ON EXACTLY 3-DEFICIENT-PERFECT NUMBERS

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ABSTRACT. Let n and k be positive integers and $\sigma(n)$ the sum of all positive divisors of n. We call n an exactly k-deficient-perfect number with deficient divisors $d_1, d_2, \ldots, d_k$ if $d_1, d_2, \ldots, d_k$ are distinct proper divisors of n and $\sigma(n) = 2n - (d_1 + d_2 + \cdots + d_k)$. In this article, we show that the only odd exactly 3-deficient-perfect number with at most two distinct prime factors is $1521 = 3^2 \cdot 13^2$.

1. INTRODUCTION

Throughout this article, let n be a positive integer, $\sigma(n)$ the sum of all positive divisors of n, and $\omega(n)$ the number of distinct prime factors of n. We say that n is perfect if $\sigma(n) = 2n$. It is well-known that n is even and perfect if and only if $n = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are primes. It has also been a longstanding conjecture that there are infinitely many even perfect numbers and that an odd perfect number does not exist. Attempting to understand perfect numbers, mathematicians have studied other closely related concepts. Recall that if $\sigma(n) < 2n$, then n is said to be deficient; if $\sigma(n) > 2n$, then n is abundant; if $\sigma(n) = 2n + 1$, then n is quasiperfect; if $\sigma(n) = 2n - 1$, then n is almost perfect. For more information on this topic, see for example the work of Cohen [5, 6], Hagis and Cohen [11], Kishore [14], Ochem and Rao [18], Yamada [36], and the online databases GIMPS [10] and OEIS [30].

Sierpiński [29] called n pseudoperfect if n can be written as a sum of some of its proper divisors. Pollack and Shevelev [21] have recently initiated the study of a subclass of pseudoperfect numbers leading to an active investigation. We summarize this work in the following definition.

Definition 1.1. *Let n and k be positive integers. We say that n is near-perfect if n is the sum of all of its proper divisors except one of them. In addition, n is k-near-perfect if n can be written as a sum of all of its proper divisors with at most k exceptions. Moreover, n is exactly k-near-perfect if n is expressible as a sum of all of its proper divisors with exactly k exceptions. The exceptional divisors are said to be redundant. In other words,*

n is near-perfect with a redundant divisor $d \Leftrightarrow 1 \leq d < n$, $d \mid n$, and $\sigma(n) = 2n + d$;

n is 1-near-perfect $\Leftrightarrow$ n is perfect or n is near-perfect;

n is exactly k-near-perfect with redundant divisors $d_1, d_2, \ldots, d_k \Leftrightarrow$
$d_1, d_2, \ldots, d_k$ are distinct proper divisors of n and $\sigma(n) = 2n + d_1 + d_2 + \cdots + d_k$.

Motivated by the concept of near-perfect numbers, Tang, Ren, and Li [35] define the notion of deficient-perfect numbers, which also leads to an interesting research problem.

Definition 1.2. *Let $n, k \in \mathbb{N}$. Then, n is called a deficient-perfect number with a deficient divisor d if d is a proper divisor of n and $\sigma(n) = 2n - d$. Furthermore, n is exactly k-deficient-perfect with deficient divisors $d_1, d_2, \ldots, d_k$ if $d_1, d_2, \ldots, d_k$ are distinct proper divisors of n*

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and $\sigma(n) = 2n - (d_1 + d_2 + \cdots + d_k)$*. In addition,* n *is* k*-deficient-perfect if* n *is perfect or* n *is exactly* ℓ*-deficient-perfect for some* $\ell = 1, 2, \ldots, k$.

In 2012, Pollack and Shevelev [21] showed that the number of near-perfect numbers not exceeding x is $\ll x^{5/6+o(1)}$ as $x \to \infty$, and that if k is fixed and is large enough, then there are infinitely many exactly k-near-perfect numbers. A year later, Ren and Chen [27] determined all near-perfect numbers n that have $\omega(n) = 2$, and we can see from this classification that all such n are even. In the same year, Tang, Ren, and Li [35] proved that there is no odd near-perfect number n with $\omega(n) = 3$ and found all deficient-perfect numbers m with $\omega(m) \leq 2$. After that, Tang and Feng [33] extended this result by showing that there is no odd deficient-perfect number n with $\omega(n) = 3$. In 2016, Tang, Ma, and Feng [34] found the only odd near-perfect number with $\omega(n) = 4$, namely, $n = 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2$, whereas in 2019, Sun and He [32] asserted that the only odd deficient-perfect number n with $\omega(n) = 4$ is $n = 3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2$. Cohen, et al. [7] have recently improved the estimate of Pollack and Shevelev [21] on the number of near-perfect numbers $\leq x$. Hence, most results in the literature are devoted to characterizing, only when $k = 1$, the exactly k-near-perfect or exactly k-deficient-perfect numbers. Chen [4] started a slightly new direction by determining all 2-deficient-perfect numbers n with $\omega(n) \leq 2$.

In this article, we continue the investigation on odd 3-deficient-perfect numbers n with $\omega(n) \leq 2$. We found that the only such n is $n = 1521 = 3^2 \cdot 13^2$. For other articles related to the divisor functions or divisibility problems, see examples in [1, 2, 3, 8, 9, 12, 13, 15, 16, 17, 19, 20, 22, 23, 24, 25, 26, 28, 31, 36].

2. Main Results

By the definition, n is deficient-perfect if and only if n is exactly 1-deficient-perfect. Tang and Feng [33, Lemma 2.1] showed that if n is deficient-perfect and n is odd, then n is a square. We can extend their result to the following form.

Lemma 2.1. *Let* n *and* k *be positive integers. Suppose that* n *is exactly* k*-deficient-perfect and* n *is odd. Then,* n *is a square if and only if* k *is odd. In particular, if* n *is odd and exactly* 3*-deficient-perfect, then* n *is a square.*

Proof. Because 1 has no proper divisor, we can assume that $n > 1$ and write $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_1, \ldots, p_r$ are distinct odd primes and $\alpha_1, \alpha_2, \ldots, \alpha_r$ are positive integers. Let $d_1, d_2, \ldots, d_k$ be distinct proper divisors of n such that

$$2n - d_1 - d_2 - \cdots - d_k = \sigma(n) = \prod_{i=1}^{r} \sigma(p_i^{\alpha_i}) = \prod_{i=1}^{r} (1 + p_i + \cdots + p_i^{\alpha_i}). \tag{2.1}$$

Because n is odd, d_i and p_j are odd for every $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, r$. Reducing (2.1) mod 2, we obtain $k \equiv \prod_{i=1}^{r} (\alpha_i + 1) \pmod 2$. From this, we have the equivalence k is odd $\Leftrightarrow$ α_i is even for all i $\Leftrightarrow$ n is a square, which proves our lemma. □

Tang, Ren, and Li [35] determine all deficient-perfect numbers n with $\omega(n) \leq 2$. In particular, they show that if $\omega(n) = 1$ and n is deficient-perfect, then n is a power of 2. We can extend this for exactly k-deficient-perfect numbers as follows.

Lemma 2.2. *Let* $n \geq 2$, $k \geq 1$ *be integers. If* n *is exactly* k*-deficient-perfect and* $\omega(n) = 1$, *then* $k = 1$ *and* n *is a power of* 2*. Consequently, if* n *is exactly* k*-deficient-perfect and* $k \geq 2$, *then* n *has at least two distinct prime divisors. In particular, every exactly* 3*-deficient-perfect number* n *has* $\omega(n) \geq 2$.

