VARIOUS SEQUENCES FROM COUNTING SUBSETS

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Abstract. As n varies, we count the number of subsets of $\{1, 2, \ldots, n\}$ under different conditions and study the sequences formed by these numbers.

1. Introduction

We define the α-Schreier condition. Given a natural number α, a set S is said to be *α-Schreier* if $\min S/\alpha \geq |S|$, where $|S|$ is the cardinality of S. Schreier used 1-Schreier sets to solve a problem in Banach space theory [3]. These sets were also independently discovered in combinatorics and are connected to Ramsey-type theorems for subsets of $\mathbb{N}$. Next, we define the β-Zeckendorf condition. In 1972, Zeckendorf proved that every positive integer can be uniquely written as a sum of nonconsecutive Fibonacci numbers [4]. We focus on the important requirement for uniqueness of the Zeckendorf decomposition; that is, our set contains no two consecutive Fibonacci numbers. We generalize this condition to a finite set of natural numbers.

Definition 1.1. *Let $S = \{s_1, s_2, \ldots, s_k\}$ $(s_1 < s_2 < \cdots < s_k)$ for some $k \in \mathbb{N}_{\geq 2}$. The difference set of S, denoted by $D(S)$, is $\{s_2 - s_1, s_3 - s_2, \ldots, s_k - s_{k-1}\}$. The difference set of the empty set and a set with exactly one element is empty.*

Definition 1.2. *Fix a natural number β. A finite set S of natural numbers is β-Zeckendorf if $\min D(S) \geq \beta$; that is, each pair of numbers in S is at least β apart. The empty set, and a set with exactly one element, vacuously satisfy this condition.*

Chu, et al. proved the linear recurrence of the sequence obtained by counting subsets of $\{1, 2, \ldots, n\}$ that are α-Schreier [2]. In particular, [2, Theorem 1.1] states that the recurrence has order $\alpha + 1$. On the other hand, it is well known that the sequence obtained by counting subsets of $\{1, 2, \ldots, n\}$ that are β-Zeckendorf has a linear recurrence of order β. A notable example is $\beta = 2$, which gives the Fibonacci sequence. A natural extension of these results is to consider sets that are both α-Schreier and β-Zeckendorf. For each $n \in \mathbb{N}$, define

$$a_{\alpha,\beta,n} \;=\; \#\{S \subset \{1, 2, \ldots, n\} \,:\, S \text{ is } \alpha\text{-Schreier and } \beta\text{-Zeckendorf}\}.$$

Our first result shows a linear recurrence for this sequence $(a_{\alpha,\beta,n})$.

Theorem 1.3. *Fix natural numbers α and β. For $n \geq 1$, we have*

$$a_{\alpha,\beta,n} \;=\; \begin{cases} 1, & \text{for } n \leq \alpha - 1; \\ n - \alpha + 2, & \text{for } \alpha \leq n \leq 2\alpha + \beta - 1; \\ a_{\alpha,\beta,n-1} + a_{\alpha,\beta,n-(\alpha+\beta)}, & \text{for } n \geq 2\alpha + \beta. \end{cases}$$

Remark 1.4. *Theorem 1.3 says that the order of our recurrence relation is the sum $\alpha + \beta$. Substituting $\beta = 1$, we have* [2, Theorem 1.1]. *Interestingly, the number of 1's in the sequence is independent of β.*

The next results involve the Fibonacci sequence. Let the Fibonacci sequence be $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Let $(H_n)_{n\geq 0}$ be the sequence obtained by applying the partial sum operator twice to the Fibonacci sequence. In particular,

$$H_n \;=\; \sum_{i=0}^{n}(n+1-i)F_i.$$

The first few terms of (H_n) are 0, 1, 3, 7, 14, 26, and 46. We prove the following identity.

Proposition 1.5. *For $n \geq 0$, we have*

$$F_{n+4} \;=\; H_n + n + 3. \tag{1.1}$$

We then use the identity to prove the following theorem.

Theorem 1.6. *Let $(a_n)_{n\geq 1}$ be the number of subsets of $\{1, 2, \ldots, n\}$ that*

(i) *have at least two elements; and*
(ii) *have their difference sets only contain odd numbers.*

Then $a_n = H_{n-1}$.

Our last result is a companion of [1, Theorem 8], which considers subsets of $\{1, 2, \ldots, n\}$ whose difference set only contains odd numbers. Surprisingly, the number of such subsets is related to the Fibonacci sequence. For convenience, we include the theorem below.

Theorem 1.7. *Fix $n \in \mathbb{N}$. The number of subsets of $\{1, 2, \ldots, n\}$*

(1) *that contain n and whose difference sets only contain odd numbers is F_{n+1},*
(2) *whose difference sets only contain odd numbers (the empty set and sets with exactly one element vacuously satisfy this requirement) is $F_{n+3} - 1$.*

To complete the picture, we consider subsets whose difference set only contains even numbers.

