PARTIAL SUMS OF THE FIBONACCI SEQUENCE

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ABSTRACT. Let $(F_n)_{n\geq 1}$ be the Fibonacci sequence. Define $P(F_n) = (\sum_{i=1}^n F_i)_{n\geq 1}$; that is, the function P gives the sequence of partial sums of (F_n). In this paper, we first give an identity involving $P^k(F_n)$, which is the resulting sequence by applying P to (F_n) k times. Second, we provide a combinatorial interpretation of the numbers in $P^k(F_n)$.

1. INTRODUCTION

Let $(F_n)_{n\geq 1}$ be the Fibonacci sequence with $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. Define $P(F_n) = (\sum_{i=1}^n F_i)_{n\geq 1}$; that is, the function P gives the sequence of partial sums of (F_n). Generally, we can apply the function P to (F_n) k times to have the sequence $P^k(F_n)$. For ease of notation, let $a_k(n)$ denote the nth number in the sequence $P^k(F_n)$. Table 1 summarizes some initial numbers $a_k(n)$ for different values of k and n.

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	2	3	5	8	13	21	34	55	89	144
1	1	2	4	7	12	20	33	54	88	143	232	376
2	1	3	7	14	26	46	79	133	221	364	596	972
3	1	4	11	25	51	97	176	309	530	894	1490	2462
4	1	5	16	41	92	189	365	674	1204	2098	3588	6050
5	1	6	22	63	155	344	709	1383	2587	4685	8273	14323

Table 1. Initial numbers $a_k(n)$ for different values of k and n.

We compute a few values of $a_k(n)$ to illustrate how Table 1 is obtained. For $k = 0$, we do not apply the partial sum operator to the first row, so we have the Fibonacci sequence. For $k = 1$, we apply the partial sum operator once to the first row to have the second row. Thus,

$$
\begin{aligned}
a_2(1) &= a_1(1) = 1, \\
a_2(2) &= a_1(1) + a_1(2) = 1 + 1 = 2, \\
a_2(3) &= a_1(1) + a_1(2) + a_1(3) = 1 + 1 + 2 = 4,
\end{aligned}
$$

and so on. For $k = 2$, we apply the partial sum operator twice to the first row; equivalently, we apply the partial sum operator once to the second row. Repeat the procedure to fill in the table.

We now consider the first and second rows. Observe that each number in the second row is one less than a Fibonacci number; in particular, $a_0(n + 2) - 1 = a_1(n)$ for all $n \geq 1$. A natural question to ask is about the relationship between any two consecutive rows of Table 1. Continuing to look at pairs of numbers whose columns differ by two, we compute several

differences

$$
\begin{aligned}
a_1(11) - a_2(9) &= 11, \\
a_2(10) - a_3(8) &= 55, \\
a_3(8) - a_4(6) &= 309 - 189 = 120, \\
a_4(12) - a_5(10) &= 6050 - 4685 = 1365.
\end{aligned}
$$

If we check these numbers in the Pascal triangle, we see a pattern. Call the outmost left edge of the Pascal triangle the *first line* (consisting of 1's); move to the right 1 unit to have the *second line* (consisting of 1, 2, 3, 4, ...), and so on. Then 11 lies on the second line; 55 lies on the third line; 120 lies on the fourth line; 1365 lies on the fifth line. This might lead to the suspicion that $a_{k-1}(n+2) - a_k(n)$ lies on the kth line. The following theorem shows that our suspicion is indeed well-founded.

Theorem 1.1. *For $k \geq 1$, we have*

$$
a_k(n) = a_{k-1}(n+2) - \binom{n+k}{k-1}.
$$

By definition, the identity is equivalent to

$$
\sum_{m=1}^{n} a_{k-1}(m) = a_{k-1}(n+2) - \binom{n+k}{k-1}.
$$

Remark 1.2. If we let $k = 1$, then Theorem 1.1 gives the identity $\sum_{m=1}^{n} a_0(m) = a_0(n+2) - 1$, which is the well-known identity $\sum_{m=1}^{n} F_m = F_{n+2} - 1$.

Our next result gives a combinatorial interpretation of $a_k(n)$. For $n \in \mathbb{N}$, define

$$
s_k(n) = \#\{S \subset \{1, 2, \ldots, n\} : |S| \geq k \text{ and } \min S \geq |S|\}.
$$

Sets with $\min S \geq |S|$ are called Schreier sets. Schreier [2] used these sets to solve a problem in Banach space theory, and they were also independently discovered in combinatorics and are connected to Ramsey-type theorems for subsets of $\mathbb{N}$. Table 2 summarizes some initial numbers $s_k(n)$ for different values of k and n.

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12
0	2	3	5	8	13	21	34	55	89	144	233	377
1	1	2	4	7	12	20	33	54	88	143	232	376
2	0	0	1	3	7	14	26	46	79	133	221	364
3	0	0	0	0	1	4	11	25	51	97	176	309
4	0	0	0	0	0	0	1	5	16	41	92	189
5	0	0	0	0	0	0	0	0	1	6	22	63

Table 2. Initial numbers $s_k(n)$ for different values of k and n.

We obtain Table 2 with the help of a simple program. Comparing Tables 1 and 2, we see that $s_k(n)$ is a shift of $a_k(n)$. The following theorem describes this relationship.

Theorem 1.3. *For $k \geq 0$, we have*

$$
s_k(n) = a_k(n - 2(k - 1)).
$$

Remark 1.4. In Theorem 1.3, if $n - (2k - 1) \leq 0$, then $a_k(n - 2(k - 1)) = 0$ by convention.

