EXTENDED GIBONACCI SUMS OF POLYNOMIAL PRODUCTS OF ORDERS 4 AND 5 REVISITED

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Abstract. We explore two gibonacci sums of polynomial products of orders 4 and 5, and their Pell, Jacobsthal, Vieta, and Chebyshev implications. We also confirm two gibonacci and two Jacobsthal results using graph-theoretic tools.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary complex variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 7, 10]. *Pell polynomials* $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [7].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial* [3, 10]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$ and $j_n(1) = L_n$.

Let $a(x) = x$ and $b(x) = -1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = V_n(x)$, the nth *Vieta polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = v_n(x)$, the nth *Vieta-Lucas polynomial* [4, 10].

Finally, let $a(x) = 2x$ and $b(x) = -1$. When $z_0(x) = 1$ and $z_1(x) = x$, $z_n(x) = T_n(x)$, the nth *Chebyshev polynomial of the first kind*; and when $z_0(x) = 1$ and $z_1(x) = 2x$, $z_n(x) = U_n(x)$, the nth *Chebyshev polynomial of the second kind* [4, 10].

The Jacobsthal, Vieta, and Chebyshev subfamilies are closely related by the relationships in Table 1, where $i = \sqrt{-1}$ [4, 6, 10].

$J_n(x)$	$=$	$x^{(n-1)/2} f_n(1/\sqrt{x})$	$j_n(x)$	$=$	$x^{n/2} l_n(1/\sqrt{x})$
$V_n(x)$	$=$	$i^{n-1} f_n(-ix)$	$v_n(x)$	$=$	$i^n l_n(-ix)$
$V_n(2x)$	$=$	$U_{n-1}(x)$	$v_n(2x)$	$=$	$2T_n(x)$

Table 1: Relationships Among the Subfamilies

In the interest of clarity, concision, and convenience, we omit the argument in the functional notation, when there is no ambiguity; so z_n will mean $z_n(x)$. We let $g_n = f_n$ or l_n, and $c_n = J_n(x)$ or $j_n(x)$; we also omit much of the basic algebra.

A *gibonacci polynomial product of order m* is a product of gibonacci polynomials g_{n+k} of the form $\prod_{k \geq 0} g_{n+k}^{s_j}$, where $\sum_{s_j \geq 1} s_j = m$ [6, 11].

1.1. **Some Gibonacci and Jacobsthal Sums of Polynomial Products of Orders 2 and 3.** Table 2 shows some well-known identities involving sums of products of orders 2 and 3 of gibonacci and Jacobsthal polynomials, where $\Delta^2 = x^2+4$ and $D^2 = 4x+1$ [7, 9, 10]. They will be useful in our discourse.

g_{m+n}	$=$	$f_{m+1}g_n + f_m g_{n-1}$	j_{m+n}	$=$	$J_{m+1}c_n + xJ_m c_{n-1}$
xf_{2n}	$=$	$f_{n+1}^2 - f_{n-1}^2$	J_{2n}	$=$	$J_{n+1}^2 - x^2J_{n-1}^2$
$2l_{2n}$	$=$	$l_n^2 + \Delta^2 f_n^2$	$2j_{2n}$	$=$	$j_n^2 + D^2J_n^2$
xl_{2n+1}	$=$	$l_{n+1}^2 - \Delta^2 f_n^2$	j_{2n+1}	$=$	$j_{n+1}^2 - xD^2J_n^2$
$\Delta^2 f_{2n+1}$	$=$	$l_{n+1}^2 + l_n^2$	D^2J_{2n+1}	$=$	$j_{n+1}^2 + xj_n^2$
$x\Delta^2 f_{2n}$	$=$	$l_{n+1}^2 - l_{n-1}^2$	D^2J_{2n}	$=$	$j_{n+1}^2 - x^2j_{n-1}^2$
$x\Delta^2 l_{3n}$	$=$	$l_{n+1}^3 + xl_n^3 - l_{n-1}^3$	D^2j_{3n}	$=$	$J_{n+1}^3 + xJ_n^3 - x^3J_{n-1}^3$

Table 2: Some Gibonacci and Jacobsthal Sums of Polynomial Products of Orders 2 and 3

With this background, we now explore similar formulas for l_{4n} and l_{5n} as sums of gibonacci polynomial products of order 4 and 5, respectively; and similarly for j_{4n} and j_{5n}.

