-1}^2 + 2x f_{n+1}^2 f_n^2 f_{n-1} \\ &= x^2 f_n^3 (x f_n + f_{n-1})^2 + f_{n+1}^2 f_n f_{n-1}^2 + \left(2f_{n+1}^2 f_n f_{n-1}^2 + 2x^2 f_{n+1} f_n^3 f_{n-1} - 2f_{n+1} f_n f_{n-1}^3\right) \\ &= 3f_{n+1}^2 f_n f_{n-1}^2 + 2x^2 f_{n+1} f_n^3 f_{n-1} - 2f_{n+1} f_n f_{n-1}^3 + x^4 f_n^5 + 2x^3 f_n^4 f_{n-1} + x^2 f_n^3 f_{n-1}^2. \end{aligned}$$

So,

$$\begin{aligned} xB &= \left(3f_{n+1}^2 f_n f_{n-1}^2 + 2x^2 f_{n+1} f_n^3 f_{n-1} - 2f_{n+1} f_n f_{n-1}^3 + x^4 f_n^5 + 2x^3 f_n^4 f_{n-1} + x^2 f_n^3 f_{n-1}^2\right) \\ &\quad + x^2 f_n^5 + (x^2+1) f_n f_{n-1}^4 - 2f_{n+1}^2 f_n f_{n-1}^2 + 2x^2 f_{n+1} f_n^3 f_{n-1} - 2x^3 f_n^4 f_{n-1} \\ &= 4x^2 f_{n+1} f_n^3 f_{n-1} + (x^4 + x^2) f_n^5 + x^2 f_n f_{n-1}^4 + C, \end{aligned}$$

where

$$\begin{aligned}
C &= f_{n+1}^2 f_n f_{n-1}^2 - 2f_{n+1} f_n f_{n-1}^3 + x^2 f_n^3 f_{n-1}^2 + f_n f_{n-1}^4 \\
&= (x f_{n+1} f_n^2 f_{n-1}^2 + f_{n+1} f_n f_{n-1}^3) + f_n f_{n-1}^4 - 2f_{n+1} f_n f_{n-1}^3 + (x^2 f_{n+1} f_n^3 f_{n-1} - x^3 f_n^4 f_{n-1}) \\
&= \left(x f_{n+1} f_n^2 f_{n-1}^2 - f_{n+1} f_n f_{n-1}^3\right) + f_n f_{n-1}^4 + \left(x^2 f_{n+1} f_n^3 f_{n-1} - x^3 f_n^4 f_{n-1}\right) \\
&= x^2 f_{n+1} f_n^3 f_{n-1} + x f_{n+1} f_n^2 f_{n-1}^2 - f_{n+1} f_n f_{n-1}^3 - x^3 f_n^4 f_{n-1} + f_n f_{n-1}^4.
\end{aligned}$$

Thus,

$$\begin{aligned}
xA + xB &= x\left(f_{n+1}^5 + 2x^3 f_n^5 + 4x f_{n+1} f_n^3 f_{n-1} - f_{n-1}^5\right) + 2x^3 f_n^4 f_{n-1} - x^2 f_n f_{n-1}^4 \\
&\quad + \left[(x^4 + x^2) f_n^5 + 4x^2 f_{n+1} f_n^3 f_{n-1} + x^2 f_n f_{n-1}^4\right] + \left(x f_{n+1} f_n^2 f_{n-1}^2 - f_{n+1} f_n f_{n-1}^3\right) \\
&\quad + f_n f_{n-1}^4 + \left(x^2 f_{n+1} f_n^3 f_{n-1} - x^3 f_n^4 f_{n-1}\right) \\
&= x\left[f_{n+1}^5 + (3x^3 + x) f_n^5 - f_{n-1}^5 + 8x f_{n+1} f_n^3 f_{n-1}\right] + x f_{n+1} f_n^2 f_{n-1}^2 - f_{n+1} f_n f_{n-1}^3 \\
&\quad + x^2 f_{n+1} f_n^3 f_{n-1} + x^3 f_n^4 f_{n-1} + f_n f_{n-1}^4 \\
&= x\left[f_{n+1}^5 + 9x f_{n+1} f_n^3 f_{n-1} + (3x^3 + x) f_n^5 - f_{n-1}^5\right] + D,
\end{aligned}$$