Proof. Suppose $n = p^\alpha$ and the deficient divisors of n are $d_1 = p^{\beta_1}$, $d_2 = p^{\beta_2}$, ..., $d_k = p^{\beta_k}$, where $\alpha > \beta_1 > \beta_2 > \cdots > \beta_k \geq 0$. Because $(p^{\alpha+1} - 1)/(p-1) = \sigma(n) = 2n - d_1 - \cdots - d_k$, we obtain

$$(d_1 + d_2 + \cdots + d_k)(p-1) - 1 = p^\alpha(p-2). \tag{2.2}$$

If $p \geq 3$, then

$$\begin{aligned} p^\alpha \leq p^\alpha(p-2) &= (d_1 + d_2 + \cdots + d_k)(p-1) - 1 \\ &\leq (p^{\alpha-1} + p^{\alpha-2} + \cdots + p^{\alpha-k})(p-1) - 1 = p^\alpha - p^{\alpha-k} - 1, \end{aligned}$$

which is impossible. Therefore, $p = 2$ and n is a power of 2. By (2.2), we also obtain $d_1 + \cdots + d_k = 1$, which implies $k = 1$ and $\beta_1 = 0$. □

We now give the main result of this paper.

Theorem 2.3. *The only odd exactly* 3*-deficient-perfect number that has* $\omega(n) = 2$ *is* $1521 = 3^2 \cdot 13^2$*, with three deficient divisors* $d_1 = 507$*,* $d_2 = 117$*, and* $d_3 = 39$*.*

Proof. It is easy to check that if $n = 1521$ and d_1, d_2, d_3 are as above, then $\omega(n) = 2$, n is odd, d_1, d_2, d_3 are proper divisors of n, $\sigma(n) = 2n - d_1 - d_2 - d_3$, and so n is exactly 3-deficient-perfect. For the other direction, assume that n is odd, $\omega(n) = 2$, and n is exactly 3-deficient-perfect. By Lemma 2.1, n is a square, so we can write $n = p_1^{2\alpha} p_2^{2\beta}$, where $2 < p_1 < p_2$ and $\alpha, \beta \geq 1$. In addition, let $d_1 > d_2 > d_3$ be the deficient divisors of n, and let $D_1 = n/d_1$, $D_2 = n/d_2$, $D_3 = n/d_3$. Then $p_1 \leq D_1 < D_2 < D_3 \leq n$. Because $\sigma(n) = 2n - d_1 - d_2 - d_3$, we obtain

$$\begin{aligned} 2 &= \frac{\sigma(n)}{n} + \frac{d_1}{n} + \frac{d_2}{n} + \frac{d_3}{n} \\ &= \frac{({p_1}^{2\alpha+1} - 1)({p_2}^{2\beta+1} - 1)}{(p_1 - 1)(p_2 - 1)p_1^{2\alpha} p_2^{2\beta}} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} \\ &< \frac{p_1 p_2}{(p_1 - 1)(p_2 - 1)} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3}. \end{aligned} \tag{2.3}$$

If $p_1 \geq 5$, then $p_1/(p_1 - 1) \leq 5/4$, $p_2 \geq 7$, $p_2/(p_2 - 1) \leq 7/6$, $D_1 \geq 5$, $D_2 \geq 7$, $D_3 \geq 25$, and (2.3) implies that

$$2 < \frac{5}{4} \cdot \frac{7}{6} + \frac{1}{5} + \frac{1}{7} + \frac{1}{25} = 1.8411\ldots,$$

which is a contradiction. So, $p_1 = 3$. For convenience, let $p_2 = p$. Then, $n = 3^{2\alpha} p^{2\beta}$ and (2.3) becomes

$$2 < \frac{3p}{2(p-1)} + \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3}. \tag{2.4}$$

If $p \geq 83$, then (2.4) leads to $2 < (3/2)(83/82) + 1/3 + 1/9 + 1/27 = 1.9997\ldots$, which is impossible. So, $5 \leq p \leq 79$. Recall that the primes in $[5, 79]$ are 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79. If $p \geq 11$ and $D_1 > 3$, then $D_1 \geq 9$, $D_2 \geq 11$, $D_3 \geq 27$, and (2.4) gives $2 < (3/2)(11/10) + 1/9 + 1/11 + 1/27 = 1.8890\ldots$, which is false. Therefore,

$$\text{if } p \geq 11, \text{ then } D_1 = 3. \tag{2.5}$$

Similarly, if $p \geq 23$ and $D_2 > 9$, then $2 < (3/2)(23/22) + 1/3 + 1/23 + 1/27 = 1.9820\ldots$, which is not true. Thus,

$$\text{if } p \geq 23, \text{ then } D_2 = 9. \tag{2.6}$$

Next, we divide our calculations into 11 cases according to the value of p. In addition, we write the possible values of D_1, D_2, D_3 in increasing order.

Case 1. $47 \le p \le 79$. By (2.5) and (2.6), we have $D_1 = 3$, $D_2 = 9$, and the possible values of D_3 in increasing order are $D_3 = 27, p, 81, \ldots$. If $D_3 \ge p$, then (2.4) implies $2 < (3/2)(47/46) + 1/3 + 1/9 + 1/47 = 1.9983\ldots$, which is false. So, $D_3 = 27$. Then, $2\alpha \ge 3$, $d_1 = n/D_1 = 3^{2\alpha-1}p^{2\beta}$, $d_2 = 3^{2\alpha-2}p^{2\beta}$, $d_3 = 3^{2\alpha-3}p^{2\beta}$, and

$$\begin{aligned}\frac{\left(3^{2\alpha+1}-1\right)\left(p^{2\beta+1}-1\right)}{2(p-1)} &= \sigma(3^{2\alpha}p^{2\beta}) = 2\cdot 3^{2\alpha}p^{2\beta} - d_1 - d_2 - d_3 \\ &= 3^{2\alpha-3}p^{2\beta}(2\cdot 3^3 - 3^2 - 3 - 1) = 41\cdot 3^{2\alpha-3}p^{2\beta}.\end{aligned}$$

This leads to

$$3^{2\alpha-3} = \frac{p^{2\beta+1}-1}{(82-p)p^{2\beta}-81}. \tag{2.7}$$

The left side of (2.7) is an integer, and we get a contradiction by showing that the right side of (2.7) is not an integer. From this point on, let A be the number on the right side of (2.7). If $p = 47$, then A is equal to

$$\frac{47\cdot 47^{2\beta}-1}{35\cdot 47^{2\beta}-81} = 1 + \frac{12\cdot 47^{2\beta}+80}{35\cdot 47^{2\beta}-81} = 1 + \frac{12+(80/47^{2\beta})}{35-(81/47^{2\beta})} \in (1,2),$$

and so $A \notin \mathbb{Z}$. Similarly,

$$\begin{aligned}&\text{if } p = 53, \text{ then } A = 1 + \frac{24p^{2\beta}+80}{29p^{2\beta}-81} \in (1,2);\\ &\text{if } p = 59, \text{ then } A = 2 + \frac{13p^{2\beta}+161}{23p^{2\beta}-81} \in (2,3);\\ &\text{if } p = 61, \text{ then } A = 2 + \frac{19p^{2\beta}+161}{21p^{2\beta}-81} \in (2,3);\\ &\text{if } p = 67, \text{ then } A = 4 + \frac{7p^{2\beta}+323}{15p^{2\beta}-81} \in (4,5).\end{aligned}$$

The remaining cases $p = 71, 73, 79$ lead to $A \in (6,7)$, $A \in (8,9)$, and $A \in (26,27)$, respectively. In any case, $A \notin \mathbb{Z}$ and we have a contradiction. Hence, this case does not lead to a solution.