Theorem 1.8. *Fix $n \in \mathbb{N}$. The number of subsets of $\{1, 2, \ldots, n\}$*

1. *that contain n and whose difference sets only contain even numbers is $2^{\lfloor (n-1)/2 \rfloor}$,*
2. *whose difference sets only contain even numbers (the empty set and sets with exactly one element vacuously satisfy this requirement) is*

$$\begin{cases} 3 \cdot 2^{(n-1)/2} - 1, & \text{if } n \text{ is odd;} \\ 2 \cdot 2^{n/2} - 1, & \text{if } n \text{ is even.} \end{cases}$$

The following corollary is immediate.

Corollary 1.9. *Let*

$$\mathcal{S}_n = \{S \subset \{1, 2, \ldots, n\} : D(S) \text{ only has odd numbers or only has even numbers}\};$$
$$\mathcal{S}_{n,1} = \{S \subset \{1, 2, \ldots, n\} : D(S) \text{ only has odd numbers}\};$$
$$\mathcal{S}_{n,2} = \{S \subset \{1, 2, \ldots, n\} : D(S) \text{ only has even numbers}\}.$$

Then $\lim_{n\to\infty} \frac{|\mathcal{S}_{n,1}|}{|\mathcal{S}_n|} = 1$; that is, as $n \to \infty$, almost all sets in S_n have their difference sets only contain odd numbers.

Proof. Because $3 \cdot 2^{(n-1)/2} > 2 \cdot 2^{n/2}$ and by Theorem 1.7, it suffices to prove that

$$\lim_{n\to\infty} \frac{3 \cdot 2^{(n-1)/2} - 1}{F_{n+3} - 1} \;=\; 1,$$

which we can prove by using Binet's formula for F_{n+3}. □

Remark 1.10. *Intuitively, the above corollary says that sets in $\mathcal{S}_{n,1}$ dominate sets in $\mathcal{S}_{n,2}$. The reason is that for a set in $\mathcal{S}_1$, the difference between consecutive elements can be as small as 1, which gives us much more freedom in constructing such a set than a set in $\mathcal{S}_2$.*

Section 2 is devoted to proofs of our main results, whereas Section 3 generalizes Proposition 1.5 and raises several questions for future research.

2. Proofs

Proof of Theorem 1.3. For $n \le \alpha - 1$, the only subset of $\{1, 2, \ldots, n\}$ that is α-Schreier is the empty set, which is also β-Zeckendorf. Hence, $a_{\alpha,\beta,n} = 1$.

Consider $\alpha \le n \le 2\alpha+\beta-1$. Let $S \subset \{1, 2, \ldots, n\}$ be α-Schreier and β-Zeckendorf. Suppose that $|S| \ge 2$. Because $\min S/\alpha \ge |S| \ge 2$, we have $\min S \ge 2\alpha$. Because S is β-Zeckendorf, the other elements in S must be at least $2\alpha + \beta$, which contradicts $n \le 2\alpha + \beta - 1$. Hence, either $S = \emptyset$ or $S = \{k\}$ for $\alpha \le k \le n$. Therefore, $a_{\alpha,\beta,n} = n - \alpha + 2$.

Finally, consider $n \ge 2\alpha + \beta$. Let

$$\begin{aligned}\mathcal{A} &= \{S \subset \{1, 2, \ldots, n\} : S \text{ is } \alpha\text{-Schreier and } \beta\text{-Zeckendorf and } \max S < n\};\\ \mathcal{B} &= \{S \subset \{1, 2, \ldots, n\} : S \text{ is } \alpha\text{-Schreier and } \beta\text{-Zeckendorf and } \max S = n\}.\end{aligned}$$

Clearly, $\mathcal{A} = \{S \subset \{1, 2, \ldots, n - 1\} : S \text{ is } \alpha\text{-Schreier and } \beta\text{-Zeckendorf}\}$. Hence, $|\mathcal{A}| = a_{\alpha,\beta,n-1}$. It suffices to prove $|\mathcal{B}| = a_{\alpha,\beta,n-(\alpha+\beta)}$. We show a bijection between $\mathcal{B}$ and $\mathcal{C} = \{S \subset \{1, 2, \ldots, n - (\alpha + \beta)\} : S \text{ is } \alpha\text{-Schreier and } \beta\text{-Zeckendorf}\}$.