where

$$\begin{aligned}
D &= x f_{n+1} f_n^2 f_{n-1}^2 - f_{n+1} f_n f_{n-1}^3 + x^3 f_n^4 f_{n-1} + f_n f_{n-1}^4 \\
&= x f_{n+1} f_n^2 f_{n-1}^2 + x^2 f_n^3 f_{n-1}(f_{n+1} - f_{n-1}) - f_n f_{n-1}^3 (f_{n+1} - f_{n-1}) \\
&= x f_{n+1} f_n^2 f_{n-1}^2 + x^2 f_{n+1} f_n^3 f_{n-1} - x^2 f_n^3 f_{n-1}^2 - x f_n^2 f_{n-1}^3 \\
&= x f_n^2 f_{n-1}^2 (x f_n + f_{n-1}) + x^2 f_{n+1} f_n^3 f_{n-1} - x^2 f_n^3 f_{n-1}^2 - x f_n^2 f_{n-1}^3 \\
&= x^2 f_{n+1} f_n^3 f_{n-1}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
xA + xB &= x\left[f_{n+1}^5 + 10x f_{n+1} f_n^3 f_{n-1} + x(3x^2 + 1) f_n^5 - f_{n-1}^5\right]; \\
x f_{5n} &= A + B \\
&= f_{n+1}^5 + 10x f_{n+1} f_n^3 f_{n-1} + x(3x^2 + 1) f_n^5 - f_{n-1}^5. \qquad (3.1)
\end{aligned}$$

In particular, we have

$$F_{5n} = F_{n+1}^5 + 10F_{n+1} F_n^3 F_{n-1} + 4F_n^5 - F_{n-1}^5.$$

For example, $F_{11}^5 + 10F_{11}F_{10}^3F_9 + 4F_{10}^5 - F_9^5 = 12,586,269,025 = F_{50}$.

Because $5|F_{5n}$, it follows from the identity that $F_{n+1}^5 \equiv F_n^5 + F_{n-1}^5 \pmod 5$.

3.1. **Pell Byproducts.** It follows from identity (3.1) that

$$\begin{aligned}
2x p_{5n} &= p_{n+1}^5 + 20x p_{n+1} p_n^3 p_{n-1} + 2x(12x^2 + 1) p_n^5 - p_{n-1}^5; \\
2P_{5n} &= P_{n+1}^5 + 20P_{n+1} P_n^3 P_{n-1} + 26P_n^5 - P_{n-1}^5.
\end{aligned}$$

4. Jacobsthal Consequences

Next, we investigate the Jacobsthal implications of identities (2.1) and (3.1). In both cases, we employ the relationship $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ in Table 1, and omit a lot of basic algebra in the interest of brevity.

Replacing x with $1/\sqrt{x}$ in identity (2.1) and multiplying the resulting equation with $x^{(4n-1)/2}$, we get

$$\begin{aligned}\frac{1}{\sqrt{x}}\left[x^{(4n-1)/2}f_{4n}\right] &= \frac{1}{\sqrt{x}}\left(x^{n/2}f_{n+1}\right)^4 + \frac{2}{x}\cdot x^{3/2}\left[x^{(n-1)/2}f_n\right]^4 \\ &\quad + \frac{4}{\sqrt{x}}\cdot x^2\left[x^{(n-1)/2}f_n\right]^3\left[x^{(n-2)/2}f_{n-1}\right] - x^{7/2}\left[x^{(n-2)/2}f_{n-1}\right]^4 \\ J_{4n}(x) &= J_{n+1}^4(x) + 2xJ_n^4(x) + 4x^2J_n^3(x)J_{n-1}(x) - x^4J_{n-1}^4(x), \qquad (4.1)\end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$.