Case 2. $p \in \{37, 41, 43\}$. By (2.5) and (2.6), we have $D_1 = 3$, $D_2 = 9$, and $D_3 = 27, p, 81, \ldots$. If $D_3 \ge 81$, then (2.4) implies $2 < (3/2)(37/36) + 1/3 + 1/9 + 1/81 = 1.9984\ldots$, which is not possible. So, $D_3 = \{27, p\}$.

Case 2.1. $D_1 = 3$, $D_2 = 9$, and $D_3 = 27$. Then $2\alpha \ge 3$, (2.7) holds, and the calculations in Case 1 work in this case too. Because (2.7) holds, we still let A be the right side of (2.7). Therefore, if $p = 37$, then $A \in (0,1)$ and if $p \in \{41, 43\}$, then $A \in (1,2)$, which is a contradiction.

Case 2.2. $D_1 = 3$, $D_2 = 9$, and $D_3 = p$. Then,

$$\begin{aligned}\frac{\left(3^{2\alpha+1}-1\right)\left(p^{2\beta+1}-1\right)}{2(p-1)} &= \sigma(3^{2\alpha}p^{2\beta}) = \sigma(n) = 2n - d_1 - d_2 - d_3 \\ &= 2\cdot 3^{2\alpha}p^{2\beta} - 3^{2\alpha-1}p^{2\beta} - 3^{2\alpha-2}p^{2\beta} - 3^{2\alpha}p^{2\beta-1} \\ &= 3^{2\alpha-2}p^{2\beta-1}(14p-9),\end{aligned}$$

which implies

$$3^{2\alpha-2} = \frac{p^{2\beta+1}-1}{(46p-p^2-18)p^{2\beta-1}-27}. \tag{2.8}$$

Equality (2.8) can be used in the same way as (2.7). So, let B be the number on the right side of (2.8). Similar to the previous computation, we see that if $p = 37$, then $B \in (4, 5)$ and if $p = 43$, then $B \in (16, 17)$, which contradicts that $B = 3^{2\alpha-2} \in \mathbb{Z}$. Suppose $p = 41$. Then, $B \in (8, 10)$, which implies $B = 9$. Equating the right side of (2.8) with $B = 9$, substituting $p = 41$, and performing a straightforward manipulation leads to $41^{2\beta-1} = 121$, which is not possible. Hence, there is no solution in this case.

Remark 2.4. *Before going further, we note that the calculations similar to* (2.7) *and* (2.8) *and their applications occur throughout the proof, and we give less details than those in* (2.7) *and* (2.8)*.*

Case 3. $p \in \{29, 31\}$. Then by (2.5) and (2.6), $D_1 = 3$, $D_2 = 9$, and $D_3 = 27, p, 81, 3p, 243, 9p, 729, \ldots$. If $p = 31$ and $D_3 \geq 243$, then (2.4) implies $2 < (3/2)(31/30)+1/3+1/9+1/243 = 1.9985\ldots$, which is false. Similarly, assuming $p = 29$ and $D_3 \geq 729$ leads to a false inequality. Therefore,

$$\text{if } p = 31, \text{ then } D_3 \in \{27, 31, 81, 93\}, \tag{2.9}$$

$$\text{if } p = 29, \text{ then } D_3 \in \{27, 29, 81, 87, 243, 261\}. \tag{2.10}$$

Next, we divide our calculations according to the value of D_3.

Case 3.1. $D_3 = 27$. Then, (2.7) holds and the same method still works. We obtain

$$\text{if } p = 29, \text{ then } A = \left(29p^{2\beta} - 1\right) / \left(53p^{2\beta} - 81\right) \in (0, 1);$$

$$\text{if } p = 31, \text{ then } A = \left(31p^{2\beta} - 1\right) / \left(51p^{2\beta} - 81\right) \in (0, 1).$$

So, $A \notin \mathbb{Z}$ and we get a contradiction.

Case 3.2. $D_3 = p \in \{29, 31\}$. Then, (2.8) holds and

$$\text{if } p = 29, \text{ then } B = \left(841p^{2\beta-1} - 1\right) / \left(475p^{2\beta-1} - 27\right) \in (1, 2);$$

$$\text{if } p = 31, \text{ then } B = \left(961p^{2\beta-1} - 1\right) / \left(447p^{2\beta-1} - 27\right) \in (1, 2),$$

which is a contradiction.

Case 3.3. $D_3 = 81$. Similar to the calculations for (2.7) and (2.8), we write $\sigma(n) = 2n - d_1 - d_2 - d_3$, where d_1, d_2 are the same as before, but $d_3 = n/D_3 = 3^{2\alpha-4}p^{2\beta}$ and $2\alpha \geq 4$. After a similar algebraic manipulation, we get

$$3^{2\alpha-4} = \frac{p^{2\beta+1} - 1}{(250 - 7p)p^{2\beta} - 243}. \tag{2.11}$$

When $p = 29$ or 31, the right side of (2.11) is in the interval $(0, 1)$, which is impossible.

Case 3.4. $D_3 = 93$. By (2.9) and (2.10), we know that $p = 31$. Similar to Case 3.3 but with $d_3 = n/D_3 = 3^{2\alpha-1}p^{2\beta-1}$, we start with $\sigma(n) = 2n - d_1 - d_2 - d_3$ and perform an algebraic manipulation to obtain

$$3^{2\alpha-2} = \frac{p^{2\beta+1} - 1}{(34p - p^2 - 6)p^{2\beta-1} - 27} = \frac{961p^{2\beta-1} - 1}{87p^{2\beta-1} - 27} \in (11, 12),$$

which is false.

Case 3.5. $D_3 \in \{87, 243, 261\}$. By (2.9) and (2.10), we have $p = 29$. Similar to Case 3.3 but with different values of $d_3 = n/D_3 = 3^{2\alpha-1}p^{2\beta-1}$, $3^{2\alpha-5}p^{2\beta}$, or $3^{2\alpha-2}p^{2\beta-1}$ when $D_3 = 87$, 243, or 261, respectively. These lead to

$$2\alpha \geq 2 \text{ and } 3^{2\alpha-2} = \frac{p^{2\beta+1}-1}{(34p-p^2-6)p^{2\beta-1}-27} = \frac{841p^{2\beta-1}-1}{139p^{2\beta-1}-27} \in (6,7), \text{ if } D_3 = 87;$$

$$2\alpha \geq 5 \text{ and } 3^{2\alpha-5} = \frac{p^{2\beta+1}-1}{(754-25p)p^{2\beta}-729} = \frac{29^{2\beta+1}-1}{29^{2\beta+1}-729} \in (1,2), \text{ if } D_3 = 243;$$

$$2\alpha \geq 2 \text{ and } 3^{2\alpha-2} = \frac{p^{2\beta+1}-1}{(30p-p^2-2)p^{2\beta-1}-27} = \frac{841p^{2\beta-1}-1}{27p^{2\beta-1}-27} \in (31,33), \text{ if } D_3 = 261.$$

In any case, we get a contradiction.

