A FAMILY OF SUMS OF GIBONACCI POLYNOMIAL PRODUCTS OF ORDER 4

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Abstract. We explore sums of gibonacci polynomial products of order 4 for g_{4n-1}, g_{4n}, g_{4n+1}, g_{4n+2}, and g_{4n+3} in terms of g_{n-2}^i, g_n^j, and g_{n+2}^k, where g_n denotes the nth gibonacci polynomial, $0 \le i, j, k \le 4$, and $i+j+k=4$. Correspondingly, they yield formulas for G_{4n-1}, G_{4n}, G_{4n+1}, G_{4n+2}, and G_{4n+3}, where G_n denotes the nth gibonacci number. In addition, they have Pell implications.

1. Introduction

Gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x)+b(x)z_n(x)$, where x is an arbitrary complex variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials; and $n \ge 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 5, 6].

The nth *Pell polynomial* $p_n(x)$ and the nth *Pell-Lucas polynomial* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. Correspondingly, the nth Pell number P_n and the nth Pell-Lucas number Q_n are given by $P_n = p_n(1)$ and $2Q_n = q_n(1)$, respectively [5].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. We let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $G_n = F_n$ or L_n, $B_n = P_n$ or Q_n, and omit a lot of basic algebra.

A *gibonacci polynomial product of order* m is a product of gibonacci polynomials g_{n+k} of the form $\prod_{k\in\mathcal{Z}} g_{n+k}^{s_j}$, where $\sum_{s_j\ge 1} s_j = m$ [7, 8].

1.1. Sums of Fibonacci Polynomial Products of Order 3.

The following sums of gibonacci polynomial products of order 3 play a pivotal role in our discourse [3]:

$$x^2 f_{3n} = 3f_{n+2}^2 f_n - (2x^2+5)f_{n+2}f_n^2 + (x^2+1)f_n^3 + (x^2+1)f_n^2 f_{n-2}; \tag{1.1}$$

$$x^3 f_{3n+1} = f_{n+2}^3 - 3f_{n+2}^2 f_n + (2x^2+3)f_{n+2}f_n^2 - (x^2+1)f_n^3 - x^2 f_n^2 f_{n-2}; \tag{1.2}$$

$$x^2 f_{3n+2} = f_{n+2}^3 - 2f_{n+2}f_n^2 + f_n^2 f_{n-2}, \tag{1.3}$$

where $f_n = f_n(x)$.

2. Sums of Gibonacci Polynomial Products of Order 4

In addition to the above identities, our discussion also hinges on gibonacci recurrence identities $f_{n+2}+f_{n-2} = (x^2+2)f_n$, $f_{n+2}-f_{n-2} = xl_n$, $f_{n+1}+f_{n-1} = l_n$, $f_n = (x^2+1)f_{n-2}+xf_{n-3}$, $f_{2n} = f_n l_n$, $f_{2n+1} = f_{n+1}^2 + f_n^2$, and the *gibonacci addition formula* [5]

$$g_{a+b} = f_{a+1}g_b + f_a g_{b-1}.$$

With this background, we begin our exploration with $x^3 f_{4n+2}$.

2.1. **A Gibonacci Sum for $x^3 f_{4n+2}$.** By the Fibonacci addition formula, and identities (1.2) and (1.3), we have

$$\begin{aligned}
f_{4n+2} &= f_{3n+2}f_{n+1} + f_{3n+1}f_n;\\
x^3 f_{4n+2} &= (x^2 f_{3n+2})(xf_{n+1}) + (x^3 f_{3n+1})f_n\\
&= \left(f_{n+2}^3 - 2f_{n+2}f_n^2 + f_n^2 f_{n-2}\right)(f_{n+2} - f_n)\\
&\quad + \left[f_{n+2}^3 - 3f_{n+2}^2 f_n - (2x^2+3)f_{n+2}f_n^2 - (x^2+1)f_n^3 - x^2 f_n^2 f_{n-2}\right] f_n\\
&= f_{n+2}^4 - 5f_{n+2}^2 f_n^2 + (2x^2+5)f_{n+2}f_n^3 + f_{n+2}f_n^2 f_{n-2} - (x^2+1)f_n^4\\
&\quad - (x^2+1)f_n^3 f_{n-2}. \qquad (2.1)
\end{aligned}$$

Next, we develop an alternate formula for $x^3 f_{4n+2}$.