Now, replace x with $1/\sqrt{x}$ in identity (3.1) and multiply the ensuing equation with $x^{(5n-1)/2}$. This yields

$$\begin{aligned}&\frac{1}{\sqrt{x}}\left[x^{(5n-1)/2}f_{5n}\right] = \frac{1}{\sqrt{x}}\left(x^{n/2}f_{n+1}\right)^5 + \frac{10}{\sqrt{x}}\cdot x^2\left(x^{n/2}f_{n+1}\right)\left[x^{(n-1)/2}f_n\right]^3\left[x^{(n-2)/2}f_{n-1}\right] \\ &\quad + \frac{1}{\sqrt{x}}\left(\frac{x+3}{x}\right)\cdot x^2\left[x^{(n-1)/2}f_n\right]^5 - x^4\sqrt{x}\left[x^{(n-2)/2}f_{n-1}\right]^5 \\ &J_{5n}(x) = J_{n+1}^5(x) + 10x^2J_{n+1}(x)J_n^3(x)J_{n-1}(x) + x(x+3)J_n^5(x) - x^5J_{n-1}^5(x), \qquad (4.2)\end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$.

It follows from identities (4.1) and (4.2) that

$$J_{4n} = J_{n+1}^4 + 4J_n^4 + 16J_n^3J_{n-1} - 16J_{n-1}^4; \qquad (4.3)$$

$$J_{5n} = J_{n+1}^5 + 40J_{n+1}J_n^3J_{n-1} + 10J_n^5 - 32J_{n-1}^5. \qquad (4.4)$$

For example,

$$\begin{aligned}J_{11}^4 + 4J_{10}^4 + 16J_{10}^3J_9 - 16J_9^4 &= 366,503,875,925 = J_{40}; \\ J_8^5 + 40J_8J_7^3J_6 + 10J_7^5 - 32J_6^5 &= 11,453,246,123 = J_{35}.\end{aligned}$$

Identities (4.3) and (4.4) imply that $J_{4n} \equiv J_{n+1}^4 \pmod 4$, $J_{5n} \equiv J_{n+1}^5 + 2J_n^5 \pmod 8$, and $J_{5n} \equiv J_{n+1}^5 - 2J_n^5 \pmod{10}$.

5. Vieta and Chebyshev Implications

The relationships $V_n(x) = i^{n-1}f_n(-ix)$ and $U_n(x) = V_{n+1}(2x)$ in Table 1 imply that identities (2.1) and (3.1) have Vieta and Chebyshev companions:

$$\begin{aligned}xV_{4n} &= V_{n+1}^4 - 2x^2V_n^4 + 4xV_n^3V_{n-1} - V_{n-1}^4; \\ xV_{5n} &= V_{n+1}^5 + 10xV_{n+1}V_n^3V_{n-1} - x(3x^2-1)V_n^5 + V_{n-1}^5; \\ 2xU_{4n} &= U_{n+1}^4 - 8x^2U_n^4 + 8xU_n^3U_{n-1} - U_{n-1}^4; \\ 2xU_{5n} &= U_{n+1}^5 + 20xU_{n+1}U_n^3U_{n-1} - 2x(12x^2-1)U_n^5 + U_{n-1}^5,\end{aligned}$$

where $V_n = V_n(x)$ and $U_n = U_n(x)$. In the interest of brevity, we omit their confirmations also.

6. Graph-Theoretic Models

Next, we confirm identities (2.1) and (4.1) with graph-theoretic tools. To this end, consider the *Fibonacci digraph* D_1 in Figure 1 with vertices v_1 and v_2, where a *weight* is assigned to each edge [5].

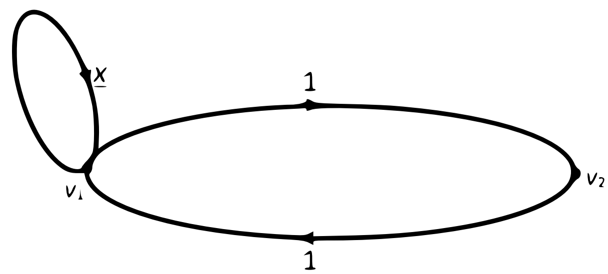

Figure 1. Weighted Fibonacci Digraph D_1

It follows by induction from its *weighted adjacency matrix* $Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ that

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $Q = Q(x)$ and $n \geq 1$ [5].

A *walk* from vertex v_i to vertex v_j is a sequence v_i-e_i-v_{i+1}-$\cdots$-v_{j-1}-e_{j-1}-v_j of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is *closed* if $v_i = v_j$; otherwise, it is *open*. The *length* of a walk is the number of edges in the walk. The *weight* of a walk is the product of the weights of the edges along the walk.