2. A Sum of Gibonacci Polynomial Products of Order 4

By the gibonacci addition formula in Table 1, we have

$$\begin{aligned}
l_{4n} &= f_{2n+1}l_{2n} + f_{2n}l_{2n-1};\\
2x^2\Delta^2 l_{4n} &= x^2(\Delta^2 f_{2n+1})(2l_{2n}) + 2\Delta^2(xf_{2n})(xl_{2n-1})\\
&= x^2(l_{n+1}^2 + l_n^2)(l_n^2 + \Delta^2 f_n^2) + 2x\Delta^2 l_n f_n(l_n^2 - \Delta^2 f_{n-1}^2)\\
&= x^2\left(l_{n+1}^2 l_n^2 + \Delta^2 l_{n+1}^2 f_n^2 + l_n^4 + \Delta^2 l_n^2 f_n^2\right) + 2x\Delta^2\left(l_n^3 f_n - \Delta^2 l_n f_n f_{n-1}^2\right)\\
&= l_{n+1}^2(l_{n+1} - l_{n-1})^2 + x^2\Delta^2 l_{n+1}^2 f_n^2 + x^2 l_n^4 + x^2\Delta^2 l_n^2 f_n^2 + 2x\Delta^2 l_n^3 f_n\\
&\quad - 2x\Delta^2 l_n f_n f_{n-1}^2(2l_{n+1} - xl_n)\\
&= l_{n+1}^4 - 2l_{n+1}^3 l_{n-1} + l_{n+1}^2 l_{n-1}^2 + x^2\Delta^2 l_{n+1}^2 f_n^2 - 4x\Delta^2 l_{n+1} l_n f_{n-1}^2 + x^2 l_n^4\\
&\quad + 2x\Delta^2 l_n^3 f_n + x^2\Delta^2 l_n^2 f_n^2 + 2x^2\Delta^2 l_n^2 f_{n-1}^2\\
&= l_{n+1}^4 - l_{n+1}^3 l_{n-1} - xl_{n+1}^2 l_n l_{n-1} + x^2\Delta^2 l_{n+1}^2 f_n^2 - 4x\Delta^2 l_{n+1} l_n f_{n-1}^2 + x^2 l_n^4\\
&\quad + 2x\Delta^2 l_n^3 f_n + x^2\Delta^2 l_n^2 f_n^2 + 2x^2\Delta^2 l_n^2 f_{n-1}^2. \qquad (2.1)
\end{aligned}$$

In particular, we have

$$\begin{aligned}
10L_{4n} &= L_{n+1}^4 - L_{n+1}^3 L_{n-1} - L_{n+1}^2 L_n L_{n-1} + 5L_{n+1}^2 F_n^2 - 20L_{n+1}L_n F_{n-1}^2 + L_n^4\\
&\quad + 10L_n^3 F_n + 5L_n^2 F_n^2 + 10L_n^2 F_{n-1}^2. \qquad (2.2)
\end{aligned}$$

2.1. **Interesting Consequences.** It follows from equation (2.2) that $L_{n+1}^4 + L_n^4 \equiv L_{n+1}^3 L_{n-1} + L_{n+1}^2 L_n L_{n-1} \pmod 5$. Consequently, $L_{n+1}^3 L_n \equiv -L_n(L_n^3 - L_{n+1}^2 L_{n-1}) \pmod 5$. But $(L_n, 5) = 1$; so $L_{n+1}^3 + L_n^3 \equiv L_{n+1}^2 L_{n-1} \pmod 5$.

Because $(L_n, 5) = 1$, it follows by the well-known *Fermat's Little Theorem* [5] that $L_n^4 \equiv 1 \pmod 5$. This implies $L_{n+1}^4 + L_n^4 \equiv 2 \pmod 5$. As a result, $L_{n+1}^3 L_{n-1} + L_{n+1}^2 L_n L_{n-1} \equiv 2 \pmod 5$. Fermat's Little Theorem also implies that $L_{n+1}^p + L_n^p \equiv L_{n+2} \pmod p$, where p is a prime.

It is known that if a is a positive integer and $p > 3$ is a prime, then $a^p \equiv a \pmod{6p}$ [2, 5]. This implies $L_{n+1}^p + L_n^p \equiv L_{n+2} \pmod{6p}$, where $p > 3$.

2.2. **Pell-Lucas Byproducts.** It follows from identity (2.1) that

$$\begin{aligned} 32x^2(x^2+1)q_{4n} &= q_{n+1}^4 - q_{n+1}^3 q_{n-1} - 2xq_{n+1}^2 q_n q_{n-1} + 16x^2(x^2+1)q_{n+1}^2 p_n^2 \\ &\quad - 32x(x^2+1)q_{n+1}q_n p_{n-1}^2 + 4x^2 q_n^4 + 16x(x^2+1)q_n^3 p_n \\ &\quad + 16x^2(x^2+1)q_n^2 p_n^2 + 32x^2(x^2+1)q_n^2 p_{n-1}^2. \end{aligned}$$

This yields

$$\begin{aligned} 8Q_{4n} &= Q_{n+1}^4 - Q_{n+1}^3 Q_{n-1} - 2Q_{n+1}^2 Q_n Q_{n-1} + 8Q_{n+1}^2 P_n^2 - 16Q_{n+1}Q_n P_{n-1}^2 + 4Q_n^4 \\ &\quad + 16Q_n^3 P_n + 8Q_n^2 P_n^2 + 16Q_n^2 P_{n-1}^2. \end{aligned}$$

Consequently, $Q_{n+1}^4 + 4Q_n^4 \equiv Q_{n+1}^3 Q_{n-1} + 2Q_{n+1}^2 Q_n Q_{n-1} \pmod 8$.

Because Q_n is odd, it follows by *Euler's theorem* [5] that $Q_n^{\varphi(8)} = Q_n^4 \equiv 1 \pmod 8$, where φ denotes *Euler's phi function*. Consequently, $Q_{n+1}^4 + 4Q_n^4 \equiv 5 \pmod 8$.

Next, we explore the Jacobsthal-Lucas consequences of identity (2.1).