Case 4. $p = 23$. By (2.5) and (2.6), we have $D_1 = 3$ and $D_2 = 9$. We start from

$$(3^{2\alpha+1}-1)(p^{2\beta+1}-1) = 2(p-1)\sigma(n) = 2(p-1)(2n-d_1-d_2-d_3)$$
$$= 28(p-1)3^{2\alpha-2}p^{2\beta} - 2(p-1)d_3.$$

Writing $(3^{2\alpha+1}-1)(p^{2\beta+1}-1) = 27p3^{2\alpha-2}p^{2\beta} - 3^{2\alpha+1} - p^{2\beta+1} + 1$, the above leads to

$$(28-p)3^{2\alpha-2}p^{2\beta} - 3^{2\alpha+1} - p^{2\beta+1} + 1 + 2(p-1)d_3 = 0. \tag{2.12}$$

Multiplying both sides of (2.12) by $28-p$ and factoring a part of it gives us

$$\left((28-p)3^{2\alpha-2}-p\right)\left((28-p)p^{2\beta}-27\right) = 28(p-1) - 2(28-p)(p-1)d_3. \tag{2.13}$$

Substituting $p = 23$, the equation (2.13) becomes

$$(5 \cdot 3^{2\alpha-2} - 23)(5 \cdot 23^{2\beta} - 27) = 616 - 220d_3. \tag{2.14}$$

Let A_1 and A_2 be the expressions on the left and the right side of (2.14), respectively. If $\alpha \geq 2$, then $A_1 > 616$, while $A_2 < 616$, which is not the case. So, $\alpha = 1$ and $A_1 = -18(5 \cdot 23^{2\beta} - 27)$. Because $3 \mid A_1$ and $3 \nmid 616$, we see that $3 \nmid d_3$. Because $d_3 \mid n$ and $n = 3^{2\alpha}23^{2\beta}$, we obtain $d_3 = 23^{b_3}$ for some $b_3 \geq 0$. If $b_3 = 0$, then $A_2 = 616 - 220 \equiv 5 \pmod{23}$; if $b_3 \geq 1$, then $A_2 \equiv 18 \pmod{23}$. But, $A_1 \equiv 3 \pmod{23}$, and so $A_1 = A_2$ and $A_1 \not\equiv A_2 \pmod{23}$, which is not possible.

Case 5. $p = 19$. By (2.5), $D_1 = 3$. So, $\{D_2, D_3\} \subseteq \{9, 19, 27, 57, \ldots\}$. If $D_2 \geq 19$ and $D_3 \geq 57$, then (2.4) implies that $2 < (3/2)(19/18) + 1/3 + 1/19 + 1/57 = 1.9868\ldots$, which is not true. Therefore, ($D_2 = 9$) or ($D_2 = 19$ and $D_3 = 27$).

Case 5.1. $D_2 = 9$. Then, the computation in Case 4 still works and (2.13) holds. Substituting $p = 19$ in (2.13) and dividing both sides by 9, we obtain

$$(3^{2\alpha} - 19)(19^{2\beta} - 3) = 56 - 36d_3. \tag{2.15}$$

Let A_3, A_4 be the expressions on the left and the right side of (2.15), respectively. If $\alpha \geq 2$, then $A_3 > 56$, while $A_4 < 56$, which is not true. Therefore, $\alpha = 1$. Then, $11 \equiv A_3 \equiv A_4 \equiv -1 + 2d_3 \pmod{19}$, and so $19 \nmid d_3$. Because $d_3 \mid n$ and $n = 3^{2\alpha}p^{2\beta} = 3^2 \cdot 19^{2\beta}$, we see that $d_3 = 1, 3, 9$. Substituting $d_3 = 1, 3, 9$ in (2.15) leads to $5 \cdot 19^{2\beta} = 5, 41, 149$, respectively, which has no solution.

Case 5.2. $D_2 = 19$ and $D_3 = 27$. Similar to the calculations for (2.7) and (2.14) but with different values of d_2 and d_3, we obtain, after an algebraic manipulation, that

$$3^{2\alpha-3} = \frac{361 \cdot 19^{2\beta-1} - 1}{117 \cdot 19^{2\beta-1} - 81} \in (3,4),$$

which is not possible.

Case 6. $p \in \{11, 13, 17\}$. Then by (2.5), we have $D_1 = 3$. The possible values of D_2 and D_3 listed in increasing order are 9, p, 27, $3p$, 81, $9p$, $\min\{p^2, 243\}$, $\max\{p^2, 243\}$, …. We can eliminate some cases by using (2.4) as before. If $p = 17$ and $D_2 \geq 27$, then (2.4) implies $2 < (3/2)(17/16) + 1/3 + 1/27 + 1/51 < 2$; if $p = 17, D_2 \geq 17$, and $D_3 \geq 81$, then (2.4) leads to $2 < (3/2)(17/16) + 1/3 + 1/17 + 1/81 < 2$. Similarly, if $p = 13$, then we must have $D_2 < 39$; if $p = 13$ and $D_2 \geq 27$, then it forces $D_3 < 243$; if $p = 11$, then $D_2 < 81$ or $D_3 < 243$. Therefore, we obtain

$$\text{if } p = 17, \text{ then } (D_2 = 9) \text{ or } (D_2 = 17 \text{ and } D_3 \in \{27, 51\}); \tag{2.16}$$

$$\text{if } p = 13, \text{ then } (D_2 \in \{9, 13\}) \text{ or } (D_2 = 27 \text{ and } D_3 \in \{39, 81, 117, 169\}); \tag{2.17}$$

$$\text{if } p = 11, \text{ then } (D_2 \in \{9, 11, 27, 33\}) \text{ or } (D_2 = 81 \text{ and } D_3 \in \{99, 121\}) \text{ or } (D_2 = 99 \text{ and } D_3 = 121). \tag{2.18}$$

We divide our calculations according to the values of D_2 and D_3 listed in (2.16), (2.17), and (2.18).

Case 6.1. $D_2 = 9$ (so p can be any of 11, 13, or 17). Because $D_1 = 3$ and $D_2 = 9$, equation (2.13) holds. Substituting $p = 11, 13, 17$ in (2.13), we obtain, respectively

$$(17 \cdot 3^{2\alpha-2} - 11)(17 \cdot 11^{2\beta} - 27) = 280 - 340d_3 \text{ (if } p = 11), \tag{2.19}$$

$$(15 \cdot 3^{2\alpha-2} - 13)(15 \cdot 13^{2\beta} - 27) = 336 - 360d_3 \text{ (if } p = 13), \tag{2.20}$$

$$(11 \cdot 3^{2\alpha-2} - 17)(11 \cdot 17^{2\beta} - 27) = 448 - 352d_3 \text{ (if } p = 17), \tag{2.21}$$

where d_3 in (2.19) is a proper divisor of $3^{2\alpha}11^{2\beta}$, d_3 in (2.20) is a proper divisor of $3^{2\alpha}13^{2\beta}$, and d_3 in (2.21) is a proper divisor of $3^{2\alpha}17^{2\beta}$. Because $\alpha, \beta \geq 1$, the left side of (2.19) and (2.20) are positive, whereas the right side of (2.19) and (2.20) are negative. So, (2.19) and (2.20) do not lead to a solution. For (2.21), we have $448 - 352d_3 \leq 96$, which implies $\alpha = 1$. Then, (2.21) reduces to $3 \cdot 17^{2\beta} + 13 - 16d_3 = 0$. Reducing this mod 3 and mod 17, we see that $d_3 \equiv 1 \pmod 3$ and $d_3 \equiv 4 \pmod{17}$. Because $d_3 \mid 3^{2\alpha}17^{2\beta}$, $3 \nmid d_3$, and $17 \nmid d_3$, we obtain $d_3 = 1$, which contradicts that $d_3 \equiv 4 \pmod{17}$. Thus, there is no solution in this case.