An Alternate Formula. Because $l_{2n+1} = f_{2n+2} + f_{2n}$, we have

$$\begin{aligned}
xl_{2n+1} &= (f_{n+2} - f_n)(f_{n+2} + f_n) + f_n(f_{n+2} - f_{n-2})\\
&= f_{n+2}^2 + f_{n+2}f_n - f_n^2 - f_n f_{n-2}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
xf_{4n+2} &= f_{2n+1}(xl_{2n+1})\\
&= \left(f_{n+1}^2 + f_n^2\right)(xl_{2n+1});\\
x^3 f_{4n+2} &= \left[(f_{n+2} - f_n)^2 + x^2 f_n^2\right]\left[f_{n+2}^2 + f_{n+2}f_n - f_n^2 - f_n f_{n-2}\right]\\
&= f_{n+2}^4 + A + B + C + E + F + G,
\end{aligned}$$

where

$$\begin{aligned}
A &= -f_{n+2}^3 f_n + (x^2-2)f_{n+2}^2 f_n^2 - f_{n+2}^2 f_n f_{n-2}\\
&= -f_{n+2}^2 f_n(f_{n+2} + f_{n-2}) + (x^2-2)f_{n+2}^2 f_n^2\\
&= -f_{n+2}^2 f_n\left[(x^2+2)f_n\right] + (x^2-2)f_{n+2}^2 f_n^2\\
&= -4f_{n+2}^2 f_n^2;\\
B &= (x^2+2)f_{n+2}f_n^3\\
&= xf_{n+2}f_n^2(f_{n+1} - f_{n-1}) + f_{n+2}f_n^2(f_{n+2} - xf_{n+1}) + f_{n+2}f_n^3\\
&= -f_{n+2}f_n^2(xf_{n-1}) + f_{n+2}^2 f_n^2 + f_{n+2}f_n^3\\
&= -f_{n+2}f_n^2(f_n - f_{n-2}) + f_{n+2}^2 f_n^2 + f_{n+2}f_n^3\\
&= f_{n+2}^2 f_n^2 + f_{n+2}f_n^2 f_{n-2};\\
C &= 2f_{n+2}f_n^2 f_{n-2};\\
E &= f_{n+2}f_n^3 - x^2 f_n^4\\
&= xf_{n+1} + f_n)f_n^3 - x^2 f^4\\
&= f_n^4 + xf_n^3 f_{n-1}\\
&= f_n^4 + x^2 f_n^2 f_{n-1}^2 + xf_n^2 f_{n-} f_{n-2};
\end{aligned}$$

$$\begin{aligned}
F &= -(x^2+1)f_n^3 f_{n-2} \\
&= -f_n^2 f_{n-2}(f_{n+2}-xf_{n+1}) - x^2 f_n^3 f_{n-2} \\
&= -f_{n+2}f_n^2 f_{n-2} + xf_{n+1}f_n^2 f_{n-2} - x^2 f_n^3 f_{n-2} \\
&= -f_{n+2}f_n^2 f_{n-2} + xf_n^2 f_{n-1}f_{n-2}; \\
G &= -f_n^4 \\
&= -f_n^2(xf_{n-1}+f_{n-2})^2 \\
&= -x^2 f_n^2 f_{n-1}^2 - 2xf_n^2 f_{n-1}f_{n-2} - f_n^2 f_{n-2}^2.
\end{aligned}$$

Thus,

$$x^3 f_{4n+2} = f_{n+2}^4 - 3f_{n+2}^2 f_n^2 + 2f_{n+2}f_n^2 f_{n-2} + f_n^4 - f_n^2 f_{n-2}^2. \tag{2.2}$$