2. Proofs

We use induction repeatedly and several straightforward observations to prove Theorems 1.1 and 1.3. The following lemma is useful.

Lemma 2.1. *For $k \geq 0$, we have $a_k(3) = \binom{k+2}{k} + 1$.*

Proof. We prove the lemma by induction.

Base case. For $k = 0$, $a_0(3) = F_3 = 2 = \binom{2}{0} + 1$.

Inductive hypothesis. Suppose that the formula holds for all $k \leq \ell$ for some $\ell \geq 0$. We show that it holds for $k = \ell + 1$. We have

$$
\begin{aligned}
a_{\ell+1}(3) &= a_{\ell+1}(2) + a_\ell(3) && \text{by how we construct Table 1} \\
&= (\ell + 2) + \left(\binom{\ell + 2}{\ell} + 1 \right) && \text{by the inductive hypothesis} \\
&= \binom{\ell + 2}{\ell + 1} + \left(\binom{\ell + 2}{\ell} + 1 \right) \\
&= \binom{\ell + 3}{\ell + 1} + 1.
\end{aligned}
$$

This completes our proof. □

Proof of Theorem 1.1. We prove the theorem by induction on k.

Base case. For $k = 1$, the identity is $\sum_{m=1}^n a_0(m) = a_0(n + 2) - 1$, which is true because $\sum_{m=1}^n F_m = F_{n+2} - 1$.

Inductive hypothesis. Suppose that the identity is true for $k \leq \ell$ for some $\ell \geq 1$. We show that

$$
\sum_{m=1}^n a_\ell(m) = a_\ell(n + 2) - \binom{n + \ell + 1}{\ell}. \tag{2.1}
$$

We prove (2.1) by induction on n.

Base case. If $n = 1$, then the left side is $a_\ell(1) = 1$, whereas the right side is $a_\ell(3) - \binom{\ell+2}{\ell} = 1$ by Lemma 2.1.

Inductive hypothesis. Suppose that (2.1) holds for $n \leq j$ for some $j \geq 1$. We show that it holds for $n = j + 1$; that is,

$$
\sum_{m=1}^{j+1} a_\ell(m) = a_\ell(j + 3) - \binom{j + \ell + 2}{\ell}.
$$

We have

$$
\begin{aligned}
\sum_{m=1}^{j+1} a_\ell(m) &= a_\ell(j + 1) + \sum_{m=1}^{j} a_\ell(m) \\
&= \left(a_{\ell-1}(j + 3) - \binom{j + \ell + 1}{\ell - 1} \right) + \left(a_\ell(j + 2) - \binom{j + \ell + 1}{\ell} \right) && \text{by our inductive hypotheses} \\
&= (a_{\ell-1}(j + 3) + a_\ell(j + 2)) - \binom{j + \ell + 2}{\ell} \\
&= a_\ell(j + 3) - \binom{j + \ell + 2}{\ell}.
\end{aligned}
$$

This completes our proof. □

Next, we prove Theorem 1.3. We will need the following lemma.

Lemma 2.2. *Fix $n \geq 1$. We have*

$$\#\{S \subset \{1, 2, \ldots, n\} \; : \; |S| = k \text{ and } \min S \geq k\} \; = \; \binom{n - k + 1}{k}.$$

Proof. Observe that to form a set $S \subset \{1, 2, \ldots, n\}$ with $|S| = k$ and $\min S \geq k$, we choose k numbers in $\{1, 2, \ldots, n\}$ such that the smallest number is at least k. To satisfy these requirements, we need to choose k numbers from k to n inclusive. Therefore, the number of ways is $\binom{n-k+1}{k}$. $\square$

The following corollary is immediate.

Corollary 2.3. *Fix $\ell \geq 0$ and $n \geq 1$. We have*

$$s_{\ell+1}(n) \; = \; s_\ell(n) - \binom{n - \ell + 1}{\ell}.$$

Proof of Theorem 1.3. We prove the theorem by induction on k.

Base case. For $k = 0$, we want to show that $s_0(n) = a_0(n + 2) = F_{n+2}$. This follows from [1, Theorem 1].

Inductive hypothesis. Suppose that the formula holds for $k \leq \ell$ for some $\ell \geq 0$. We want to show that

$$s_{\ell+1}(n) \; = \; a_{\ell+1}(n - 2\ell). \tag{2.2}$$

For $n \leq 2\ell$, the right side of (2.2) is 0 by convention. The left side counts the number of subsets $S \subset \{1, 2, \ldots, n\}$ such that $|S| \geq \ell + 1$ and $\min S \geq |S| \geq \ell + 1$. Because $|S| \geq \ell + 1 > n/2$, we know that S must contain a number smaller than or equal to $n/2 \leq \ell$. Hence, $\min S \leq \ell$, which contradicts that $\min S \geq \ell + 1$. Therefore, the left side is also 0.

For $n > 2\ell$, we have

$$a_{\ell+1}(n - 2\ell) = a_\ell(n - 2\ell + 2) - \binom{n - \ell + 1}{\ell}$$

by Theorem 1.1. On the other hand, we have

$$s_{\ell+1}(n) = s_\ell(n) - \binom{n - \ell + 1}{\ell}$$

by Corollary 2.3. By the inductive hypothesis, $s_\ell(n) = a_\ell(n - 2\ell + 2)$; therefore, we have $s_{\ell+1}(n) = a_{\ell+1}(n - 2\ell)$. This completes our proof. $\square$

REFERENCES

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