2.3. **Jacobsthal-Lucas Implications.** Replacing x with $1/\sqrt{x}$ and multiplying the resulting equation with $x^{4n/2}$, we get

$$\begin{aligned} 2D^2\left(x^{4n/2}l_{4n}\right) &= \left[x^{(n+1)/2}l_{n+1}\right]^4 - x\left[x^{(n+1)/2}l_{n+1}\right]^3\left[x^{(n-1)/2}l_{n-1}\right] \\ &\quad - x\left[x^{(n+1)/2}l_{n+1}\right]^2\left[x^{n/2}l_n\right]\left[x^{(n-1)/2}l_{n-1}\right] \\ &\quad + D^2\left[x^{(n+1)/2}l_{n+1}\right]^2\left[x^{(n-1)/2}f_n\right]^2 \\ &\quad - 4D^2x^2\left[x^{(n+1)/2}l_{n+1}\right]\left[x^{n/2}l_n\right]\left[x^{(n-2)/2}f_{n-1}\right]^2 \\ &\quad + x\left[x^{n/2}l_n\right]^4 + 2D^2x\left[x^{n/2}l_n\right]^3\left[x^{(n-1)/2}f_n\right] \\ &\quad + D^2x\left[x^{n/2}l_n\right]^2\left[x^{(n-1)/2}f_n\right]^2 + 2D^2x^2\left[x^{n/2}l_n\right]^2\left[x^{(n-2)/2}f_{n-1}\right]^2; \\ 2D^2 j_{4n} &= j_{n+1}^4 - xj_{n+1}^3 j_{n-1} - xj_{n+1}^2 j_n j_{n-1} + D^2 j_{n+1}^2 J_n^2 - 4D^2x^2 j_{n+1} j_n J_{n-1}^2 \\ &\quad + xj_n^4 + 2D^2xj_n^3 J_n + D^2xj_n^2 J_n^2 + 2D^2x^2 j_n^2 J_{n-1}^2, \end{aligned} \tag{2.3}$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Identity (2.3) yields

$$\begin{aligned} 18j_{4n} &= j_{n+1}^4 - 2j_{n+1}^3 j_{n-1} - 2j_{n+1}^2 j_n j_{n-1} + 9j_{n+1}^2 J_n^2 - 144j_{n+1} j_n J_{n-1}^2 \\ &\quad + 2j_n^4 + 36j_n^3 J_n + 18j_n^2 J_n^2 + 72j_n^2 J_{n-1}^2. \end{aligned} \tag{2.4}$$

It follows from identity (2.4) that

$$\begin{aligned} j_{n+1}^4 + 2j_n^4 &\equiv 2j_{n+1}^3 j_{n-1} + 2j_{n+1}^2 j_n j_{n-1} \pmod 9 \\ &\equiv 2j_{n+1}^2 j_{n-1}(j_{n+1} + j_n) \pmod 9 \\ &\equiv 3\cdot 2^{n+1} j_{n+1}^2 j_{n-1} \pmod 9. \end{aligned}$$

Because $(j_n, 9) = 1$, Euler's theorem implies that $j_n^6 \equiv 1 \pmod 9$. Consequently, $j_{n+1}^6 + j_n^6 \equiv 2 \pmod 9$ and $j_{n+1}^6 + 2j_n^6 \equiv 3 \pmod 9$.

Next, we pursue a gibonacci sum for $x^2\Delta^4 l_{5n}$.

3. A Sum of Gibonacci Polynomial Products of Order 5

First, notice by the gibonacci addition formula that

$$\begin{aligned} l_{3n-1} &= f_{2n}l_n + f_{2n-1}l_{n-1}; \\ x\Delta^2 l_{3n-1} &= (x\Delta^2 f_{2n})l_n + x(\Delta^2 f_{2n-1})l_{n-1} \\ &= \left(l_{n+1}^2 - l_{n-1}^2\right) l_n + \left({}_n^2 + l_{n-1}^2\right) x l_{n-1} \\ &= l_{n+1}^2 l_n + x l_n^2 l_{n-1} - l_n l_{n-1}^2 + x l_{n-1}^3. \end{aligned}$$

Again by the addition formula, we then get

$$\begin{aligned} l_{5n} &= f_{2n+1}l_{3n} + f_{2n}l_{3n-1}; \\ x^2\Delta^4 l_{5n} &= x(\Delta^2 f_{2n+1})(x\Delta^2 l_{3n}) + (x\Delta^2 f_{2n})(x\Delta^2 l_{3n-1}) \\ &= x\left(l_{n+1}^2 + l_n^2\right)\left(l_{n+1}^3 + x l_n^3 - l_{n-1}^3\right) \\ &\quad + \left(l_{n+1}^2 - l_{n-1}^2\right)\left(l_{n+1}^2 l_n + x l_n^2 l_{n-1} - l_n l_{n-1}^2 + x l_{n-1}^3\right) \\ &= x l_{n+1}^5 + l_{n+1}^4 l_n + x l_{n+1}^3 l_n^2 + x^2 l_{n+1}^2 l_n^3 + x l_{n+1}^2 l_n^2 l_{n-1} - 2 l_{n+1}^2 l_n l_{n-1}^2 \\ &\quad + x^2 l_n^5 - 2x l_n^2 l_{n-1}^3 + l_n l_{n-1}^4 - x l_{n-1}^5. \end{aligned} \tag{3.1}$$