It follows from identities (2.1) and (2.2) that

$$2f_{n+2}^2 f_n^2 + f_{n+2}f_n^2 f_{n-2} + (x^2+2)f_n^4 + (x^2+1)f_n^3 f_{n-2} = (2x^2+5)f_{n+2}f_n^3 + f_n^2 f_{n-2}^2. \tag{2.3}$$

This can be confirmed independently. To this end, let

$$\begin{aligned}
A^* &= f_{n+2}f_n^2 f_{n-2} - (2x^2+5)f_{n+2}f_n^3 + f_{n+2}f_n^2 f_{n-2} \\
&= f_{n+2}f_n^2[2f_{n+2} - (2x^2+5)f_n + f_{n-2}] \\
&= f_{n+2}f_n^2[f_{n+2} + (x^2+2)f_n - (2x^2+5)f_n] \\
&= f_{n+2}f_n^2[f_{n+2} - (x^2+3)f_n] \\
&= -f_{n+2}f_n^2(f_n + f_{n-2}); \\
B^* &= (x^2+2)f_n^4 + (x^2+1)f_n^3 f_{n-2} - f_n^2 f_{n-2}^2 \\
&= f_n^2\left[(x^2+2)f_n^2 + (x^2+1)f_n f_{n-2} - f_{n-2}^2\right] \\
&= f_n^2\left\{f_n(f_{n+2}+f_{n-2}) + f_{n-2}\left[(x^2+1)f_n - f_{n-2}\right]\right\} \\
&= f_n^2\left[f_n(f_{n+2}+f_{n-2}) + f_{n-2}\left(x^2 f_n + xf_{n-1}\right)\right] \\
&= f_n^2\left(f_{n+2}f_n + f_n f_{n-2} + xf_{n+1}f_{n-2}\right) \\
&= f_{n+2}f_n^2(f_n + f_{n-2}).
\end{aligned}$$

Then $A^* + B^* = 0$; this confirms identity (2.3), as desired.

Identities (2.1) and (2.2) yield

$$\begin{aligned}
F_{4n+2} &= F_{n+2}^4 - 5F_{n+2}^2 F_n^2 + 7F_{n+2}F_n^2 + F_{n+2}F_n^2 F_{n-2} - 2F_n^4 - 2F_n^3 F_{n-2}; &(2.4)\\
F_{4n+2} &= F_{n+2}^4 - 3F_{n+2}^2 F_n^2 + 2F_{n+2}F_n^2 F_{n-2} + F_n^4 - F_n^2 F_{n-2}^2, &(2.5)
\end{aligned}$$

respectively; see [2, 4] for a graph-theoretic proof of identity (2.5) using path graphs.

Next, we find a similar formula for $x^4 f_{4n+1}$.

2.2. **A Gibonacci Sum for $x^4 f_{4n+1}$.** By the gibonacci addition formula, and identities (1.1) and (1.2), we have

$$\begin{aligned}
x^4 f_{4n+1} &= (x^3 f_{3n+1})(xf_{n+1}) + x^2(x^2 f_{3n})f_n \\
&= (f_{n+2}-f_n)\left[f_{n+2}^3 - 3f_{n+2}^2 f_n + (2x^2+3)f_{n+2}f_n^2 - (x^2+1)f_n^3 - x^2 f_n^2 f_{n-2}\right] \\
&\quad + x^2 f_n\left[2f_{n+2}^2 f_n - (x^2+3)f_{n+2}f_n^2 - f_{n+2}f_n f_{n-2} + (x^2+1)f_n^3 + (x^2+1)f_n^2 f_{n-2}\right] \\
&= f_{n+2}^4 - 4f_{n+2}^3 f_n + 2(2x^2+3)f_{n+2}^2 f_n^2 - (x^4+6x^2+4)f_{n+2}f_n^3 - 2x^2 f_{n+2}f_n^2 f_{n-2} \\
&\quad + (x^2+1)^2 f_n^4 + (x^4+2x^2)f_n^3 f_{n-2}.
\end{aligned} \tag{2.6}$$

In particular, we have

$$\begin{aligned} F_{4n+1} &= F_{n+2}^4 - 4F_{n+2}^3F_n + 10F_{n+2}^2F_n^2 - 11F_{n+2}F_n^3 - 2F_{n+2}F_n^2F_{n-2} \\ &\quad + 4F_n^4 + 3F_n^3F_{n-2}. \end{aligned} \tag{2.7}$$

Next, we explore a formula for $x^3 f_{4n}$.

