A NOTE ON LEDIN'S SUMMATION PROBLEM

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Abstract. This paper takes a historical view of some long-standing issues associated with polynomials developed from sums of Fibonacci numbers in which the latter have powers of integers as coefficients. The sequences of coefficients of these polynomials are arrayed in matrices with links to *The On-Line Encyclopedia of Integer Sequences*. Problems for further study are conjectured, including inhomogeneous gibonacci difference equations.

1. Introduction

Ledin [9] studied Fibonacci sums of the form

$$S(m,n)=\sum_{k=1}^{n}k^mF_k \tag{1.1}$$

for integers $m,n\geq 0$ and where F_k is the kth Fibonacci number, and generalized the calculations of them up to a point. Number sums in n for the particular cases $S(0,n)$ [2], $S(1,n)$ [3], and $S(3,n)$ [7] were known at the time of publication of his paper. Ledin's method was generalizable to somewhat analogous problems, such as Fibonacci convolutions, but there were aspects of the computations that were limited, possibly because the current version of *The On-Line Encyclopedia of Integer Sequences* (OEIS) [10] was not then available.

It is the purpose of this note to open up, from Ledin's original paper, some number and combinatorial issues for further exploration: the results presented in this note can be used as undergraduate project exercises [12] and ideas for further research [11]. In relation to the former, there are tables here that invite further exploration, and in relation to the latter, there are a number of conjectures that are plausible enough to invite further exploration.

2. Notation

Ledin showed that (1.1) can be expressed in the form

$$S(m,n)=P_1(m,n)F_n+P_2(m,n)F_{n+1}+C(m), \tag{2.1}$$

in which $C(m)$ is a constant depending only on m, and $P_1(m,n)$ and $P_2(m,n)$ are polynomials in n of degree m of the form

$$P_i(m,n)=\sum_{j=0}^{m}(-1)^j\binom{m}{j}M_{i,j}n^{m-j},\quad i=1,2, \tag{2.2}$$

where "$M_{1,j}$ and $M_{2,j}$ are certain numbers, the law of formation of which is yet to be determined" [9]. The following examples, from Ledin for $m=0,1,2,3$, will make the subsequent discussion clearer.

$$S(0,n) = (1)F_n + (1)F_{n+1} - 1;$$
$$S(1,n) = (n-1)F_n + (n-2)F_{n+1} + 2;$$
$$S(2,n) = (n^2-2n+5)F_n + (n^2-4n+8)F_{n+1} - 8;$$
$$S(3,n) = (n^3-3n^2+15n-31)F_n + (n^3-6n^2+24n-50)F_{n+1} + 50.$$

3. Patterns

We can see that $M_{1,j}$ and $M_{2,j}$ follow the patterns set out in Table 1.

j	0	1	2	3	4	5	6	7	OEIS
$M_{1,j}$	1	1	5	31	257	2671	33305	484471	A000556
$M_{2,j}$	1	2	8	50	416	4322	53888	783890	A000557

Table 1: $M_{1,j}$ and $M_{2,j}$

We now express $P_i(m,n)$ in an alternative manner, so that we can focus on the coefficients in the polynomials. Let

$$P_i(m,n) = \sum_{r=0}^{m} Q_i(m,r)n^r, \quad i = 1,2, \tag{3.1}$$

in other words, the $Q_i(m,r)$ are the coefficients of n^r in $P_i(m,n)$. Examples of the $Q_i(m,r)$ are set out in Tables 2 and 3 (The values shown below have been calculated using modern software. Ledin provides formulas for $S(m,n)$ up to $m = 10$, but some of the coefficients given for $S(10,n)$ are incorrect: $\pm$5064**8**92768 $\to$ $\pm$5064**9**92768, 5864**8**7120 $\to$ 5864**3**7120, 3130**28**7705 $\to$ 3130**33**7705.)

m/r	0	1	2	3	4	5	6	7	8
0	1								
1	−1	1							
2	5	−2	1						
3	−31	15	−3	1					
4	257	−124	30	−4	1				
5	−2671	1285	−310	50	−5	1			
6	33305	−16026	3855	−620	75	−6	1		
7	−484471	233135	−56091	8995	−1085	105	−7	1	
8	8054177	−3875768	932540	−149576	17990	−1736	140	−8	1