In particular, this yields

$$\begin{aligned} 25L_{5n} &= L_{n+1}^5 + L_{n+1}^4 L_n + L_{n+1}^3 L_n^2 + L_{n+1}^2 L_n^3 + L_{n+1}^2 L_n^2 L_{n-1} - 2L_{n+1}^2 L_n L_{n-1}^2 \\ &\quad + L_n^5 - 2L_n^2 L_{n-1}^3 + L_n L_{n-1}^4 - L_{n-1}^5. \end{aligned} \tag{3.2}$$

For example,

$$\begin{aligned} &L_6^5 + L_6^4 L_5 + L_6^3 L_5^2 + L_6^2 L_5^3 + L_6^2 L_5^2 L_4 \\ &- 2L_6^2 L_5 L_4^2 + L_5^5 - 2L_5^2 L_4^3 + L_5 L_4^4 - L_4^5 \;=\; 4{,}194{,}025 = 25L_{25}. \end{aligned}$$

Identity (3.2) implies that $25L_{5n} \equiv L_{n+1}^5 - L_{n-1}^5 \pmod{L_n}$. Because $L_{n+1}^5 \equiv L_{n-1}^5 \pmod{L_n}$ by the binomial theorem, it follows that $25L_{5n} \equiv 0 \pmod{L_n}$. But $(L_n, 5) = 1$, so $L_{5n} \equiv 0 \pmod{L_n}$. This also follows from the property that $l_{5n} = l_n^5 - 5(-1)^n l_n^3 + 5l_n$ [7].

Next, we investigate the Jacobsthal implications of identity (3.1).

3.1. Jacobsthal Byproducts.

Replacing x with $1/\sqrt{x}$ in equation (3.1) and multiplying the resulting equation with $x^{(5n+5)/2}$, we get

$$\begin{aligned} D^4\left[x^{(5n/2)} l_{5n}\right] &= \left[x^{(n+1)/2} l_{n+1}\right]^5 + x\left[x^{(n+1)/2} l_{n+1}\right]^4\left[x^{n/2} l_n\right] + x\left[x^{(n+1)/2} l_{n+1}\right]^3\left[x^{n/2} l_n\right]^2 \\ &\quad + x\left[x^{(n+1)/2} l_{n+1}\right]^2\left[x^{n/2} l_n\right]^3 + x^2\left[x^{(n+1)/2} l_{n+1}\right]^2\left[x^{n/2} l_n\right]^2\left[x^{(n-1)/2} l_{n-1}\right] \\ &\quad - 2x^3\left[x^{(n+1)/2} l_{n+1}\right]^2\left[x^{n/2} l_n\right]\left[x^{(n-1)/2} l_{n-1}\right]^2 \\ &\quad - x^2\left[x^{n/2} l_n\right]^5 - 2x^4\left[x^{n/2} l_n\right]^2\left[x^{(n-1)/2} l_{n-1}\right]^3 \\ &\quad + x^5\left[x^{n/2} l_n\right]\left[x^{(n-1)/2} l_{n-1}\right]^4 - x^5\left[x^{(n-1)/2} l_{n-1}\right]^5; \\ D^4 j_{5n} &= j_{n+1}^5 + x j_{n+1}^4 j_n + x j_{n+1}^3 j_n^2 + x^2 j_{n+1}^2 j_n^3 + x^2 j_{n+1}^2 j_n^2 j_{n-1} \\ &\quad - 2x^3 j_{n+1}^2 j_n j_{n-1}^2 - x^2 j_n^5 - 2x^4 j_n^2 j_{n-1}^3 + x^5 j_n j_{n-1}^4 - x^5 j_{n-1}^5, \end{aligned} \tag{3.3}$$

where $l_n = l_n(1/\sqrt{x})$, $j_n = j_n(x)$, and $D^2 = 4x + 1$.

In particular, we have

$$\begin{aligned}81j_{5n} &= j_{n+1}^5 + 2j_{n+1}^4 j_n + 2j_{n+1}^3 j_n^2 + 4j_{n+1}^2 j_n^3 + 4j_{n+1}^2 j_n^2 j_{n-1}\\ &\quad - 16j_{n+1}^2 j_n j_{n-1}^2 - 4j_n^5 - 32j_n^2 j_{n-1}^3 + 32j_n j_{n-1}^4 - 32j_{n-1}^5. \qquad (3.4)\end{aligned}$$

Next, we confirm identities (2.1) and (2.3) using graph-theoretic tools.

4. Graph-Theoretic Confirmations

Consider the *Fibonacci digraph* D_1 in Figure 1 with vertices v_1 and v_2, where a *weight* is assigned to each edge [7, 8].

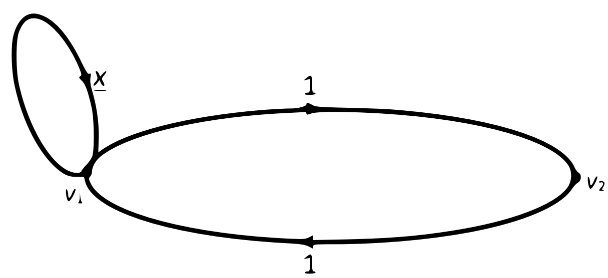

Figure 1. Weighted Fibonacci Digraph D_1

It follows, by induction, from its *weighted adjacency matrix* $Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ that

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \geq 1$.