We can employ the matrix Q^n to compute the weight of a walk of length n from any vertex v_i to any vertex v_j, as the following theorem shows [5].

Theorem 6.1. *Let M be the weighted adjacency matrix of a weighted, connected digraph with vertices $v_1, v_2, \ldots, v_k$. Then the ijth entry of the matrix M^n gives the sum of the weights of all walks of length n from v_i to v_j, where $n \geq 1$.*

The next result follows from this theorem.

Corollary 6.2. *The ijth entry of Q^n gives the sum of the weights of all walks of length n from v_i to v_j in the weighted digraph D_1, where $1 \leq i, j \leq n$.*

It follows by this corollary that the sum of the weights of closed walks of length n originating at v_1 in the digraph is f_{n+1}. This fact plays a central role in the graph-theoretic proofs.

6.1. **Proof of Identity** (2.1)**.**

Proof. Let A, B, and C denote the sets of closed walks of lengths n, $n-1$, and $n-2$, all originating at v_1. The sums of the weights of all walks in them are f_{n+1}, f_n, and f_{n-1}, respectively.

We define the sum S_1 of the weights of the elements in the product set $A \times A \times A \times A$ to be the product of the sums of weights in each component; so $S_1 = f_{n+1}^4$. Similarly, the sum S_2 of the weights in $B \times B \times B \times B$ equals $S_2 = f_n^4$, and the sum S_3 in $B \times B \times B \times C$ equals $S_3 = f_n^3 f_{n-1}$. Consequently, the sum $S = S_1 + 2x^2 S_2 + 4x S_3$ is given by

$$S = f_{n+1}^4 + 2x^2 f_n^4 + 4x f_n^3 f_{n-1}.$$

We will now establish that $S = x f_{4n} + f_{n-1}^4$ in a different way. To this end, let (u, v, w, z) be an arbitrary element of the product set $A \times A \times A \times A$. Table 3 shows the possible cases for such quadruples and the corresponding sums of weights.

Table 3: Sum of the Weights of the Quadruples

u begins with a loop?	v begins with a loop?	w begins with a loop?	z begins with a loop?	sum of the weights of the quadruples (u,v,w,z)
yes	yes	yes	yes	$x^4 f_n^4$
yes	yes	yes	no	$x^3 f_n^3 f_{n-1}$
yes	yes	no	yes	$x^3 f_n^3 f_{n-1}$
yes	yes	no	no	$x^2 f_n^2 f_{n-1}^2$
yes	no	yes	yes	$x^3 f_n^3 f_{n-1}$
yes	no	yes	no	$x^2 f_n^2 f_{n-1}^2$
yes	no	no	yes	$x^2 f_n^2 f_{n-1}^2$
yes	no	no	no	$x f_n f_{n-1}^3$
no	yes	yes	yes	$x^3 f_n^3 f_{n-1}$
no	yes	yes	no	$x^2 f_n^2 f_{n-1}^2$
no	yes	no	yes	$x^2 f_n^2 f_{n-1}^2$
no	yes	no	no	$x f_n f_{n-1}^3$
no	no	yes	yes	$x^2 f_n^2 f_{n-1}^2$
no	no	yes	no	$x f_n f_{n-1}^3$
no	no	no	yes	$x f_n f_{n-1}^3$
no	no	no	no	f_{n-1}^4

It follows from the table that

$$\begin{aligned} S_1 &= x^4 f_n^4 + 4x^3 f_n^3 f_{n-1} + 6x^2 f_n^2 f_{n-1}^2 + 4x f_n f_{n-1}^3 + f_{n-1}^4 \\ &= f_{n+1}^4. \end{aligned}$$

This implies $S_2 = f_n^4$.

To compute S_3, we let (u,v,w,z) be an arbitrary element of the product set $B \times B \times B \times C$. Table 4 shows the possible cases for such quadruples and the corresponding sums of weights.