Table 2: $Q_1(m,r)$

m/r	0	1	2	3	4	5	6	7	8
0	1								
1	-2	1							
2	8	-4	1						
3	-50	24	-6	1					
4	416	-200	48	-8	1				
5	-4322	2080	-500	80	-10	1			
6	53888	-25932	6240	-1000	120	-12	1		
7	-783890	377216	-90762	14560	-1750	168	-14	1	
8	13031936	-6271120	1508864	-242032	29120	-2800	224	-16	1

Table 3: $Q_2(m, r)$

We observe several properties. By $abs(\cdot)$, we mean the absolute value of the elements in that set, as we see in the first row of Table 4.

$abs(Q_1(m, 0)) =$ A000556	$abs(Q_2(m, 0)) =$ A000557
$Q_1(m, m) = 1$	$Q_2(m, m) = 1$
$Q_1(m, m-1) = -m$	$Q_2(m, m-1) = -2m$
$Q_1(m, m-2) = 5\binom{m}{2}$	$Q_2(m, m-2) = 8\binom{m}{2}$

Table 4: Some properties of Q

that suggest

$Q_1(m, m-n) = (-1)^n M_{1,n}\binom{m}{n}$	$Q_2(m, m-n) = (-1)^n M_{2,n}\binom{m}{n}$

which can be related to polygonal numbers [13]. Other obvious row properties include

$$\sum_{r=0}^{m} abs(Q_1(m, r)) = abs(Q_2(m, 0)), \quad \sum_{r=0}^{m} Q_2(m, r) = Q_1(m, 0),$$

$$\sum_{r=0}^{m} abs(Q_2(m, r)) \to \text{A005923},$$

to which we shall refer in the next section. There are diagonal and column properties that may also be explored.

4. Related Ideas

If we consider $S(0,2)$ and $S(1,3)$, we see that they can be rewritten as

$$F_3 = F_2 + F_1 \quad \text{and} \quad F_4 = F_3 + F_2,$$

respectively, which raises the question whether equation (2.1) can generally be turned into (inhomogeneous) gibonacci recurrence relations [8].

Let $C(m) = C_1 + C_2$ such that $F_{n+2} \mid (S(m,n) - C_1)$. Then, (2.1) can be expressed as

$$P_3(m,n)F_{n+2} = P_1(m,n)F_n + P_2(m,n)F_{n+1} + C_2, \tag{4.1}$$

which has the form of an inhomogeneous recurrence relation. This leads into extending some of the work of Asveld [1], who solved the inhomogeneous difference equation

$$G_n = G_{n-1} + G_{n-2} + \sum_{j=0}^{k} \alpha_j p_j(n) \tag{4.2}$$

with the expression

$$G_n = (1 + \Lambda_k)F_n + \lambda_k F_{n-1} - \sum_{j=0}^{k} \alpha_j p_j(n),$$

in which

$$\Lambda_k = \sum_{j=0}^{k} a_{0,j}\alpha_j, \quad \lambda_k = \sum_{j=1}^{k}\sum_{i=1}^{j} a_{i,j}\alpha_j, \quad p_j(n) = \sum_{i=0}^{j} a_{i,j} n^i,$$

which have been explored further by Horadam and Shannon [6]. What is relevant to the theme of this note can be seen in the constant term $a_{0,j}$ in $p_j(n)$:

$$\sum_{r=0}^{m} abs(Q_2(m,r)) \to \{a_{0,j}\}.$$

The coefficients are set out in Table 5, in which the second column ($a_{0,j}$) is A005923 [10].

j	$a_{0,j}$	$a_{1,j}$	$a_{2,j}$	$a_{3,j}$	$a_{4,j}$	$a_{5,j}$	$a_{6,j}$
0	1						
1	3	1					
2	13	6	1				
3	81	39	9	1			
4	673	324	78	12	1		
5	6993	3365	810	130	15	1	
6	87193	41958	10095	1620	195	18	1