The ijth entry of Q^n gives the sum of the weights of all walks of length n from v_i to v_j in the weighted digraph D_1, where $1 \leq i, j \leq n$ [6, 7]. So, the sum of the weights of closed walks of length n originating at v_1 in the digraph is f_{n+1} and that of those originating at v_2 is f_{n-1}. Consequently, the sum of the weights of all closed walks of length n is $f_{n+1} + f_{n-1} = l_n$. These facts play a major role in the graph-theoretic proof of identity (2.1).

4.1. **Confirmation of Identity** (2.1)**.** Let A and B denote the sets of closed walks of lengths $n+1$ and n in the digraph, respectively; and C and D the sets of those of lengths $n-1$ and $n-2$ originating at v_1, respectively. We define the sum S_1 of the weights of the elements in $A \times A \times A \times A$ to be the product of the sums of the weights in each component. This implies $S_1 = l_{n+1}^4$. Analogously, let S_2, S_3, S_4, S_5, and S_6 denote the sum of the weights in $A \times A \times C \times C$, $B \times B \times B \times B$, $B \times B \times B \times C$, $B \times B \times C \times C$, and $B \times B \times D \times D$, respectively. Then, $S_2 = l_{n+1}^2 f_n^2$, $S_3 = l_n^4$, $S_4 = l_n^3 f_n$, $S_5 = l_n^2 f_n^2$, and $S_6 = l_n^2 f_{n-1}^2$. Thus, the sum

$$S = S_1 + x^2\Delta^2 S_2 + x^2 S_3 + 2x\Delta^2 S_4 + x^2\Delta^2 S_5 + 2x^2\Delta^2 S_6$$

is given by

$$S = l_{n+1}^4 + x^2\Delta^2 l_{n+1}^2 f_n^2 + x^2 l_n^4 + 2x\Delta^2 l_n^3 f_n + x^2\Delta^2 l_n^2 f_n^2 + 2x^2\Delta^2 l_n^2 f_{n-1}^2. \qquad (4.1)$$

We will now compute the sum S in a different way in six steps.

Step 1. Let v be an arbitrary walk in A. Suppose it originates at v_1. If v begins with a loop, the sum of the weights of such walks is xf_{n+1}, then the sum is $1 \cdot 1 \cdot 1 \cdot f_n = f_n$. Thus, the sum of such walks v is $xf_{n+1} + f_n = f_{n+2}$. On the other hand, if v originates at v_2, then the sum of the weights of such walks is f_n. Combining the two cases, the sum of the weights of all elements in A is l_{n+1}; so $S_1 = l_{n+1}^4$.

Step 2. Let v be an arbitrary walk in C. If v begins with a loop, the sum of the weights of such walks is xf_{n-1}; otherwise, the sum is $1 \cdot 1 \cdot 1 \cdot f_{n-2} = f_{n-2}$. So, the sum of the walks in C is $xf_{n-1} + f_{n-2} = f_n$. Consequently, the sum of the weights of quadruples in $A \times A \times C \times C$ is given by $S_2 = l_{n+1}^2 f_n^2$.

Step 3. It follows by Step 1 that $S_3 = l_n^4$.

Step 4. The sum of the weights of walks in B is l_n and that in C is f_n. So, the sum of the weights of elements in $B \times B \times B \times C$ is given by $S_4 = l_n^3 f_n$.

Step 5. The sum of the elements in $B \times B \times C \times C$ is given by $S_5 = l_n^2 f_n^2$.

Step 6. Because the sum of the weights of walks in B is l_n and that in D is f_{n-1}, it follows that $S_6 = l_n^2 f_{n-1}^2$.

Thus, by identity (2.1),

$$\begin{aligned} S &= S_1 + x^2\Delta^2 S_2 + x^2 S_3 + 2x\Delta^2 S_4 + x^2\Delta^2 S_5 + 2x^2\Delta^2 S_6 \\ &= l_{n+1}^4 + x^2\Delta^2 l_{n+1}^2 f_n^2 + x^2 l_n^4 + 2x\Delta^2 l_n^3 f_n + x^2\Delta^2 l_n^2 f_n^2 + 2x^2\Delta^2 l_n^2 f_{n-1}^2 \\ &= 2x^2\Delta^2 l_{4n} + l_{n+1}^3 l_{n-1} + x l_{n+1}^2 l_n l_{n-1} + 4x\Delta^2 l_{n+1} l_n f_{n-1}^2. \end{aligned} \tag{4.2}$$

Equating the values of S in equations (3.1) and (4.2) yields the desired result. □

4.2. **Confirmation of Identity** (2.3)**.** This time, consider the *weighted Jacobsthal digraph* D_2 in Figure 2.