Case 6.2. $D_2 = p$, where $p \in \{11, 13\}$. Similar to the calculation for (2.13), we have

$$\begin{aligned}(3^{2\alpha+1} - 1)(p^{2\beta+1} - 1) &= 2(p-1)\sigma(n) = 2(p-1)(2n - d_1 - d_2 - d_3) \\ &= 2(p-1)(2 \cdot 3^{2\alpha}p^{2\beta} - 3^{2\alpha-1}p^{2\beta} - 3^{2\alpha}p^{2\beta-1} - d_3).\end{aligned}$$

Let $B_p = 16p - p^2 - 6$. Following a straightforward algebraic manipulation and multiplying both sides by B_p, the above leads to

$$(B_p 3^{2\alpha-1} - p^2)(B_p p^{2\beta-1} - 9) = 9p^2 - B_p - 2B_p(p-1)d_3. \tag{2.22}$$

Substituting $p = 11$ in (2.22), we obtain

$$(49 \cdot 3^{2\alpha-1} - 121)(49 \cdot 11^{2\beta-1} - 9) = 1040 - 980d_3. \tag{2.23}$$

Because $\alpha, \beta \geq 1$, the left side of (2.23) is larger than 60, whereas the right side of (2.23) is at most 60, so (2.23) does not give a solution. Next, substituting $p = 13$ in (2.22) and dividing both sides by 3, we obtain

$$(33 \cdot 3^{2\alpha-1} - 169)(11 \cdot 13^{2\beta-1} - 3) = 496 - 264d_3. \tag{2.24}$$

Because the right side of (2.24) is at most 232, we obtain $\alpha = 1$ and (2.24) reduces to

$$35 \cdot 13^{2\beta-1} - 12d_3 + 13 = 0. \tag{2.25}$$

Recall that $d_3 \mid n$ and $n = 3^{2\alpha}p^{2\beta} = 3^2 \cdot 13^{2\beta}$. So, $d_3 = 3^{a_3}13^{b_3}$ for some $a_3 \in \{0, 1, 2\}$ and $b_3 \geq 0$. Reducing (2.25) modulo 7, we see that $2d_3 \equiv 1 \pmod 7$. If $a_3 = 0$, then $2d_3 = 2 \cdot 13^{b_3} \equiv$

$2(-1)^{b_3} \equiv 2, -2 \not\equiv 1 \pmod 7$. If $a_3 = 2$, then $2d_3 = 18 \cdot 13^{b_3} \equiv 4(-1)^{b_3} \equiv 4, -4 \not\equiv 1 \pmod 7$. Therefore, $a_3 = 1$ and (2.25) becomes

$$35 \cdot 13^{2\beta-1} - 36 \cdot 13^{b_3} + 13 = 0. \tag{2.26}$$

Suppose, for a contradiction, that $\beta \geq 2$. Reducing (2.26) modulo 13^2, we obtain $36 \cdot 13^{b_3} \equiv 13 \pmod{13^2}$. If $b_3 \geq 2$, then $36 \cdot 13^{b_3} \equiv 0 \not\equiv 13 \pmod{13^2}$. If $b_3 = 1$, then $36 \cdot 13^{b_3} - 13 = 35 \cdot 13 \not\equiv 0 \pmod{13^2}$. If $b_3 = 0$, then $36 \cdot 13^{b_3} = 36 \not\equiv 13 \pmod{13^2}$. In any case, we reach a contradiction. Therefore, $\beta = 1$. Substituting $\beta = 1$ in (2.26), we obtain $b_3 = 1$, and so $d_3 = 3^{a_3}13^{b_3} = 39$. This leads to $n = 3^{2\alpha}p^{2\beta} = 3^2 \cdot 13^2$ with the deficient divisors $d_1 = n/D_1 = 3 \cdot 13^2 = 507$, $d_2 = n/D_2 = 3^2 \cdot 13 = 117$, and $d_3 = 39$, which we already verified at the beginning of the proof that this is indeed a solution to our problem. The elimination for the other cases can be done in a similar way to the previous cases, so we give less details. Recall that $D_1 = 3$. The other cases are as follows:

(i) $p = 17$, $D_2 = 17$, and $D_3 \in \{27, 51\}$ (this is the remaining case from (2.16)).
(ii) $p = 13$, $D_2 = 27$, and $D_3 \in \{39, 81, 117, 169\}$ (this is the remaining case from (2.17)).
(iii) $p = 11$, $D_2 \in \{27, 33\}$.
(iv) $p = 11$, $D_2 = 81$, and $D_3 \in \{99, 121\}$.
(v) $p = 11$, $D_2 = 99$, and $D_3 = 121$.

In (i), (ii), (iv), and (v), we know the values of D_1, D_2, D_3, and so we have the values of d_1, d_2, d_3. We start from the equality $\sigma(n) = 2n - d_1 - d_2 - d_3$, perform the usual algebraic manipulation, and try to write the minimum nonnegative power of 3 appearing among d_1, d_2, d_3 in terms of the other variables. We obtain the following results. For (i), we have $p = 17$, $D_1 = 3$, $D_2 = 17$, and

$$\text{if } D_3 = 27, \text{ then } 2\alpha \geq 3 \text{ and } 3^{2\alpha-3} = \frac{289 \cdot 17^{2\beta-1} - 1}{337 \cdot 17^{2\beta-1} - 81} \in (0, 1);$$
$$\text{if } D_3 = 51, \text{ then } 3^{2\alpha-1} = \frac{289 \cdot 17^{2\beta-1} - 1}{9 \cdot 17^{2\beta-1} - 9} \in (32, 35),$$

which is a contradiction. For (ii), we have $p = 13$, $D_1 = 3$, $D_2 = 27$, $2\alpha \geq 3$, and

$$\text{if } D_3 = 39, \text{ then } 3^{2\alpha-3} = \frac{169 \cdot 13^{2\beta-1} - 1}{177 \cdot 13^{2\beta-1} - 81} \in (0, 1);$$
$$\text{if } D_3 = 81, \text{ then } 2\alpha \geq 4 \text{ and } 3^{2\alpha-4} = \frac{13 \cdot 13^{2\beta} - 1}{15 \cdot 13^{2\beta} - 243} \in (0, 1);$$
$$\text{if } D_3 = 117, \text{ then } 3^{2\alpha-3} = \frac{169 \cdot 13^{2\beta-1} - 1}{33 \cdot 13^{2\beta-1} - 81} \in (5, 7);$$
$$\text{if } D_3 = 169, \text{ then } 3^{2\alpha-3} = \frac{2197 \cdot 13^{2\beta-2} - 1}{141 \cdot 13^{3\beta-2} - 81} \in (15, 37).$$

The first three cases above give a contradiction. The last case implies that

$$2197 \cdot 13^{2\beta-2} - 1 = 27(141 \cdot 13^{2\beta-2} - 81),$$

which leads to $1610 \cdot 13^{2\beta-2} = 2186$, which is impossible. For (iv), we have $p = 11$, $D_1 = 3$, $D_2 = 81$, $2\alpha \geq 4$, and

$$\text{if } D_3 = 99, \text{ then } 3^{2\alpha-4} = \frac{121 \cdot 11^{2\beta-1} - 1}{103 \cdot 11^{2\beta-1} - 243} \in (1, 2);$$
$$\text{if } D_3 = 121, \text{ then } 3^{2\alpha-4} = \frac{1331 \cdot 11^{2\beta-2} - 1}{773 \cdot 11^{2\beta-2} - 243} \in (1, 3),$$

which is false. For (v), we have $p = 11$, $D_1 = 3$, $D_2 = 99$, $D_3 = 121$, which leads to

$$3^{2\alpha-2} = \frac{1331 \cdot 11^{2\beta-2} - 1}{37 \cdot 11^{2\beta-2} - 27} \in (35, 37) \cup \{133\},$$

which is not possible. We now consider (iii). We have $p = 11$, $D_1 = 3$, $D_2 \in \{27, 33\}$. We know the values of d_1, d_2 but not d_3. We start with $\sigma(n) = 2n - d_1 - d_2 - d_3$ and write d_3 in terms of the product of the other variables. Similar to the calculation for (2.13), we obtain

$$\text{if } D_2 = 27, \text{ then } 2\alpha \geq 3 \text{ and } (3^{2\alpha-3} - 1)(11^{2\beta+1} - 81) = 80 - 20d_3; \tag{2.27}$$

$$\text{if } D_2 = 33, \text{ then } (3^{2\alpha+1} - 121)(11^{2\beta-1} - 1) = 120 - 20d_3. \tag{2.28}$$

