A SIMPLE BIJECTIVE PROOF OF A FAMILIAR DERANGEMENT RECURRENCE

SERGI ELIZALDE

ABSTRACT. It is well known that the derangement numbers d_n, which count permutations of length n with no fixed points, satisfy the recurrence $d_n = nd_{n-1} + (-1)^n$ for $n \geq 1$. Combinatorial proofs of this formula have been given by Remmel, Wilf, Désarménien, and Benjamin-Ornstein. Here, we present yet another, arguably simpler bijective proof.

Let $\mathcal{S}_n$ denote the set of permutations of $\{1, 2, \ldots, n\}$. A fixed point of $\pi \in \mathcal{S}_n$ is an element i such that $\pi(i) = i$. Let $\mathcal{D}_n \subseteq \mathcal{S}_n$ denote the subset of permutations with no fixed points, often called *derangements*, and let $d_n = |\mathcal{D}_n|$. Let $\mathcal{F}_n \subseteq \mathcal{S}_n$ denote the subset of permutations with exactly one fixed point. Clearly, $|\mathcal{F}_n| = nd_{n-1}$, because permutations in $\mathcal{F}_n$ are determined by choosing the fixed point among n possibilities, and then taking a derangement of the remaining $n-1$ elements.

It is well known [5, Eq. (2.13)] that the derangement numbers d_n satisfy the recurrence

$$d_n = nd_{n-1} + (-1)^n \tag{1}$$

for $n \geq 1$. This equation states that the number of $\pi \in \mathcal{S}_n$ with no fixed points and the number of $\pi \in \mathcal{S}_n$ with one fixed point differ by one. Stanley [5] acknowledges that proving recurrence (1) combinatorially requires "considerably more work" than proving the other well-known recurrence for derangement numbers, $d_n = (n-1)(d_{n-1} + d_{n-2})$. Bijective proofs of (1) have been given by Remmel, Wilf, Désarménien, and, more recently, Benjamin and Ornstein.[1] Remmel's bijection [4] is not simple, and it proves a q-analogue of (1). Désarménien's elegant bijection [2] first maps derangements to another set of permutations, namely those whose first valley is in an even position. Wilf's bijection [6] is easy to program, but it is recursive. Benjamin and Ornstein's bijection [1] is perhaps the simplest of these four, but its description still requires four different cases.

In this note, we present a new, arguably simpler bijective proof of equation (1). We describe a bijection $\psi : \mathcal{D}_n^* \to \mathcal{F}_n^*$, where $\mathcal{D}_n^* = \mathcal{D}_n \setminus \{(1,2)(3,4)\cdots(n-1,n)\}$ and $\mathcal{F}_n^* = \mathcal{F}_n$ when n is even, and $\mathcal{D}_n^* = \mathcal{D}_n$ and $\mathcal{F}_n^* = \mathcal{F}_n \setminus \{(1)(2,3)\cdots(n-1,n)\}$ when n is odd.

We write derangements in cycle notation so that each cycle begins with its smallest element, and cycles are ordered by increasing first element. On the other hand, we write permutations in $\mathcal{F}_n$ with their fixed point at the beginning.

Let $\pi \in \mathcal{D}_n^*$, and let k be the largest nonnegative integer such that the cycle notation of π starts with $(1,2)(3,4)\cdots(2k-1,2k)$. Note that $0 \leq k < n/2$, because $\pi \neq (1,2)(3,4)\cdots(n-1,n)$. To define $\psi(\pi) \in \mathcal{F}_n^*$, consider two cases:

[1] Another bijection is described by Rakotondrajao [3, Sec. 3], but it appears to be flawed: for $n = 5$, both $(5, (13)(24))$ and $(5, (1324))$ are mapped to $(124)(35)$.

(i) If the cycle containing $2k+1$ has at least three elements, change the first $k+1$ cycles of π as follows:

$$\begin{aligned} \pi &= (1,2)(3,4)\cdots(2k-1,2k)(2k+1,a_1,a_2,\ldots,a_j)\cdots \\ \psi(\pi) &= (1)(2,3)(4,5) \quad \cdots \quad (2k,a_1)(2k+1,a_2,\ldots,a_j)\cdots \end{aligned}$$

Note that, if $k=0$, then $\{1,2,\ldots,2k\}=\emptyset$ and the fixed point in $\psi(\pi)$ is a_1.

(ii) Otherwise, change the first $k+2$ cycles of π as follows:

$$\begin{aligned} \pi &= (1,2)(3,4)\cdots(2k-1,2k)(2k+1,a_1)(2k+2,a_2,\ldots,a_j)\cdots \\ \psi(\pi) &= (1)(2,3)(4,5) \quad \cdots \quad (2k,2k+1)(2k+2,a_1,a_2,\ldots,a_j)\cdots \end{aligned}$$

The inverse map ψ^{-1} has a similar description. Given $\sigma \in \mathcal{F}_n^*$, let ℓ be the fixed point of σ, and consider two cases. If $\ell \neq 1$, merge the cycles containing ℓ and 1 as follows:

$$\sigma = (\ell)(1,a_2,\ldots,a_j)\cdots \quad \mapsto \quad \psi^{-1}(\sigma) = (1,\ell,a_2,\ldots,a_j)\cdots$$

Otherwise, let σ' be the derangement of $\{2,\ldots,n\}$ obtained by removing the fixed point 1 from σ; apply ψ to σ' (using the above description, but identifying $\{2,\ldots,n\}$ with $\{1,\ldots,n-1\}$ in an order-preserving fashion); and replace its fixed point (ℓ) with the 2-cycle $(1,\ell)$ to get $\psi^{-1}(\sigma)$.

As an example, below are the images by ψ of all the derangements in $\mathcal{D}_4$ and some in $\mathcal{D}_5$, with the entry a_1 colored in **boldface** in case (i) and in *italics* in case (ii).

π	(12)(34)	(1*3*)(24)	(1*4*)(23)	(1**2**34)	(1**2**43)	(1**3**24)	(1**3**42)	(1**4**23)	(1**4**32)
$\psi(\pi)$	–	(1)(2*3*4)	(1)(2*4*3)	(**2**)(134)	(**2**)(143)	(**3**)(124)	(**3**)(142)	(**4**)(123)	(**4**)(132)

π	–	(12)(3*4*5)	(12)(3*5*4)	(1**2**3)(45)	(1*3*)(245)	(1*4*)(235)	(1**5**4)(23)	...
$\psi(\pi)$	(1)(23)(45)	(1)(2*4*)(35)	(1)(2*5*)(34)	(**2**)(13)(45)	(1)(2*3*45)	(1)(2*4*35)	(**5**)(14)(23)	...

References

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Department of Mathematics, Dartmouth College, Hanover, NH 03755, USA
Email address: sergi.elizalde@dartmouth.edu