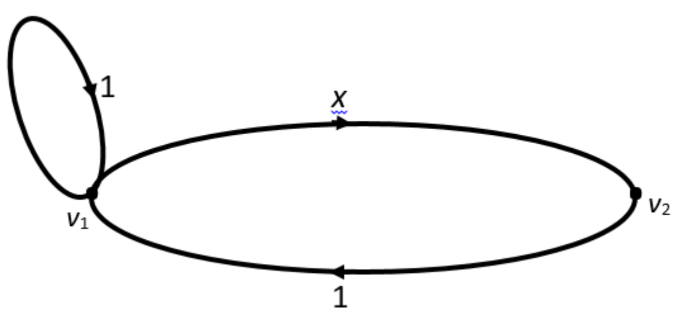

Figure 2. Weighted Jacobsthal Digraph D_2

It follows from its weighted adjacency matrix $M = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix}$ that

$$M^n = \begin{bmatrix} J_{n+1}(x) & xJ_n(x) \\ J_n(x) & xJ_{n-1}(x) \end{bmatrix},$$

where $n \geq 1$.

Consequently, the sum of the weights of closed walks of length n from v_1 to itself is $J_{n+1}(x)$, and that from v_2 to itself is $xJ_{n-1}(x)$. So, the sum of the weights of length n in the digraph is $J_{n+1}(x) + xJ_{n-1}(x) = j_n(x)$.

We are now ready for the graph-theoretic proof. In the interest of brevity and clarity, we again omit the argument in the functional notation.

Proof. Let A and B denote the sets of closed walks of lengths $n+1$ and n in the digraph, respectively; and C and D the sets of those of lengths $n-1$ and $n-2$ originating at v_1, respectively. The sums of the walks in them are j_{n+1}, j_n, J_n, and J_{n-1}, respectively.

Let S_1, S_2, S_3, S_4, S_5, and S_6 denote the sums of the weights of elements in $A \times A \times A \times A$, $A \times A \times C \times C$, $B \times B \times B \times B$, $B \times B \times B \times C$, $B \times B \times C \times C$, and $B \times B \times D \times D$, respectively. Then, $S_1 = j_{n+1}^4$, $S_2 = j_{n+1}^2 j_n^2$, $S_3 = j_n^4$, $S_4 = j_n^3 J_n$, $S_5 = j_n^2 J_n^2$, and $S_6 = j_n^2 J_{n-1}^2$.

Then, the sum

$$S = S_1 + D^2 S_2 + xS_3 + 2xD^2 S_4 + xD^2 S_5 + 2x^2 D^2 S_6$$

is given by

$$S = j_{n+1}^4 + D^2 j_{n+1}^2 J_n^2 + xj_n^4 + 2xD^2 j_n^3 J_n + xD^2 j_n^2 J_n^2 + 2x^2 D^2 j_n^2 J_{n-1}^2. \tag{4.3}$$

We will now recompute the sum S in a different way in six steps.

Step 1. Let w be an arbitrary walk in A. Suppose it originates at v_1. If w begins with a loop, the sum of the weights of such walks is $1 \cdot J_{n+1}$; otherwise, the sum is $x \cdot 1 \cdot J_{n-1} = xJ_{n-1}$. The sum of such walks v is $J_{n+1} + xJ_{n-1} = j_{n+1}$. So, the sum S_1 of the weights of the elements in $A \times A \times C \times C$ is given by $S_1 = j_{n+1}^4$.

Step 2. The sum of the weights of walks in C is J_n. So, the sum of the weights of elements in $A \times A \times C \times C$ equals $j_{n+1}^2 J_n^2$; that is, $S_2 = j_{n+1}^2 J_n^2$.

Step 3. It follows by Step 1 that the sum of the weights of elements in $B \times B \times B \times B$ is j_n^4; so $S_3 = j_n^4$.

Step 4. The sum of the weights of quadruples in $B \times B \times B \times C$ is $j_n^3 J_n$; so $S_4 = j_n^3 J_n$.

Step 5. The sum S_5 of the weights of elements in $B \times B \times C \times C$ is given by $S_5 = j_n^2 J_n^2$.

Step 6. The sum of the weights of the walks in D is J_{n-1}; so $S_6 = j_n^2 J_{n-1}^2$.

Using identity (2.3), we then have

$$\begin{aligned} S &= S_1 + D^2 S_2 + xS_3 + 2xD^2 S_4 + xD^2 S_5 + 2x^2 D^2 S_6 \\ &= j_{n+1}^4 + D^2 j_{n+1}^2 J_n^2 + xj_n^4 + 2xD^2 j_n^3 J_n + xD^2 j_n^2 J_n^2 + 2x^2 D^2 j_n^2 J_{n-1}^2 \\ &= 2D^2 j_{4n} + xj_{n+1}^3 j_{n-1} + xj_{n+1}^2 j_n j_{n-1} + 4D^2 x^2 j_{n+1} j_n J_{n-1}^2. \end{aligned} \tag{4.4}$$

Equating the values of S in equations (4.3) and (4.4) yields the desired result. □

Next, we confirm identity (3.1) using graph-theoretic techniques

4.3. **Confirmation of Identity** (3.1)**.**

Proof. The sum S of the weights of closed walks of length $5n$ in digraph D_1 is given by $S = l_{5n}$; so $x^2 \Delta^4 S = x^2 \Delta^4 l_{5n}$.

We will now compute $x^2 \Delta^4 S$ in a different way. To this end, let w be an arbitrary closed walk of length $5n$.