In (2.27), 2α is an even integer ≥ 3, so $2\alpha \geq 4$, and thus, the left side of (2.27) is larger than 80, whereas the right side of (2.27) is less than 80, which is a contradiction. Because the right side of (2.28) is less than 120, we see that $\alpha = 1$ and (2.28) reduces to $47 \cdot 11^{2\beta+1} - 10d_3 + 13 = 0$. Reducing this modulo 11, we see that $10d_3 \equiv 2 \pmod{11}$, and therefore, $d_3 \equiv 9 \pmod{11}$. So, $11 \nmid d_3$. Because $d_3 \mid n$ and $n = 3^{2\alpha}p^{2\beta} = 3^2 \cdot 11^{2\beta}$, we have $d_3 = 1, 3, 9$. Because $d_3 \equiv 9 \pmod{11}$, $d_3 = 9$ only. Then $47 \cdot 11^{2\beta+1} - 90 + 13 = 0$. This leads to $47 \cdot 11^{2\beta+1} = 77$, which has no solution.

Case 7. $p = 7$. Then, $\{D_1, D_2, D_3\} \subseteq \{3, 7, 9, 21, \ldots\}$. If $D_1 \geq 7$ and $D_2 \geq 21$, then (2.4) implies $2 < (3/2)(7/6) + 1/7 + 1/21 + 1/21 < 2$, which is impossible. So, ($D_1 = 3$) or ($D_1 = 7$ and $D_2 = 9$). If $D_1 = 3$, then $d_1 = 3^{2\alpha-1}7^{2\beta}$ and we have

$$\begin{aligned}
0 &= 12\left(\sigma(n) - 2n + d_1 + d_2 + d_3\right) \\
&= (3^{2\alpha+1} - 1)(7^{2\beta+1} - 1) - 24n + 12(d_1 + d_2 + d_3) \\
&= 3^{2\alpha}7^{2\beta}\left(21 - 3/7^{2\beta} - 7/3^{2\alpha} - 24\right) + 1 + 12(d_1 + d_2 + d_3) \\
&= 1 + 12d_1\left(1 + d_2/d_1 + d_3/d_1\right) - 3^{2\alpha}7^{2\beta}(3 + 3/7^{2\beta} + 7/3^{2\alpha}) \\
&> 1 + 12d_1 - 3^{2\alpha}7^{2\beta}(3 + 3/7^2 + 7/3^2) \\
&> 12d_1 - 3^{2\alpha}7^{2\beta}(4) = 0,
\end{aligned}$$

which is a contradiction. So, $D_1 = 7$ and $D_2 = 9$. We start with $\sigma(n) = 2n - d_1 - d_2 - d_3$, substitute $d_1 = 3^{2\alpha}7^{2\beta-1}$, $d_2 = 3^{2\alpha-2}7^{2\beta}$, and do the usual algebraic manipulation to obtain

$$(3^{2\alpha-1} - 49)(7^{2\beta-1} - 9) = 440 - 12d_3. \tag{2.29}$$

If $\alpha \geq 3$ and $\beta \geq 2$, then the left side of (2.29) is larger than 440, whereas the right side of (2.29) is smaller than 440. Therefore, ($\alpha \in \{1, 2\}$) or ($\alpha \geq 3$ and $\beta = 1$). Because $d_3 \mid n$ and $n = 3^{2\alpha}7^{2\beta}$, $d_3 = 3^{a_3}7^{b_3}$ for some $a_3, b_3 \geq 0$.

Case 7.1. $\alpha \geq 3$ and $\beta = 1$. Then, (2.29) reduces to

$$3^{2\alpha-1} + 171 = 6 \cdot 3^{a_3}7^{b_3}. \tag{2.30}$$

Because $3^{2\alpha-1} + 171 = 3^2(3^{2\alpha-3} + 19)$, we obtain $3^2 \parallel 6d_3$, which implies $a_3 = 1$. Dividing both sides of (2.30) by 9, we obtain $3^{2\alpha-3} + 19 = 2 \cdot 7^{b_3}$. Reducing this modulo 3, we have a contradiction.

Case 7.2. $\alpha \in \{1, 2\}$. If $\alpha = 2$, then (2.29) leads to $d_3 \equiv 0 \pmod{11}$, which contradicts that $d_3 = 3^{a_3}7^{b_3}$. So, $\alpha = 1$. Then, $a_3 \in \{0, 1, 2\}$ and (2.29) reduces to $23 \cdot 7^{2\beta-1} - 6d_3 + 13 = 0$. From this, we see that $7 \nmid d_3$. So, $b_3 = 0$, $d_3 = 3^{a_3}$, and the above equation becomes $23 \cdot 7^{2\beta-1} - 6 \cdot 3^{a_3} + 13 = 0$. Substituting $a_3 = 0, 1, 2$, we obtain $23 \cdot 7^{2\beta-1} = -7, 5, 41$, which is not possible. Hence, there is no solution in this case.

Case 8. $p = 5$. Then, the possible values of D_1, D_2, D_3 listed in increasing order are $3, 5, 9, 15, 25, \ldots$. If $D_1 \geq 25$, then (2.4) implies $2 < (3/2)(5/4) + 1/25 + 1/25 + 1/25 < 2$, which is false. Therefore, $D_1 \in \{3, 5, 9, 15\}$. It is possible to obtain bounds for D_2 and D_3 as in the other cases, but the same method will lead to a longer calculation. In this case, it is better to get a bound only for D_1 and go back to d_1, d_2, d_3. Let $d_1 = 3^{a_1}5^{b_1}$, $d_2 = 3^{a_2}5^{b_2}$, and $d_3 = 3^{a_3}5^{b_3}$, where $a_i, b_i \geq 0$, and recall that $n > d_1 > d_2 > d_3 \geq 1$ and d_1, d_2, d_3 are the deficient divisors of $n = 3^{2\alpha}5^{2\beta}$. In addition, from $\sigma(n) = 2n - (d_1 + d_2 + d_3)$, we get

$$\begin{aligned}(3^{2\alpha+1} - 1)(5^{2\beta+1} - 1) &= 16 \cdot 3^{2\alpha}5^{2\beta} - 8(d_1 + d_2 + d_3)\\ &= 16 \cdot 3^{2\alpha}5^{2\beta} - 8(3^{a_1}5^{b_1} + 3^{a_2}5^{b_2} + 3^{a_3}5^{b_3}).\end{aligned} \tag{2.31}$$

From (2.31), we see that $8(d_1 + d_2 + d_3) = 3^{2\alpha}5^{2\beta} + 3^{2\alpha+1} + 5^{2\beta+1} - 1$, which implies

$$1 < \frac{8}{3^{2\alpha}5^{2\beta}}(d_1 + d_2 + d_3) < 1 + \frac{3}{5^2} + \frac{5}{3^2} < 2. \tag{2.32}$$

Because $D_1 \in \{3, 5, 9, 15\}$ and $d_1 = n/D_1$, we see that

$$(a_1, b_1) = (2\alpha - 1, 2\beta), (2\alpha, 2\beta - 1), (2\alpha - 2, 2\beta), \text{ or } (2\alpha - 1, 2\beta - 1). \tag{2.33}$$