Table 5: Coefficients in Asveld's $p_j(n)$

5. Recurrence Relation

A recurrence relation for $S(m,n)$ may also be developed. For integers $m,n \geq 0$, let

$$p(m,n) = \sum_{k=1}^{n} k^m = \sum_{j=0}^{m+1} a_{m+1,j} n^j,$$

where $a_{m+1,j} = \frac{1}{(m+1)}\binom{m+1}{j} B^+_{m+1-j}$ $(j > 0)$ and B^+_i are the appropriately signed Bernoulli numbers. The constant term $a_{m+1,0}$ of the polynomial in n is zero, but is included for convenience in the derivation. The recurrence relation is then

$$\frac{1}{m+1} S(m+1,n) = p(m,n+1)F_n + p(m,n)F_{n+1} - \sum_{j=0}^{m} b_{m+1,j} S(j,n) \tag{5.1}$$

with $S(0,n) = F_n + F_{n+1} - 1$, and where

$$b_{m+1,j} = \sum_{r=j}^{m+1} \binom{r}{j} a_{m+1,r} \quad (j \neq m)$$

with, for $j = m$, $b_{1,0} = 2$ $(m = 0)$ and $b_{m+1,m} = \frac{5}{2}$ $(m > 0)$.

Proof.

$$\begin{aligned}
S(m,n) &= \sum_{k=1}^{n} k^m F_k = \sum_{k=1}^{2} k^m F_k + \sum_{k=3}^{n} k^m \left(1 + \sum_{i=1}^{k-2} F_i\right) \\
&= 1 + 2^m + \sum_{k=3}^{n} k^m + \sum_{k=1}^{n-2} (k+2)^m \sum_{i=1}^{k} F_i \\
&= p(m,n) + \sum_{i=1}^{n-2} F_i \sum_{k=i}^{n-2} (k+2)^m = p(m,n) + \sum_{i=1}^{n-2} F_i \sum_{k=i+2}^{n} k^m \\
&= p(m,n) + \sum_{i=1}^{n-2} F_i (p(m,n) - p(m,i+1)) \\
&= p(m,n)\left(1 + \sum_{i=1}^{n-2} F_i\right) - \sum_{i=1}^{n-2} F_i p(m,i+1) \\
&= p(m,n)F_n - \sum_{i=1}^{n} F_i p(m,i+1) + F_{n-1} p(m,n) + F_n p(m,n+1) \\
&= p(m,n+1)F_n + p(m,n)F_{n+1} - \sum_{i=1}^{n} F_i \sum_{j=0}^{m+1} a_{m+1,j} (i+1)^j \\
&= p(m,n+1)F_n + p(m,n)F_{n+1} - \sum_{i=1}^{n} F_i \sum_{j=0}^{m+1} a^*_{m+1,j} i^j \\
&= p(m,n+1)F_n + p(m,n)F_{n+1} - \sum_{j=0}^{m+1} a^*_{m+1,j} S(j,n),
\end{aligned}$$

where

$$a^*_{m+1,j} = \sum_{r=j}^{m+1} a_{m+1,r}\binom{r}{j}.$$

Therefore,

$$a^*_{m+1,m+1}S(m+1,n) = p(m,n+1)F_n + p(m,n)F_{n+1} - \sum_{j=0}^{m} a^*_{m+1,j}S(j,n) - S(m,n).$$

Therefore,

$$\frac{1}{m+1}S(m+1,n) = p(m,n+1)F_n + p(m,n)F_{n+1} - \sum_{j=0}^{m} b_{m+1,j}S(j,n),$$

where $b_{m+1,j} = a^*_{m+1,j}$ $(j \neq m)$, $b_{m+1,m} = a^*_{m+1,m} + 1$. □

Table 6 shows the initial rows of the $b_{m+1,j}$ coefficient array.

m	$(m+1)/j$	0	1	2	3	4	5
0	1	$b_{1,0} = 2$	1				
1	2	1	5/2	1/2			
2	3	1	13/6	5/2	1/3		
3	4	1	3	13/4	5/2	1/4	
4	5	1	119/30	6	13/3	5/2	1/5

