VALUES OF BERNOULLI AND EULER POLYNOMIALS AT RATIONAL POINTS

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ABSTRACT. A problem proposed by E. Lehmer about Bernoulli polynomials is solved, using a classic theorem of D. H. Lehmer. A similar result is obtained for Euler polynomials.

1. INTRODUCTION

The Bernoulli numbers B_n and the Bernoulli polynomials $B_n(x)$ are defined, respectively, by

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty} B_n \frac{z^n}{n!}; \tag{1}$$

$$\frac{ze^{xz}}{e^z-1}=\sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}. \tag{2}$$

Thus, $B_0(x)=1$, $B_1(x)=x-\frac{1}{2}$, $B_2(x)=x^2-x+\frac{1}{6}$, $B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x$, $B_4(x)=x^4-2x^3+x^2-\frac{1}{30}$, etc.

From the above definitions, we have

$$B_n(x)=\sum_{r=0}^{n}\binom{n}{r}B_r x^{n-r}. \tag{3}$$

In particular, $B_n(0)=B_n$. Note that $B_n=0$, whenever $n>1$ is odd.

The following evaluations of $B_n(x)$ are well-known (cf. [5, Section 24.4]) and can be derived directly from (1) and (2):

$$B_n(\tfrac{1}{2})=(2^{1-n}-1)B_n, \tag{4}$$

$$B_{2n}(\tfrac{1}{3})=B_{2n}(\tfrac{2}{3})=\tfrac{1}{2}(3^{1-2n}-1)B_{2n}, \tag{5}$$

$$B_{2n}(\tfrac{1}{4})=B_{2n}(\tfrac{3}{4})=\tfrac{1}{2}(4^{1-2n}-2^{1-2n})B_{2n}, \tag{6}$$

$$B_{2n}(\tfrac{1}{6})=B_{2n}(\tfrac{5}{6})=\tfrac{1}{2}(6^{1-2n}-3^{1-2n}-2^{1-2n}+1)B_{2n}. \tag{7}$$

In [3], E. Lehmer used (4)–(7) to derive a large class of important congruences involving arithmetic sums, Bernoulli numbers, Fermat quotients, and Wilson quotients. E. Lehmer pointed out that the number $q=1,2,3,4,6$ are characterized by $\phi(q)\leq 2$ (ϕ is Euler's totient function), and asked whether similar evaluations of $B_n(\frac{a}{q})$ exist for other q. In [1], some *mod p* evaluations of $B_{p-1}(\frac{a}{q})$ were extended to other q.

In this paper, we show that similar evaluations of $B_n(\frac{a}{q})$ do not exist for other q (see Theorem 2.1). This is closely connected with the following classic result (cf. [2] and [4, p. 37]) of D. H. Lehmer:

Theorem 1.1 (D. H. Lehmer)**.** *Let $\frac{a}{q}$ be a rational number, where $q > 2$ and $(a, q) = 1$. Then, $2\cos\frac{2a\pi}{q}$ is an algebraic integer of degree $\phi(q)/2$, and $2\sin\frac{2a\pi}{q}$ is an algebraic integer of degree $\phi(q)$, $\phi(q)/4$, or $\phi(q)/2$, according to $(q, 8) < 4$, $(q, 8) = 4$, or $(q, 8) > 4$, respectively.*

In particular, we have:

Corollary 1.2. *Let $\frac{a}{q}$ be a rational number, where $q > 0$ and $(a, q) = 1$. Then, $\cos\frac{2a\pi}{q}$ is rational if and only if $q = 1, 2, 3, 4,$ or 6.*

A similar question can be proposed for Euler numbers E_n and Euler polynomials $E_n(x)$, which are defined respectively by

$$\frac{2e^z}{e^{2z}+1} = \sum_{n=0}^{\infty} E_n \frac{z^n}{n!}, \tag{8}$$

$$\frac{2e^{xz}}{e^z+1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!}. \tag{9}$$

We note that each $E_n \in \mathbb{Z}$ and $E_n = 0$, whenever n is odd.

There are only two known evaluations of $E_n(x)$ at rational points (cf. [5, Section 24.4]):

$$E_n(\tfrac{1}{2}) = 2^{-n}E_n, \tag{10}$$

$$E_{2n}(\tfrac{1}{6}) = E_{2n}(\tfrac{5}{6}) = 2^{-2n-1}(1+3^{-2n})E_{2n}. \tag{11}$$

We show that similar evaluations of $E_n(\frac{a}{q})$ do not exist for other q (see Theorem 2.2), using another corollary of Theorem 1.1.