Given a set S and $k \in \mathbb{N}$, we let $S - k = \{s - k : s \in S\}$. Define the function $f : \mathcal{B} \to \mathcal{C}$ such that

$$f(S) = \begin{cases} \emptyset, & \text{if } S = \{n\};\\ S\backslash\{n\} - \alpha, & \text{if } |S| > 1.\end{cases}$$

We show that f is well-defined. If $|S| > 1$, we have

$$\min f(S) = \min(S\backslash\{n\} - \alpha) = \min S - \alpha \ge \alpha|S| - \alpha = \alpha|f(S)|.$$

Hence, $f(S)$ is α-Schreier. Because S is β-Zeckendorf, $f(S)$ is also β-Zeckendorf. Lastly, we have

$$\max f(S) = \max(S\backslash\{n\}) - \alpha \le (n - \beta) - \alpha = n - (\beta + \alpha).$$

Therefore, $f(S) \in \mathcal{C}$. We know that f is injective by definition, and thus, $|\mathcal{B}| \le |\mathcal{C}|$. Next, define the function $g : \mathcal{C} \to \mathcal{B}$ such that $g(S) = (S + \alpha) \cup \{n\}$. Because S is β-Zeckendorf and $\max S \le n - (\alpha + \beta)$, we know that $g(S)$ is also β-Zeckendorf. To see why $g(S)$ is α-Schreier, we observe that

$$\min g(S) = \min S + \alpha \ge \alpha(|S| + 1) = \alpha|g(S)|.$$

Hence, g is well-defined and is injective by definition. Therefore, $|\mathcal{B}| \ge |\mathcal{C}|$. We conclude that $|\mathcal{B}| = |\mathcal{C}|$, which completes our proof. □

Proof of Proposition 1.5. We prove the proposition by induction. Clearly, the identity holds for $n = 0$. Suppose the identity holds for $n = k$ for some $k \ge 0$; that is, $F_{k+4} = H_k + k + 3$.

We show that $F_{k+5} = H_{k+1} + k + 4$. We have

$$\begin{aligned} F_{k+5} = F_{k+4} + F_{k+3} &= H_k + k + 3 + F_{k+3} \\ &= \left(H_{k+1} - \sum_{i=0}^{k+1} F_i\right) + k + 3 + F_{k+3} \\ &= (H_{k+1} + k + 4) + \left(F_{k+3} - \sum_{i=0}^{k+1} F_i - 1\right). \end{aligned}$$

It is well known that $F_{k+3} - \sum_{i=0}^{k+1} F_i - 1 = 0$; therefore, $F_{k+5} = H_{k+1} + k + 4$. This completes our proof. □

Proof of Theorem 1.6. By Theorem 1.7 and (1.1), we have

$$a_n = F_{n+3} - 2 - n = (H_{n-1} + n + 2) - 2 - n = H_{n-1}.$$

This completes our proof. □

Proof of Theorem 1.8. We prove the first item by induction. Let P_n (and O_n, respectively) be the number of subsets of $\{1, 2, \ldots, n\}$ (and the set of subsets of $\{1, 2, \ldots, n\}$, resp.) that satisfy our requirement.

Base cases. For $n = 1$, $\{1\}$ is the only subset of $\{1\}$ that satisfies our requirement. Hence, $P_1 = 1 = 2^{\lfloor (1-1)/2 \rfloor}$. Similarly, $O_2 = \{\{2\}\}$ and $P_2 = 1 = 2^{\lfloor (2-1)/2 \rfloor}$.

Inductive hypothesis. Suppose that there exists a $k \geq 2$ such that for all $n \leq k$ we have $P_n = 2^{\lfloor (n-1)/2 \rfloor}$. We show that $P_{k+1} = 2^{\lfloor k/2 \rfloor}$. Observe that taking the union of sets O_{k+1-2i} for $1 \leq i < (k+1)/2$ with $k+1$ produces a set in O_{k+1}, and any set in O_{k+1} is of the form of a set in O_{k+1-2i} plus the element $k+1$. Therefore,

$$P_{k+1} = |O_{k+1}| = 1 + \sum_{1 \leq i < (k+1)/2} |O_{k+1-2i}| = 1 + \sum_{1 \leq i < (k+1)/2} P_{k+1-2i}.$$

The number 1 accounts for the set $\{k+1\}$. If k is odd, we have

$$\begin{aligned} P_{k+1} &= 1 + P_{k-1} + P_{k-3} + \cdots + P_2 \\ &= 1 + 2^{\lfloor (k-2)/2 \rfloor} + 2^{\lfloor (k-4)/2 \rfloor} + \cdots + 2^{\lfloor 1/2 \rfloor} \\ &= 1 + 2^{(k-3)/2} + 2^{(k-5)/2} + \cdots + 2^{0/2} = 2^{(k-1)/2} = 2^{\lfloor k/2 \rfloor}. \end{aligned}$$