Table 6: Values of $b_{m+1,j}$

Examples. (c.f. $S(1,n)$ and $S(2,n)$ in Section 2 above)

$$\begin{aligned} S(1,n) &= p(0,n+1)F_n + p(0,n)F_{n+1} - \sum_{j=0}^{0} b_{1,j}S(j,n) \\ &= (n+1)F_n + nF_{n+1} - b_{1,0}S(0,n) \\ &= (n+1)F_n + nF_{n+1} - 2(F_n + F_{n+1} - 1) = (n-1)F_n + (n-2)F_{n+1} + 2; \\ \frac{1}{2}S(2,n) &= p(1,n+1)F_n + p(1,n)F_{n+1} - \sum_{j=0}^{1} b_{2,j}S(j,n), \end{aligned}$$

so

$$\begin{aligned} &S(2,n) = (n+1)(n+2)F_n + n(n+1)F_{n+1} - 2b_{2,0}S(0,n) - 2b_{2,1}S(1,n) \\ &= (n+1)(n+2)F_n + n(n+1)F_{n+1} - 2(F_n + F_{n+1} - 1) - 5((n-1)F_n + (n-2)F_{n+1} + 2) \\ &= (n^2 - 2n + 5)F_n + (n^2 - 4n + 8)F_{n+1} - 8. \end{aligned}$$

Further, considering the $b_{m+1,j}$ coefficients themselves, it is well-known (although still surprising) that the inverse of the matrix of coefficients $a_{m+1,j}$ $(j > 0)$ of $p(m,n)$ is a Pascal-like matrix [4]. It is then of interest to note that the values given in Table 6 (with an additional value of $b_{0,0} = 1$ in the top-left position) produce:

$$\{b_{m+1,j}\}^{-1}_{6\times 6} = \begin{pmatrix} 1 \\ -2 & 1 \\ 8 & -5 & 2 \\ -50 & 31 & -15 & 3 \\ 416 & -257 & 124 & -30 & 4 \\ -4322 & 2671 & -1285 & 310 & -50 & 5 \end{pmatrix},$$

in which the first and second columns are related to $M_{2,j}$ (and so $C(m)$ and Table 3 for $Q_1(m,r)$) and $M_{1,j}$, respectively. Columns 2 to 5 correspond with values in the columns of Table 2 for $Q_1(m,r)$. Other than the first row, row sums are the negative of the corresponding row entry of column 2 (i.e. -1, 5, -31, 257, $\ldots$), and so are also related to $M_{1,j}$.

6. Conjectures

Equations (5.1) and (2.1) yield

$$\begin{aligned} &\frac{1}{m+1}(P_1(m+1,n)F_n + P_2(m+1,n)F_{n+1} + C(m+1)) \\ &= p(m,n+1)F_n + p(m,n)F_{n+1} - \sum_{j=0}^{m} b_{m+1,j}(P_1(j,n)F_n + P_2(j,n)F_{n+1} + C(j)) \\ &= \left(p(m,n+1) - \sum_{j=0}^{m} b_{m+1,j}P_1(j,n)\right)F_n + \left(p(m,n) - \sum_{j=0}^{m} b_{m+1,j}P_2(j,n)\right)F_{n+1} \\ &\quad - \sum_{j=0}^{m} b_{m+1,j}C(j), \end{aligned}$$

from which we conjecture the following by identifying the corresponding "constant" terms and coefficients of F_n and F_{n+1}.

Conjecture 1.

$$C(m+1) = -(m+1)\sum_{j=0}^{m} b_{m+1,j}C(j), \quad C(0) = -1. \tag{6.1}$$

Examples. (c.f. constant terms of the example $S(m,n)$ functions in Section 2)

$$C(1) = -\sum_{j=0}^{0} b_{1,j}C(j) = -b_{1,0}C(0) = -2C(0) = 2;$$

$$C(2) = -2\sum_{j=0}^{1} b_{2,j}C(j) = -2(b_{2,0}C(0) + b_{2,1}C(1)) = -2\left(C(0) + \frac{5}{2}C(1)\right) = -8;$$

$$\begin{aligned} C(3) &= -3\sum_{j=0}^{2} b_{3,j}C(j) = -3(b_{3,0}C(0) + b_{3,1}C(1) + b_{3,2}C(2)) \\ &= -3\left(C(0) + \frac{13}{6}C(1) + \frac{5}{2}C(2)\right) = 50. \end{aligned}$$