2.3. **A Gibonacci Sum for $x^3 f_{4n}$.** By identities (2.2) and (2.6), and the addition formula, we have

$$\begin{aligned} x^3 f_{4n} &= x^3 f_{4n+2} - x^4 f_{4n+1} \\ &= (f_{n+2}^4 - 3f_{n+2}^2 f_n^2 + 2f_{n+2}f_n^2 f_{n-2} + f_n^4 - f_n^2 f_{n-2}^2) \\ &\quad - [f_{n+2}^4 - 4f_{n+2}^3 f_n + 2(2x^2+3)f_{n+2}^2 f_n^2 - (x^4+6x^2+4)f_{n+2}f_n^3 \\ &\quad - 2x^2 f_{n+2}f_n^2 f_{n-2} + (x^2+1)^2 f_n^4 + (x^4+2x^2)f_n^3 f_{n-2}] \\ &= 4f_{n+2}^3 f_n - (4x^2+9)f_{n+2}^2 f_n^2 + (x^4+6x^2+4)f_{n+2}f_n^3 + 2(x^2+1)f_{n+2}f_n^2 f_{n-2} \\ &\quad - (x^4+2x^2)f_n^4 - (x^4+2x^2)f_n^3 f_{n-2} - f_n^2 f_{n-2}^2. \end{aligned} \tag{2.8}$$

An Alternate Version. Using the identity $f_{2n} = f_n l_n$, we can develop an alternate formula for $x^3 f_{4n}$:

$$\begin{aligned} f_{4n} &= f_{2n}(f_{2n+1} + f_{2n-1}) \\ &= f_n l_n \left[\left(f_{n+1}^2 + f_n^2\right) + \left(f_n^2 + f_{n-1}^2\right) \right]; \\ x^3 f_{4n} &= f_n(xl_n)\left[(xf_{n+1})^2 + 2x^2 f_n^2 + (xf_{n-1})^2\right] \\ &= f_n(f_{n+2} - f_{n-2})\left[(f_{n+2} - f_n)^2 + 2x^2 f_n^2 + (f_n - f_{n-2})^2\right] \\ &= f_{n+2}^3 f_n - 2f_{n+2}^2 f_n^2 - f_{n+2}^2 f_n f_{n-2} + 2(x^2+1)f_{n+2}f_n^3 + f_{n+2}f_n f_{n-2}^2 \\ &\quad - 2(x^2+1)f_n^3 f_{n-2} + 2f_n^2 f_{n-2}^2 - f_n f_{n-2}^3. \end{aligned} \tag{2.9}$$

It follows by identities (2.8) and (2.9) that

$$\begin{aligned} F_{4n} &= 4F_{n+2}^3 F_n - 13F_{n+2}^2 F_n^2 + 11F_{n+2}F_n^3 + 4F_{n+2}F_n^2 F_{n-2} \\ &\quad - 3F_n^4 - 3F_n^3 F_{n-2} - F_n^2 F_{n-2}^2; \end{aligned} \tag{2.10}$$

$$\begin{aligned} &= F_{n+2}^3 F_n - 2F_{n+2}^2 F_n^2 - F_{n+2}^2 F_n F_{n-2} + 4F_{n+2}F_n^3 + F_{n+2}F_n F_{n-2}^2 \\ &\quad - 4F_n^3 F_{n-2} + 2F_n^2 F_{n-2}^2 - F_n F_{n-2}^3, \end{aligned} \tag{2.11}$$

respectively.