Similarly, if k is even, we also have $P_{k+1} = 2^{\lfloor k/2 \rfloor}$. This completes our proof of the first item. The second item follows from the first by noticing that the number of subsets of $\{1, 2, \ldots, n\}$ whose difference sets only contain even numbers is equal to $1 + \sum_{k=1}^{n} |O_k| = 1 + \sum_{k=1}^{n} 2^{\lfloor (k-1)/2 \rfloor}$, where the number 1 accounts for the empty set. It is an exercise to show that this formula and the formula given in item 2 are the same. □

3. Generalizations and Questions

In this section, we generalize Proposition 1.5 and raise two questions for future research. For each $n \geq 2$, define the sequence $(F_{n,m})_{m \geq 0}$ as follows: $F_{n,0} = 0$, $F_{n,1} = \cdots = F_{n,n} = 1$, and $F_{n,m} = F_{n,m-1} + F_{n,m-n}$ for $m \geq n+1$. Let $(K_{n,m})$ and $(H_{n,m})$ be the sequence obtained by applying the partial sum operation to $(F_{n,m})$ once and twice, respectively. For example, when $n = 3$, we have Table 1.

m	0	1	2	3	4	5	6	7	8	9	10	11	12
$F_{n,m}$	0	1	1	1	2	3	4	6	9	13	19	28	41
$K_{n,m}$	0	1	2	3	5	8	12	18	27	30	49	77	118
$H_{n,m}$	0	1	3	6	11	19	31	49	76	106	155	232	350

Table 1. The sequences $(F_{3,m}), (K_{3,m})$, and $(H_{3,m})$ for $0 \leq m \leq 12$.

The following proposition generalizes Proposition 1.5.

Proposition 3.1. *For $n \geq 2$ and $m \geq 0$, we have*

(1) $\sum_{i=0}^{k+1} F_{n,i} = F_{n,k+1+n} - 1$ *for $k \geq 0$,*
(2) $F_{n,m+2n} = H_{n,m} + m + (n+1)$.

Proof. We prove item (1). Fix $n \geq 2$ and $k \geq 0$. We have

$$\begin{aligned} F_{n,k+1+n} - \sum_{i=0}^{k+1} F_{n,i} - 1 &= (F_{n,k+1+n} - F_{n,k+1}) - \sum_{i=0}^{k} F_{n,i} - 1 \\ &= F_{n,k+n} - \sum_{i=0}^{k} F_{n,i} - 1 \\ &= (F_{n,k+n} - F_{n,k}) - \sum_{i=0}^{k-1} F_{n,i} - 1 \\ &= F_{n,k+n-1} - \sum_{i=0}^{k-1} F_{n,i} - 1 \\ &= \cdots = F_{n,n-1} - 1 = 0. \end{aligned}$$

Hence, we have $F_{n,k+1+n} - \sum_{i=0}^{k+1} F_{n,i} - 1 = 0$, so $\sum_{i=0}^{k+1} F_{n,i} = F_{n,k+1+n} - 1$.

Next, we prove item (2). Fix $n \geq 2$. We proceed by induction.

Base case. For $m = 0$, the identity is equivalent to $F_{n,2n} = n + 1$, which is true.

Inductive hypothesis. Suppose that the identity is true for all $0 \leq m \leq k$ for some $k \geq 0$. We want to show that it is true for $m = k + 1$. We have

$$\begin{aligned} F_{n,k+1+2n} &= F_{n,k+2n} + F_{n,k+1+n} \\ &= (H_{n,k} + k + (n+1)) + F_{n,k+1+n} && \text{by the inductive hypothesis} \\ &= (H_{n,k} + F_{n,k+1+n} - 1) + (k+1) + (n+1) \\ &= \left(H_{n,k} + \sum_{i=0}^{k+1} F_{n,i} \right) + (k+1) + (n+1) && \text{by item (1)} \\ &= H_{n,k+1} + (k+1) + (n+1). \end{aligned}$$

This completes our proof. □

Theorem 1.6 shows that $(H_{2,m})$ is related to the number of certain subsets of $\{1, 2, \ldots, n\}$; however, the author is unable to find such a combinatorial perspective of the sequence $(H_{n,m})$ when $m > 2$. Is there a connection between the sequence $(H_{n,m})$ and the number of subsets of $\{1, 2, \ldots, n\}$ restricted to certain conditions as in Theorem 1.6?

Fix $k \geq 2$. Another way to generalize Theorem 1.6 is to look at the sequence formed by counting subsets of $\{1, 2, \ldots, n\}$ satisfying two conditions: (i) have at least k elements, and (ii) have their difference sets only contain odd numbers. When $k = 2$, Theorem 1.6 connects the sequence obtained by counting subsets to the Fibonacci sequence; however, the author is unable to find such a connection for bigger values of k. For example, when $k = 3$, the sequence we obtain is 0, 0, 1, 3, 8, 17, 34, 63, 113, 196, 334, 560, ... Is there a neat relation among terms in this sequence?

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