Table 4: Sum of the Weights of the Quadruples

u begins with a loop?	v begins with a loop?	w begins with a loop?	z begins with a loop?	sum of the weights of the quadruples (u,v,w,z)
yes	yes	yes	yes	$x^4 f_{n-1}^3 f_{n-2}$
yes	yes	yes	no	$x^3 f_{n-1}^3 f_{n-3}$
yes	yes	no	yes	$x^3 f_{n-1}^2 f_{n-2}^2$
yes	yes	no	no	$x^2 f_{n-1}^2 f_{n-2} f_{n-3}$
yes	no	yes	yes	$x^3 f_{n-1}^2 f_{n-2}^2$
yes	no	yes	no	$x^2 f_{n-1}^2 f_{n-2} f_{n-3}$
yes	no	no	yes	$x^2 f_{n-1} f_{n-2}^3$
yes	no	no	no	$x f_{n-1} f_{n-2}^2 f_{n-3}$
no	yes	yes	yes	$x^3 f_{n-1}^2 f_{n-2}^2$
no	yes	yes	no	$x^2 f_{n-1}^2 f_{n-2} f_{n-3}$
no	yes	no	yes	$x^2 f_{n-1} f_{n-2}^3$
no	yes	no	no	$x f_{n-1} f_{n-2}^2 f_{n-3}$
no	no	yes	yes	$x f_{n-1} f_{n-2}^3$
no	no	yes	no	$x f_{n-1} f_{n-2}^2 f_{n-3}$
no	no	no	yes	$x f_{n-2}^4$
no	no	no	no	$f_{n-2}^3 f_{n-3}$

It follows from the table that

$$\begin{aligned} S_3 &= (x^4 f_{n-1}^3 f_{n-2} + x^3 f_{n-1}^3 f_{n-3}) + (3x^3 f_{n-1}^2 f_{n-2}^2 + 3x^2 f_{n-1}^2 f_{n-2} f_{n-3}) \\ &\quad + (3x^2 f_{n-1} f_{n-2}^3 + 3x f_{n-1} f_{n-2}^2 f_{n-3}) + (x f_{n-2}^4 + f_{n-2}^3 f_{n-3}) \\ &= x^3 f_{n-1}^4 + 3x^2 f_{n-1}^3 f_{n-2} + 3x f_{n-1}^2 f_{n-2}^2 + f_{n-1} f_{n-2}^3 \\ &= (x^3 f_{n-1}^4 + x^2 f_{n-1}^3 f_{n-2}) + (2x^2 f_{n-1}^2 f_{n-2} + 2x f_{n-1}^2 f_{n-2}^2) + (x f_{n-1}^2 f_{n-2}^2 + f_{n-1} f_{n-2}^3) \end{aligned}$$

$$\begin{aligned}
&= x^2 f_n f_{n-1}^3 + 2x f_n f_{n-1}^2 f_{n-2} + f_n f_{n-1} f_{n-2}^2 \\
&= f_n f_{n-1}\left(x^2 f_{n-1}^2 + 2x f_{n-1} f_{n-2} + f_{n-2}^2\right) \\
&= f_n^3 f_{n-1}.
\end{aligned}$$

Notice that

$$\begin{aligned}
f_{3n-1} &= f_{2n} f_n + f_{2n-1} f_{n-1}; \\
x f_{3n-1} &= \left(f_{n+1}^2 - f_{n-1}^2\right) f_n + x\left(f_n^2 + f_{n-1}^2\right) f_{n-1}; \\
x f_{3n-1} f_n &= (x f_n + f_{n-1})^2 f_n^2 - f_n^2 f_{n-1}^2 + x f_n^3 f_{n-1} + x f_n f_{n-1}^3 \\
&= x^2 f_n^4 + 3x^2 f_n^3 f_{n-1} + x f_n f_{n-1}^3.
\end{aligned}$$

Collecting the values of S_1, S_2, and S_3, and using the identity $x f_{3n} = f_{n+1}^3 + x f_n^3 - f_{n-1}^3$ [6], we get

$$\begin{aligned}
S &= f_{n+1}^4 + 2x^2 f_n^4 + 4x f_n^3 f_{n-1} \\
&= \left(f_{n+1}^4 + x^2 f_n^4 + x f_n^3 f_{n-1} - x f_n f_{n-1}^3 - f_{n-1}^4\right) + f_{n-1}^4 + \left(x^2 f_n^4 + 3x f_{n-1}^3 + x f_n f_{n-1}^3\right) \\
&= f_{n+1}^4 + x f_n^3 (x f_n + f_{n-1}) - (x f_n + f_{n-1}) f_{n-1}^3 + f_{n-1}^4 + x f_{3n-1} f_n \\
&= \left(f_{n+1}^3 + x f_n^3 - f_{n-1}^3\right) f_{n+1} + f_{n-1}^4 + x f_{3n-1} f_n \\
&= x(f_{3n} f_{n+1} + f_{3n-1} f_n) + f_{n-1}^4 \\
&= x f_{4n} + f_{n-1}^4.
\end{aligned}$$