Corollary 1.3. *Let $\frac{a}{q}$ be a rational number, where $q > 0$ and $(a, q) = 1$. Then, $\sin\frac{a\pi}{q}$ is rational if and only if $q = 1, 2,$ or 6.*

2. Proof of the Main Results

The main result of this paper is the following:

Theorem 2.1. *Assume that there exist k nonzero real numbers $a_1, a_2, \ldots, a_k$, k distinct positive numbers $b_1, b_2, \ldots, b_k$, two even integers $s > t \geq 0$, and a rational number $\frac{a}{q}$ with $q > 0$ and $(a, q) = 1$, such that*

$$B_n(\tfrac{a}{q}) = (a_1 b_1^n + a_2 b_2^n + \cdots + a_k b_k^n)B_n, \tag{12}$$

whenever $n \equiv t \pmod{s}$. Then, we have $q = 1, 2, 3, 4,$ or 6, and $a_i, b_i \in \mathbb{Q}$ for $1 \leq i \leq k$.

Proof. Assume that (12) is valid. Then by (1) and (2), we have

$$\sum_{j=1}^{s} f(\tfrac{a}{q}, \zeta^j z) = \sum_{m=1}^{k}\sum_{j=1}^{s} a_m b_m^{t-s} f(0, \zeta^j b_m z), \tag{13}$$

where $\zeta = e^{2\pi i/s}$ and $f(x, z) = \frac{z^{s-t+1}e^{xz}}{e^z-1}$.

Hence,

$$\begin{aligned}&\int_{L_r} \frac{f(\frac{a}{q}, z)dz}{z} \\ =&\frac{1}{s}\sum_{j=1}^{s}\int_{L_r} \frac{f(\frac{a}{q}, \zeta^j z)dz}{z} \\ =&\frac{1}{s}\sum_{m=1}^{k}\sum_{j=1}^{s} a_m b_m^{t-s}\int_{L_r} \frac{f(0, \zeta^j b_m z)dz}{z} \\ =&\sum_{m=1}^{k} a_m b_m^{t-s}\int_{L_r} \frac{f(0, b_m z)dz}{z},\end{aligned} \tag{14}$$

where L_r is the circle $|z| = r$. Note that we always assume that L_r does not contain any pole of the integrand. By Cauchy's residue theorem,

$$\frac{1}{2\pi i}\int_{L_r} \frac{f(\frac{a}{q}, z)dz}{z} = \sum_{n=-[r/(2\pi)]}^{[r/(2\pi)]} e^{2an\pi i/q}(2n\pi i)^{s-t}, \tag{15}$$

and

$$\frac{1}{2\pi i}\int_{L_r} \frac{f(0, b_m z)dz}{z} = 2\sum_{n=1}^{[rb_m/(2\pi)]} (2n\pi i)^{s-t}, \tag{16}$$

where $[u]$ denotes as usual, the greatest integer not exceeding u. Combining (14), (15), and (16) immediately implies that

$$\sum_{n=-[r/(2\pi)]}^{[r/(2\pi)]} e^{2an\pi i/q} n^{s-t} = 2\sum_{m=1}^{k} a_m b_m^{t-s} \sum_{n=1}^{[rb_m/(2\pi)]} n^{s-t}. \tag{17}$$

Using (17), we shall show that $b_1, b_2, \ldots, b_k \in \mathbb{Q}$. Assume that b_l is the maximal irrational number among $b_1, b_2, \ldots, b_k$. Let $g(r)$ be the right side of (17). Then, one checks directly that

$$g(\tfrac{2\pi}{b_l} + \epsilon) - g(\tfrac{2\pi}{b_l} - \epsilon) = 2a_l b_l^{t-s} \neq 0, \tag{18}$$

when $\epsilon > 0$ is sufficiently small. Hence, $r = \frac{2\pi}{b_l}$ is a jump discontinuity of $g(r)$. Thus, we arrive at a contradiction that the left side of (17) is continuous at $r = \frac{2\pi}{b_l}$.

Note that (12) implies that $a_1 b_1^n + a_2 b_2^n + \cdots + a_k b_k^n \in \mathbb{Q}$ if $n \equiv t \pmod{s}$. By the Vandermonde determinant $|\{b_i^{sj}\}_{i,j}| \neq 0$, we also have $a_1, a_2, \ldots, a_k \in \mathbb{Q}$.

Now, we are in the last step of the proof. Assume that $q \neq 1, 2, 3, 4$, and 6. Then, the left side of (17) is $2\cos\frac{2a\pi}{q}$ for $2\pi < r < 4\pi$, whereas the right side of (17) is rational. This contradicts Corollary 1.2. □

The same proof, using Corollary 1.3, leads to a similar result about Euler polynomials.

Theorem 2.2. *Assume that there exist k nonzero real numbers $a_1, a_2, \ldots, a_k$, k distinct positive numbers $b_1, b_2, \ldots, b_k$, two even integers $s > t \geq 0$, and a rational number $\frac{a}{q}$ with $q > 0$ and $(a, q) = 1$, such that*

$$E_n(\tfrac{a}{q}) = (a_1 b_1^n + a_2 b_2^n + \cdots + a_k b_k^n)E_n, \tag{19}$$

whenever $n \equiv t \pmod{s}$. Then, $q = 1, 2$, or 6, and $a_i, b_i \in \mathbb{Q}$ for $1 \leq i \leq k$.

References

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