It also follows by identities (2.8) and (2.9) that

$$\begin{aligned} &3f_{n+2}^3 f_n - (4x^2+7)f_{n+2}^2 f_n^2 + f_{n+2}^2 f_n f_{n-2} + (x^4+4x^2+2)f_{n+2}f_n^3 \\ &\quad + 2(x^2+1)f_{n+2}f_n^2 f_{n-2} - f_{n+2}f_n f_{n-2}^2 - (x^4+2x^2)f_n^4 \\ &\quad - (x^4-2)f_n^3 f_{n-2} - 3f_n^2 f_{n-2}^2 + f_n f_{n-2}^3 = 0. \end{aligned}$$

This yields

$$\begin{aligned} &3f_{n+2}^3 - (4x^2+7)f_{n+2}^2 f_n + f_{n+2}^2 f_{n-2} + (x^4+4x^2+2)f_{n+2}f_n^2 \\ &\quad + 2(x^2+1)f_{n+2}f_n f_{n-2} - f_{n+2}f_{n-2}^2 - (x^4+2x^2)f_n^3 \\ &\quad - (x^4-2)f_n^2 f_{n-2} - 3f_n f_{n-2}^2 + f_{n-2}^3 = 0. \end{aligned} \tag{2.12}$$

This can be confirmed independently using the identity $f_{n+2} + f_{n-2} = (x^2 + 2)f_n$ [5]. To this end, we let

$$\begin{aligned}
A &= 3f_{n+2}^3 - (4x^2 + 7)f_{n+2}^2 f_n + f_{n+2}^2 f_{n-2} \\
&= f_{n+2}^2 \left[3(f_{n+2} - f_n) - (4x^2 + 4)f_n + f_{n-2}\right] \\
&= f_{n+2}^2 \left[3xf_{n+1} - (4x^2 + 3)f_n - xf_{n-1}\right] \\
&= f_{n+2}^2 \left[2xf_{n+1} - (3x^2 + 3)f_n\right] \\
&= -f_{n+2}^2 \left[(x^2 + 3)f_n - 2xf_{n-1}\right] \\
&= -f_{n+2}^2 \left[(x^2 + 1)f_n + 2f_{n-2}\right]; \\
B &= (x^4 + 4x^2 + 2)f_{n+2}f_n^2 + 2(x^2 + 1)f_{n+2}f_n f_{n-2} - f_{n+2}f_{n-2}^2 \\
&= (x^4 + 4x^2 + 2)f_{n+2}f_n^2 + f_{n+2}f_{n-2}\left[(2x^2 + 1)f_n + xf_{n-1}\right] \\
&= (x^4 + 4x^2 + 2)f_{n+2}f_n^2 + f_{n+2}f_{n-2}\left(f_{n+2} + x^2 f_n\right) \\
&= f_{n+2}^2 f_{n-2} + (x^4 + 4x^2 + 2)f_{n+2}f_n^2 + x^2 f_{n+2}f_n f_{n-2}; \\
C &= -(x^4 + 2x^2)f_n^3 - (x^4 - 2)f_n^2 f_{n-2} \\
&= -f_n^2 \left[x^4(f_n + f_{n-2}) + 2(x^2 f_n - f_{n-2})\right] \\
&= -2f_{n+2}f_n^2 - (x^4 - 4)f_n^3 - x^4 f_n^2 f_{n-2}; \\
E &= -3f_n f_{n-2}^2 + f_{n-2}^3 \\
&= -2f_n f_{n-2}^2 - f_{n-2}^2 \left[f_{n+2} - (x^2 + 1)f_n\right] \\
&= f_{n+2}f_{n-2}(-f_{n-2}) + (x^2 - 1)f_n f_{n-2}^2 \\
&= f_{n+2}f_{n-2}\left[f_{n+2} - (x^2 + 2)f_n\right] + (x^2 - 1)f_n f_{n-2}^2 \\
&= f_{n+2}^2 f_{n-2} - (x^2 + 2)f_{n+2}f_n f_{n-2} + (x^2 - 1)f_n f_{n-2}^2; \\
B + C + E &= 2f_{n+2}^2 f_{n-2} + F - 2f_{n+2}f_n f_{n-2} - (x^4 - 4)f_n^3 - x^4 f_n^2 f_{n-2} + (x^2 - 1)f_n f_{n-2}^2,
\end{aligned}$$