Equating the two values of S yields the desired result, as expected. □

6.2. **Proof of Identity** (4.1)**.** The graph-theoretic proof of identity (4.1) hinges on the *Jacobsthal digraph* D_2 in Figure 2 [4].

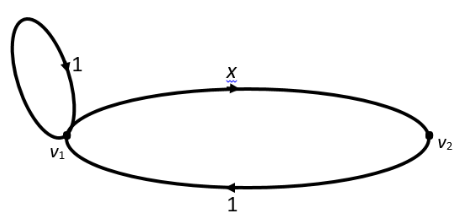

FIGURE 2. Weighted Jacobsthal Digraph D_2

It follows by induction from its weighted adjacency matrix $M = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix}$ that

$$M^n = \begin{bmatrix} J_{n+1}(x) & xJ_n(x) \\ J_n(x) & xJ_{n-1}(x) \end{bmatrix},$$

where $n \geq 1$.

Consequently, the sum of the weights of closed walks of length n originating at v_1 is $J_{n+1}(x)$. This fact plays a pivotal role in the graph-theoretic proof.

Proof. Let A, B, and C denote the sets of closed walks of lengths n, $n-1$, and $n-2$, all originating at v_1. The sums of the weights of all walks in them are J_{n+1}, J_n, and J_{n-1}, respectively. As before, we define the sum S_1 of the weights of the elements in the product set $A \times A \times A \times A$ to be the product of the sums of weights in each component; so $S_1 = J_{n+1}^4$. Similarly, the sum S_2 of the weights in $B \times B \times B \times B$ equals $S_2 = J_n^4$, and the sum S_3 in $B \times B \times B \times C$ equals $S_3 = J_n^3 J_{n-1}$. Consequently, the sum $S = S_1 + 2x^2 S_2 + 4x^2 S_3$ is given by

$$S = J_{n+1}^4 + 2x^2 J_n^4 + 4x^2 J_n^3 J_{n-1}.$$

We will now compute the sum S in a different way. To this end, let (u, v, w, z) be an arbitrary element of the product set $A \times A \times A \times A$. Table 5 shows the various possible cases for such quadruples and the corresponding sums of weights.

Table 5: Sum of the Weights of the Quadruples

u begins with a loop?	v begins with a loop?	w begins with a loop?	z begins with a loop?	sum of the weights of the quadruples (u, v, w)
yes	yes	yes	yes	J_n^4
yes	yes	yes	no	$xJ_n^3 J_{n-1}$
yes	yes	no	yes	$xJ_n^3 J_{n-1}$
yes	yes	no	no	$x^2 J_n^2 J_{n-1}^2$
yes	no	yes	yes	$xJ_n^3 J_{n-1}$
yes	no	yes	no	$x^2 J_n^2 J_{n-1}^2$
yes	no	no	yes	$x^2 J_n^2 J_{n-1}^2$
yes	no	no	no	$x^3 J_n J_{n-1}^3$
no	yes	yes	yes	$xJ_n^3 J_{n-1}$
no	yes	yes	no	$x^2 J_n^2 J_{n-1}^2$
no	yes	no	yes	$x^2 J_n^2 J_{n-1}^2$
no	yes	no	no	$x^3 J_n J_{n-1}^3$
no	no	yes	yes	$x^2 J_n^2 J_{n-1}^2$
no	no	yes	no	$x^3 J_n J_{n-1}^3$
no	no	no	yes	$x^3 J_n J_{n-1}^3$
no	no	no	no	$x^4 J_{n-1}^4$

It follows from the table that

$$\begin{aligned} S_1 &= J_n^4 + 4xJ_n^3 J_{n-1} + 6x^2 J_n^2 J_{n-1}^2 + 4x^3 J_n J_{n-1}^3 + x^4 J_{n-1}^4 \\ &= J_{n+1}^4. \end{aligned}$$

This implies $S_2 = J_n^4$.