Case 1. Suppose w originates (and ends) at v_1. Clearly, it can land at v_1 or v_2 at the $2n$th and $4n$th steps: $w = \underbrace{v_1 - \cdots - v}_{\text{subwalk of length } 2n} \; \underbrace{v - \cdots - v}_{\text{subwalk of length } 2n} \; \underbrace{v - \cdots - v_1}_{\text{subwalk of length } n}$, where $v = v_1$ or v_2.

Table 3 shows the various possible cases and the sums of weights of walks w.

w lands at v_1 at the $2n$th step?	w lands at v_1 at the $4n$th step?	w lands at v_1 at the $5n$th step?	sum of the weights of walks w
yes	yes	yes	$f_{2n+1}^2 f_{n+1}$
yes	no	yes	$f_{2n+1} f_{2n} f_n$
no	yes	yes	$f_{2n}^2 f_{n+1}$
no	no	yes	$f_{2n} f_{2n-1} f_n$

Table 3: Sums of the Weights of Closed Walks Originating at v_1

It follows from Table 3 that the sum S_1 of the weights of all such walks w is given by

$$\begin{aligned} S_1 &= \left(f_{2n+1}^2 + f_{2n}^2\right) f_{n+1} + f_{2n} f_n (f_{2n+1} + f_{2n-1}) \\ &= f_{4n+1} f_{n+1} + f_{4n} f_n \\ &= f_{5n+1}. \end{aligned}$$

Case 2. Suppose w originates (and ends) at v_2. Then also, w can land at v_1 or v_2 at the $2n$th and $4n$th steps: $w = \underbrace{v_2 - \cdots - v}_{\text{subwalk of length } 2n} \underbrace{v - \cdots - v}_{\text{subwalk of length } 2n} \underbrace{v - \cdots - v_2}_{\text{subwalk of length } n}$, where $v = v_1$ or v_2.

It follows from Table 4 that the sum S_2 of the weights of all such walks w is given by

$$\begin{aligned} S_1 &= (f_{2n+1} + f_{2n-1}) f_{2n} f_n + \left(f_{2n}^2 + f_{2n-1}^2\right) f_{n-1} \\ &= f_{4n} f_n + f_{4n-1} f_{n-1} \\ &= f_{5n-1}. \end{aligned}$$

w lands at v_1 at the $2n$th step?	w lands at v_1 at the $4n$th step?	w lands at v_2 at the $5n$th step?	sum of the weights of walks w
yes	yes	yes	$f_{2n+1} f_{2n} f_n$
yes	no	yes	$f_{2n}^2 f_{n-1}$
no	yes	yes	$f_{2n} f_{2n-1} f_n$
no	no	yes	$f_{2n-1}^2 f_{n-1}$

Table 4: Sums of the Weights of Closed Walks Originating at v_2

Thus, using identity (3.1), the cumulative sum S of the weights of all closed walks of length $5n$ in the digraph and hence, $x^2\Delta^4 S$ are given by

$$\begin{aligned} S &= S_1 + S_2 \\ &= f_{5n+1} + f_{5n-1} \\ &= l_{5n}; \\ x^2\Delta^4 S &= x^2\Delta^4 l_{5n} \\ &= x l_{n+1}^5 + l_{n+1}^4 l_n + x l_{n+1}^3 l_n^2 + x^2 l_{n+1}^2 l_n^3 + x l_{n+1}^2 l_n^2 l_{n-1} - 2 l_{n+1}^2 l_n l_{n-1}^2 \\ &\quad + x^2 l_n^5 - 2x l_n^2 l_{n-1}^3 + l_n l_{n-1}^4 - x l_{n-1}^5. \end{aligned}$$

This value of $x^2\Delta^4 S$, coupled with its original value, yields the desired result. □

Next, we confirm identity (4.3) using graph-theoretic methods.

4.4. **Confirmation of Identity** (4.3)**.**

Proof. Let S^* denote the sum of the weights of all closed walks of length $5n$ in the digraph D_2. Then, $S^* = j_{5n}$ and hence, $D^4 S^* = D^4 j_{5n}$.

To compute this sum in a different way, we let w be an arbitrary closed walk of length $5n$ in the digraph.

Case 1. Suppose w originates at v_1. It can land at v_1 or v_2 at the $2n$th and $4n$th steps:

$$w = \underbrace{v_1 - \cdots - v}_{\text{subwalk of length } 2n}\ \underbrace{v - \cdots - v}_{\text{subwalk of length } 2n}\ \underbrace{v - \cdots - v_1}_{\text{subwalk of length } n}, \text{ where } v = v_1 \text{ or } v_2.$$

It follows from Table 5 that the sum S_1^* of the weights of all such walks w is given by

$$\begin{aligned} S_1^* &= J_{n+1}\left(J_{2n+1}^2 + xJ_{2n}^2\right) + xJ_{2n}J_n(J_{2n+1} + xJ_{2n-1}) \\ &= J_{4n+1}J_{n+1} + xJ_{4n}J_n \\ &= J_{5n+1}. \end{aligned}$$

w lands at v_1 at the $2n$th step?	w lands at v_1 at the $4n$th step?	w lands at v_1 at the $5n$th step?	sum of the weights of walks w
yes	yes	yes	$J_{2n+1}^2 J_{n+1}$
yes	no	yes	$xJ_{2n+1}J_{2n}J_n$
no	yes	yes	$xJ_{2n}^2 J_{n+1}$
no	no	yes	$x^2 J_{2n}J_{2n-1}J_n$