Observe that $3^4 \equiv 1 \pmod 5$, $5^2 \equiv 1 \pmod 3$, and the exponents 4 and 2 are the smallest positive integers satisfying each congruence. From this, it is not difficult to verify that the left side of (2.31) satisfies

$$(3^{2\alpha+1} - 1)(5^{2\beta+1} - 1) \equiv \begin{cases} 3 \pmod 5, & \text{if } \alpha \text{ is even;} \\ 4 \pmod 5, & \text{if } \alpha \text{ is odd,} \end{cases} \tag{2.34}$$

$$(3^{2\alpha+1} - 1)(5^{2\beta+1} - 1) \equiv 2 \pmod 3. \tag{2.35}$$

Because 5 does not divide the left side of (2.31), at least one of d_1, d_2, d_3 is not divisible by 5, that is, at least one of b_1, b_2, b_3 is zero. By (2.33), we see that $b_1 \neq 0$. Thus,

$$b_1 \neq 0 \text{ and } \min\{b_2, b_3\} = 0. \tag{2.36}$$

Suppose, for a contradiction, that $a_1 = a_2 = a_3 = 0$. That is, $d_1 = 5^{b_1}$, $d_2 = 5^{b_2}$, $d_3 = 5^{b_3}$. Because $d_1 > d_2 > d_3$, we have $b_1 > b_2 > b_3$. So by (2.36), $b_3 = 0$ and $b_1 > b_2 > 0$. Then, the right side of (2.31) is $\equiv 2 \pmod 5$, contradicting (2.34). So, one of a_1, a_2, a_3 is not zero. By (2.35) and (2.31), one of d_1, d_2, d_3 is not divisible by 3, and so one of a_1, a_2, a_3 is zero. We conclude that

$$\max\{a_1, a_2, a_3\} \geq 1 \text{ and } \min\{a_1, a_2, a_3\} = 0. \tag{2.37}$$

The right side of (2.31) is congruent to

$$\begin{cases} (0 + 0 + 5^{b_3}) \pmod 3, & \text{if } a_1 \neq 0, a_2 \neq 0, \text{ and } a_3 = 0; \\ (0 + 5^{b_2} + 0) \pmod 3, & \text{if } a_1 \neq 0, a_2 = 0, \text{ and } a_3 \neq 0; \\ (5^{b_1} + 0 + 0) \pmod 3, & \text{if } a_1 = 0, a_2 \neq 0, \text{ and } a_3 \neq 0; \\ (5^{b_1} + 5^{b_2} + 0) \pmod 3, & \text{if } a_1 = a_2 = 0, \text{ and } a_3 \neq 0; \\ (5^{b_1} + 0 + 5^{b_3}) \pmod 3, & \text{if } a_1 = a_3 = 0, \text{ and } a_2 \neq 0; \\ (0 + 5^{b_2} + 5^{b_3}) \pmod 3, & \text{if } a_2 = a_3 = 0, \text{ and } a_1 \neq 0. \end{cases} \tag{2.38}$$

By comparing (2.31), (2.35), and (2.38), we obtain the parities of b_1, b_2, b_3 as follows. If $5^b \equiv 2 \pmod 3$, then b is odd. If $5^x + 5^y \equiv 2 \pmod 3$, then x and y are even. For convenience, for each $i \in \{1, 2, 3\}$, if b_i is odd, we write b_i' for b_i; if b_i is even, then we replace b_i by b_i''. Therefore, for each $i \in \{1, 2, 3\}$, b_i', $b_i'' \geq 0$, $b_i' = b_i$ is odd, and $b_i'' = b_i$ is even, and there are six cases to consider as follows:

Case 8.1. $d_1 = 3^{a_1}5^{b_1}$, $d_2 = 3^{a_2}5^{b_2}$, $d_3 = 5^{b_3'}$, $a_1 \neq 0$, $a_2 \neq 0$, and $a_3 = 0$,
Case 8.2. $d_1 = 3^{a_1}5^{b_1}$, $d_2 = 5^{b_2'}$, $d_3 = 3^{a_3}5^{b_3}$, $a_1 \neq 0$, $a_2 = 0$, and $a_3 \neq 0$,
Case 8.3. $d_1 = 5^{b_1'}$, $d_2 = 3^{a_2}5^{b_2}$, $d_3 = 3^{a_3}5^{b_3}$, $a_1 = 0$, $a_2 \neq 0$, and $a_3 \neq 0$,
Case 8.4. $d_1 = 5^{b_1''}$, $d_2 = 5^{b_2''}$, $d_3 = 3^{a_3}5^{b_3}$, $a_1 = a_2 = 0$, and $a_3 \neq 0$,
Case 8.5. $d_1 = 5^{b_1''}$, $d_2 = 3^{a_2}5^{b_2}$, $d_3 = 5^{b_3''}$, $a_1 = a_3 = 0$, and $a_2 \neq 0$,
Case 8.6. $d_1 = 3^{a_1}5^{b_1}$, $d_2 = 5^{b_2''}$, $d_3 = 5^{b_3''}$, $a_2 = a_3 = 0$, and $a_1 \neq 0$.

Some cases are shorter, but we will begin with Case 8.1.

Case 8.1. Because $b_3' \neq 0$, we obtain, by (2.36), that $b_1 \neq 0$ and $b_2 = 0$. By (2.33), there are four cases to consider. If $a_1 = 2\alpha - 1$ and $b_1 = 2\beta$, then

$$8(d_1 + d_2 + d_3)/(3^{2\alpha}5^{2\beta}) = 8\left(3^{2\alpha-1}5^{2\beta} + 3^{a_2} + 5^{b_3'}\right)/\left(3^{2\alpha}5^{2\beta}\right) > 8/3 > 2,$$

which contradicts (2.32). Next, suppose that $a_1 = 2\alpha$ and $b_1 = 2\beta - 1$. Because $3^{a_2} = d_2 > d_3 = 5^{b_3'} \geq 5$, we obtain $a_2 \geq 2$. Thus,

$$\begin{aligned} 0 &= 8\left(\sigma(n) - 2n + d_1 + d_2 + d_3\right) = 8(d_1 + d_2 + d_3) - 3^{2\alpha}5^{2\beta} - 3^{2\alpha+1} - 5^{2\beta+1} + 1 \\ &> 8(3^{2\alpha}5^{2\beta-1} + 3^2 + 5) - 3^{2\alpha}5^{2\beta} - 3^{2\alpha+1} - 5^{2\beta+1} \\ &= (3^{2\alpha+1} - 25)(5^{2\beta-1} - 1) + 87 > 0, \end{aligned}$$

which is false. Next, consider the case $(a_1, b_1) = (2\alpha - 2, 2\beta)$. Because $a_1 \neq 0$, $\alpha \geq 2$. If $\beta \geq 2$, then (2.32) implies that

$$\begin{aligned} 1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-2}5^{2\beta} + 3^{a_2} + 5^{b_3'}) \leq \frac{8}{3^{2\alpha}5^{2\beta}}\left(3^{2\alpha-2}5^{2\beta} + 3^{2\alpha} + 5^{2\beta-1}\right) \\ &= 8\left(\frac{1}{3^2} + \frac{1}{5^{2\beta}} + \frac{1}{3^{2\alpha}\cdot 5}\right) \leq 8\left(\frac{1}{3^2} + \frac{1}{5^4} + \frac{1}{3^4 \cdot 5}\right) < 1, \end{aligned}$$

which is a contradiction. So, $\beta = 1$. Then, $d_3 = 5$.