To compute S_3, suppose (u, v, w, z) is an arbitrary element of $B \times B \times B \times C$. It follows from Table 5 that the sum S_3 of the weights of such elements is given by

$$\begin{aligned} S_3 &= (J_{n-1}^3 J_{n-2} + xJ_{n-1}^3 J_{n-3}) + (3xJ_{n-1}^2 J_{n-2}^2 + 3x^2 J_{n-1}^2 J_{n-2} J_{n-3}) \\ &\quad + (3x^2 J_{n-1} J_{n-2}^3 + 3x^3 J_{n-1} J_{n-2}^2 J_{n-3}) + (x^3 J_{n-2}^4 + x^4 J_{n-2}^3 J_{n-3}) \\ &= J_{n-1}^4 + 3xJ_{n-1}^3 J_{n-2} + 3x^2 J_{n-1}^2 J_{n-2}^3 + x^3 J_{n-1} J_{n-2}^3 \\ &= (J_{n-1}^4 + xJ_{n-1}^3 J_{n-2}) + (2xJ_{n-1}^3 J_{n-2} + 2x^2 J_{n-1}^2 J_{n-2}^3) + (x^2 J_{n-1}^2 J_{n-2}^3 + x^3 J_{n-1} J_{n-2}^3) \\ &= J_n J_{n-1}^3 + 2xJ_n J_{n-1}^2 J_{n-2} + x^2 J_n J_{n-1} J_{n-2}^2 \\ &= J_n J_{n-1}(J_{n-1}^2 + 2xJ_{n-1} J_{n-2} + x^2 J_{n-2}^2) \\ &= J_n^3 J_{n-1}. \end{aligned}$$

Thus,

$$\begin{aligned} S &= S_1 + 2xS_2 + 4x^2 S_3 \\ &= J_{n+1}^4 + 2xJ_n^4 + 4x^2 J_n^3 J_{n-1}. \end{aligned}$$

To rewrite this value of S in a different form, consider J_{3n-1}. By the Jacobsthal addition formula, we have

$$\begin{aligned} J_{3n-1} &= J_{2n}J_n + xJ_{2n-1}J_{n-1} \\ &= \left(J_{n+1}^2 - x^2 J_{n-1}^2\right) J_n + xJ_{n-1}\left(J_n^2 + xJ_{n-1}^2\right) \\ &= J_{n+1}^2 J_n - x^2 J_n J_{n-1}^2 + xJ_n^2 J_{n-1} + x^2 J_{n-1}^3; \\ J_{3n-1}J_n &= (J_n + xJ_{n-1})^2 J_n^2 - x^2 J_n^2 J_{n-1}^2 + xJ_n^2 J_{n-1} + x^2 J_n J_{n-1}^3 \\ &= J_n^4 + 3xJ_n^3 J_{n-1} + x^2 J_n J_{n-1}^3. \end{aligned}$$

Using the identity $J_{3n} = J_{n+1}^3 + xJ_n^3 - x^3 J_{n-1}^3$ [4] and the *Jacobsthal addition formula*, we can now rewrite the value of S:

$$\begin{aligned} S &= \left(J_{n+1}^4 + xJ_n^4 + x^2 J_n^3 J_{n-1} - x^3 J_n J_{n-1}^3 - x^4 J_{n-1}^4\right) + x\left(J_n^4 + 3xJ_n^3 J_{n-1} + x^2 J_{n-1}^3\right) + x^4 J_{n-1}^4 \\ &= J_{n+1}^4 + xJ_n^3(J_n + xJ_{n-1}) - x^3 J_{n-1}^3(J_n + xJ_{n-1}) + xJ_{3n-1}J_n + x^4 J_{n-1}^4 \\ &= \left(J_{n+1}^3 + xJ_n^3 - x^3 J_{n-1}^3\right) J_{n+1} + xJ_{3n-1}J_n + x^4 J_{n-1}^4 \\ &= (J_{3n}J_{n+1} + xJ_{3n-1}J_n) + x^4 J_{n-1}^4 \\ &= J_{4n} + x^4 J_{n-1}^4. \end{aligned}$$

Equating the two values of S yields the desired result. □

7. Acknowledgment

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