Conjecture 2.

$$P_1(m+1,n) = (m+1)\left(p(m,n+1) - \sum_{j=0}^{m} b_{m+1,j} P_1(j,n)\right), \quad P_1(0,n) = 1. \tag{6.2}$$

Example. (c.f. $S(1,n)$ in Section 2)

$$P_1(1,n) = p(0,n+1) - \sum_{j=0}^{0} b_{0+1,j} P_1(j,n) = n+1 - b_{1,0} P_1(0,n) = n+1-2\times 1 = n-1.$$

Conjecture 3.

$$P_2(m+1,n) = (m+1)\left(p(m,n) - \sum_{j=0}^{m} b_{m+1,j} P_2(j,n)\right), \quad P_2(0,n) = 1. \tag{6.3}$$

Example. (c.f. $S(1,n)$ in Section 2)

$$P_2(1,n) = p(0,n) - \sum_{j=0}^{0} b_{0+1,j} P_2(j,n) = n - b_{1,0} P_2(0,n) = n - 2\times 1 = n-2.$$

Equations (6.2) and (2.2) (with $i=1$) give the following equivalent polynomials in n:

$$\sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} M_{1,k} n^{m+1-k} = (m+1)\left(p(m,n+1) - \sum_{j=0}^{m} b_{m+1,j} \sum_{k=0}^{j} (-1)^k \binom{j}{k} M_{1,k} n^{j-k}\right).$$

Comparing constant terms gives the following conjecture.

Conjecture 4.

$$M_{1,m+1} = (-1)^m (m+1)\left(\sum_{j=0}^{m} (-1)^j b_{m+1,j} M_{1,j} - 1\right), \quad M_{1,0} = 1.$$

Examples. (c.f. Table 1)

$$M_{1,1} = b_{1,0} M_{1,0} - 1 = 2 - 1 = 1;$$

$$M_{1,2} = 2\left(\sum_{j=0}^{1} (-1)^j b_{m+1,j} M_{1,j} - 1\right) = -2(b_{2,0} M_{1,0} - b_{2,1} M_{1,1} - 1)$$

$$= -2\left(1 - \frac{5}{2} - 1\right) = 5.$$

Similarly, (6.3) and (2.2) (with $i=2$) give

$$\sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} M_{2,k} n^{m+1-k} = (m+1)\left(p(m,n) - \sum_{j=0}^{m} b_{m+1,j} \sum_{k=0}^{j} (-1)^k \binom{j}{k} M_{2,k} n^{j-k}\right).$$

Therefore, we have the following conjecture.

Conjecture 5.

$$M_{2,m+1} = (-1)^m(m+1)\sum_{j=0}^{m}(-1)^j b_{m+1,j}M_{2,j}, \quad M_{2,0} = 1.$$

Examples. (c.f. Table 1)

$$M_{2,1} = b_{1,0}M_{2,0} = 2;$$

$$M_{2,2} = -2\sum_{j=0}^{1}(-1)^j b_{m+1,j}M_{2,j} = -2(b_{2,0}M_{2,0} - b_{2,1}M_{2,1}) = -2\left(1 - 2\times\frac{5}{2}\right) = 8.$$

The recurrence relations for $M_{1,m+1}$ also may be compared with those given for A000556 and A000557 in the OEIS.

7. Conclusion

Conjectures 1 and 5 indicate the following relationships:

$$C(m+1) = -(m+1)\sum_{j=0}^{m} b_{m+1,j}C(j), \quad C(0) = -1;$$

$$M_{2,m+1} = (-1)^m(m+1)\sum_{j=0}^{m}(-1)^j b_{m+1,j}M_{2,j}, \quad M_{2,0} = 1.$$

In addition to the ideas for further study raised in the foregoing, Ledin essentially asked the following three questions at the end of his paper (and in the notation of this paper):

a) Could the theory of $S(m,n)$ be extended to negative m?
b) Could the theory of $S(m,n)$ be extended to rational (and to real) m? [5]
c) What is the possibility of studying

$$S(m,r,n) = \sum_{k=1}^{n} k^m F_k^r$$

with standard techniques?

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