Starting with $0 = 8\left(\sigma(n) - 2n + d_1 + d_2 + d_3\right)$, and then simplifying leads to $2 \cdot 3^{a_2} = 13 \cdot 3^{2\alpha-2} + 21$. Because $13 \cdot 3^{2\alpha-2} + 21 > 2 \cdot 3^{2\alpha-1}$, we obtain $a_2 = 2\alpha$. But, then $21 = 2 \cdot 3^{a_2} - 13 \cdot 3^{2\alpha-2} = 5 \cdot 3^{2\alpha-2} \equiv 0 \pmod 5$, a contradiction. Next, we consider the last case: $(a_1, b_1) = (2\alpha - 1, 2\beta - 1)$. If $\alpha \geq 2$ or $\beta \geq 2$, then (2.32) implies

$$\begin{aligned} 1 &< \frac{8}{3^{2\alpha}5^{2\beta}}\left(3^{2\alpha-1}5^{2\beta-1} + 3^{a_2} + 5^{b_3'}\right) \\ &\leq 8\left(\frac{1}{15} + \max\left\{\frac{1}{25} + \frac{1}{3^4 \cdot 5}, \frac{1}{5^4} + \frac{1}{3^2 \cdot 5}\right\}\right) < 1, \end{aligned}$$

which is impossible. So, $\alpha = 1 = \beta$. Then, $a_1 = 1 = b_1$. Because $15 = d_1 > 3^{a_2} = d_2 > d_3 = 5^{b_3'} = 5$, we have $d_2 = 9$. Now, it is easy to verify that $\sigma(n) - 2n + d_1 + d_2 + d_3 = -18 \neq 0$. So, there is no solution in this case.

Case 8.2. Because $b_2 = b_2' \neq 0$, we obtain, by (2.36), that $b_3 = 0$. Similar to Case 8.1, we divide our calculation into four cases according to the values of a_1 and b_1 as given in (2.33). If $(a_1, b_1) = (2\alpha - 1, 2\beta)$, then $8(d_1 + d_2 + d_3)/(3^{2\alpha}5^{2\beta}) > 8d_1/(3^{2\alpha}5^{2\beta}) > 8/3 > 2$, contradicting (2.32). If $(a_1, b_1) = (2\alpha, 2\beta - 1)$, then $d_2 \geq 5$, $d_3 \geq 3$, and

$$\begin{aligned} 0 &= 8\left(\sigma(n) - 2n + d_1 + d_2 + d_3\right) = 8(d_1 + d_2 + d_3) - 3^{2\alpha}5^{2\beta} - 3^{2\alpha+1} - 5^{2\beta+1} + 1 \\ &\geq 3^{2\alpha+1}5^{2\beta-1} - 3^{2\alpha+1} - 5^{2\beta+1} + 65 \\ &= (3^{2\alpha+1} - 25)(5^{2\beta-1} - 1) + 40 > 0, \end{aligned}$$

which is not possible. Suppose $(a_1, b_1) = (2\alpha - 2, 2\beta)$. Because $a_1 \neq 0$, we have $\alpha \geq 2$. If $\beta \geq 2$, then (2.32) implies

$$\begin{aligned}1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(3^{2\alpha-2}5^{2\beta} + 5^{b_2'} + 3^{a_3}) \\ &\leq \frac{8}{3^{2\alpha}5^{2\beta}}\left(3^{2\alpha-2}5^{2\beta} + 5^{2\beta-1} + 3^{2\alpha}\right) \leq 8\left(\frac{1}{9} + \frac{1}{3^4 \cdot 5} + \frac{1}{5^4}\right) < 1,\end{aligned}$$

which is false. So, $\beta = 1$. Then, $d_2 = 5$ and $d_3 = 3$.

Starting from $8\,(\sigma(n) - 2n + d_1 + d_2 + d_3) = 0$ and then simplifying leads to $13 \cdot 3^{2\alpha-2} + 15 = 0$, which is impossible. The last case of (2.33) is $(a_1, b_1) = (2\alpha - 1, 2\beta - 1)$. If $\alpha \geq 2$ or $\beta \geq 2$, then (2.32) implies

$$\begin{aligned}1 &< \frac{8}{3^{2\alpha}5^{2\beta}}\left(3^{2\alpha-1}5^{2\beta-1} + 5^{b_2'} + 3^{a_3}\right) \\ &\leq 8\left(\frac{1}{15} + \frac{1}{3^{2\alpha} \cdot 5} + \frac{1}{5^{2\beta}}\right) \\ &\leq 8\left(\frac{1}{15} + \max\left\{\frac{1}{3^4 \cdot 5} + \frac{1}{5^2}, \frac{1}{3^2 \cdot 5} + \frac{1}{5^4}\right\}\right) < 1,\end{aligned}$$

which is not true. Thus, $\alpha = \beta = 1$. So, $a_1 = b_1 = 1$, $d_2 = 5$, and $d_3 = 3$. Now, it is easy to verify that $\sigma(n) - 2n + d_1 + d_2 + d_3 = -24 \neq 0$. So, there is no solution in this case.

Case 8.3. By (2.32), we obtain $1 < 8(3d_1)/(3^{2\alpha}5^{2\beta}) \leq 24 \cdot 5^{2\beta-1}/(3^{2\beta}5^{2\beta}) \leq 24/45 < 1$, a contradiction.

Case 8.4. By (2.32), we obtain

$$\begin{aligned}1 &< \frac{8}{3^{2\alpha}5^{2\beta}}(d_1 + 2d_2) = \frac{8}{3^{2\alpha}5^{2\beta}}(5^{b_1''} + 2 \cdot 5^{b_2''}) \\ &\leq \frac{8}{3^{2\alpha}5^{2\beta}}(5^{2\beta} + 2 \cdot 5^{2\beta-2}) \leq 8\left(\frac{1}{9} + \frac{2}{9 \cdot 25}\right) < 1,\end{aligned}$$

which is not possible.

Case 8.5. If $\alpha \geq 2$, then (2.32) implies that

$$1 < \frac{8}{3^{2\alpha}5^{2\beta}}(2d_1 + d_3) \leq \frac{8}{3^{2\alpha}5^{2\beta}}(2 \cdot 5^{2\beta} + 5^{2\beta-2}) \leq 8\left(\frac{2}{3^4} + \frac{1}{3^4 \cdot 5^2}\right) < 1,$$

which is false. Therefore, $\alpha = 1$. Then the left side of (2.31) is $\equiv 4 \pmod 5$, whereas the right side is $\equiv 2(d_1 + d_2 + d_3) \equiv 2(3^{a_2}5^{b_2} + 5^{b_3''}) \pmod 5$. By (2.36), $b_2 = 0$ or $b_3 = 0$. If $b_2 = 0$ and $b_3 \neq 0$, then $5^2 \leq d_3 < d_2 = 3^{a_2}$, and so $a_2 \geq 3$, contradicting that $d_2 \mid n$ and $n = 3^{2\alpha}5^{2\beta} = 3^2 \cdot 5^{2\beta}$. If $b_2 \neq 0$ and $b_3 = 0$, then $2(3^{a_2}5^{b_2} + 5^{b_3''}) \equiv 2 \pmod 5$, which is not the case. Because $\alpha = 1$, $a_2 \in \{1, 2\}$. So if $b_2 = b_3 = 0$, then $2(3^{a_2}5^{b_2} + 5^{b_3''}) \equiv 3, 0 \pmod 5$, which is not true. So there is no solution in this case.

Case 8.6. Because $5^{b_2''} = d_2 > d_3 \geq 1$, we have $b_2'' \neq 0$. By (2.36), we see that $b_3 = 0$. Then, the right side of (2.31) is $\equiv 2(3^{a_1}5^{b_1} + 5^{b_2''} + 5^{b_3''}) \equiv 2 \pmod 5$, contradicting (2.34).

This completes the proof of this theorem. □

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