Table 5: Sums of the Weights of Closed Walks Originating at v_1

Case 2. Suppose w originates at v_2. Then, w can also land at v_1 or v_2 at the $2n$th and $4n$th steps:

$$w = \underbrace{v_2 - \cdots - v}_{\text{subwalk of length } 2n}\ \underbrace{v - \cdots - v}_{\text{subwalk of length } 2n}\ \underbrace{v - \cdots - v_2}_{\text{subwalk of length } n}, \text{ where } v = v_1 \text{ or } v_2.$$

Table 6 implies that the sum S_2^* of the weights of all such walks w is given by

$$\begin{aligned} S_2^* &= xJ_{2n}J_n\left(J_{2n+1} + xJ_{2n-1}\right) + x^2 J_{n-1}(J_{2n}^2 + xJ_{2n-1}^2) \\ &= x(J_{4n}J_n + xJ_{4n-1}J_{n-1} \\ &= xJ_{5n-1}. \end{aligned}$$

w lands at v_1 at the $2n$th step?	w lands at v_1 at the $4n$th step?	w lands at v_2 at the $5n$th step?	sum of the weights of walks w
yes	yes	yes	$xJ_{2n+1}J_{2n}J_n$
yes	no	yes	$x^2 J_{2n}^2 J_{n-1}$
no	yes	yes	$x^2 J_{2n}^2 J_{2n-1}J_n$
no	no	yes	$x^3 J_{2n-1}^2 J_{n-1}$

Table 6: Sums of the Weights of Closed Walks Originating at v_2

Combining the two cases and using identity (4.3), we get

$$\begin{aligned} S^* &= S_1^* + S_2^* \\ &= J_{5n+1} + xJ_{5n-1} \\ &= j_{5n}; \\ D^4 S^* &= j_{n+1}^5 + xj_{n+1}^4 j_n + xj_{n+1}^3 j_n^2 + x^2 j_{n+1}^2 j_n^3 + x^2 j_{n+1}^2 j_n^2 j_{n-1} \\ &\quad - 2x^3 j_{n+1}^2 j_n^2 j_{n-1}^2 - x^2 j_n^5 - 2x^4 j_n^2 j_{n-1}^3 + x^5 j_n j_{n-1}^4 - x^5 j_{n-1}^5. \end{aligned}$$

This value of D^4S^*, coupled with its earlier version, yields the desired result, as expected. □

Finally, we showcase the Vieta and Chebyshev implications of equations (2.1) and (3.1); again, we omit all basic algebra involved in their justifications.

5. Vieta and Chebyshev Counterparts

Using the Vieta-gibonacci and Vieta-Chebyshev relationships in Table 1, we can find the Vieta and Chebyshev companions of identities (2.1) and (3.1):

$$\begin{aligned}
2x^2(x^2-4)v_{4n} &= v_{n+1}^4 + v_{n+1}^3 v_{n-1} + xv_{n+1}^2 v_n v_{n-1} + x^2(x^2-4)v_{n+1}^2 V_n^2 \\
&\quad + x(x^2-4)v_{n+1}v_n V_{n-1}^2 - x^2 v_n^4 + 2x(x^2-4)v_n^3 V_n \\
&\quad - x^2(x^2-4)v_n^2 V_n^2 + 2x^2(x^2-4)v_n^2 V_{n-1}^2; \\
8x^2(x^2-1)T_{4n} &= 2T_{n+1}^4 + 2T_{n+1}^3 T_{n-1} + 4xT_{n+1}^2 T_n T_{n-1} + 8x^2(x^2-1)T_{n+1}^2 U_{n-1}^2 \\
&\quad + 8x(x^2-1)T_{n+1}T_n U_{n-2}^2 - 8x^2 T_n^4 + 16x(x^2-1)T_n^3 U_{n-1} \\
&\quad - 8x^2(x^2-1)T_n^2 U_{n-1}^2 + 16x^2(x^2-1)T_n^2 U_{n-2}^2; \\
x^2(x^2-4)^2 v_{5n} &= xv_{n+1}^5 - v_{n+1}^4 v_n - xv_{n+1}^3 v_n^2 - x^2 v_{n+1}^2 v_n^3 + xv_{n+1}^2 v_n^2 v_{n-1} \\
&\quad + 2v_{n+1}^2 v_n v_{n-1}^2 + x^2 v_n^5 - 2xv_n^2 v_{n-1}^3 - v_n v_{n-1}^4 + xv_{n-1}^5; \\
4x^2(x^2-1)^2 T_{5n} &= 2T_{n+1}^5 - T_{n+1}^4 T_n - 2xT_{n+1}^3 T_n^2 - 4x^2 T_{n+1}^2 T_n^3 + 2xT_{n+1}^2 T_n^2 T_{n-1} \\
&\quad + 2T_{n+1}^2 T_n T_{n-1}^2 + 4T_n^5 - 4xT_n^2 T_{n-1}^3 - T_n T_{n-1}^4 + 2T_{n-1}^5,
\end{aligned}$$

where $v_n = v_n(x)$ and $T_n = T_n(x)$.

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