where

$$\begin{aligned}
F &= (x^4 + 4x^2)f_{n+2}f_n^2 \\
&= (x^4 + 3x^2 - 1)f_{n+2}f_n^2 + (x^2 + 1)f_{n+2}f_n(f_{n+2} - xf_{n+1}) \\
&= (x^2 + 1)f_{n+2}^2 f_n + (x^4 + 3x^2 - 1)f_{n+2}f_n^2 - (x^3 + x)f_{n+2}f_{n+1}f_n.
\end{aligned}$$

Then

$$B + C + E = f_{n+2}^2 \left[(x^2 + 1)f_n + 2f_{n-2}\right] + G + H + I + J + K + L,$$

where

$$\begin{aligned}
G &= (x^4 + 3x^2 - 1)f_{n+2}f_n^2 \\
&= f_{n+2}f_n\left[(x^4 + 2x^2)f_n\right] + (x^2 - 1)f_{n+2}f_n^2 \\
&= x^2 f_{n+2}^2 f_n + x^2 f_{n+2}f_n f_{n-2} + (x^2 - 1)f_{n+2}f_n^2; \\
H &= -(x^3 + x)f_{n+2}f_{n+1}f_n \\
&= -(x^2 + 1)f_{n+2}^2 f_n + (x^2 + 1)f_{n+2}f_n^2;
\end{aligned}$$

$$\begin{aligned}
I &= -2f_{n+2}f_nf_{n-2};\\
J &= -(x^4-4)f_n^3;\\
K &= -x^4f_n^2f_{n-2}\\
&= -x^2f_nf_{n-2}(f_{n+2}-2f_n+f_{n-2})\\
&= -x^2f_{n+2}f_nf_{n-2}+2x^2f_n^2f_{n-2}-x^2f_nf_{n-2}^2;\\
L &= (x^2-1)f_nf_{n-2}^2.
\end{aligned}$$

We then have

$$G+H+K+L=M+N+O,$$

where

$$\begin{aligned}
M &= -f_{n+2}^2f_n\\
&= -(x^2+2)f_{n+2}f_n^2+f_{n+2}f_nf_{n-2};\\
N &= 2x^2f_n^2(f_{n+2}+f_{n-2})\\
&= (2x^4+4x^2)f_n^3;\\
O &= -f_nf_{n-2}^2\\
&= f_{n+2}f_nf_{n-2}-(x^2+2)f_n^2f_{n-2}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
M+I &= -(x^2+2)f_{n+2}f_n^2-f_{n+2}f_nf_{n-2};\\
N+J &= (x^2+2)^2f_n^3;\\
G+H+I+J+K+L &= (M+I+O)+(N+J)\\
&= 0;
\end{aligned}$$

$$\begin{aligned}
B+C+E &= f_{n+2}^2\left[(x^2+1)f_n+2f_{n-2}\right]+0\\
&= -A;\\
A+B+C+E &= 0,
\end{aligned}$$

as desired.

It follows from equation (2.12) that

$$\begin{aligned}
&3F_{n+2}^3+F_{n+2}^2F_{n-2}+7F_{n+2}F_n^2\\
&+\ 4F_{n+2}F_nF_{n-2}+F_n^2F_{n-2}+F_{n-2}^3 \ =\ 11F_{n+2}^2F_n+F_{n+2}F_{n-2}^2+3F_n^3+3F_nF_{n-2}^2. \qquad (2.13)
\end{aligned}$$

Next, we investigate a formula for x^4f_{4n+3}.

2.4. **A Gibonacci Sum for x^4f_{4n+3}.** Using the Fibonacci recurrence, and identities (2.2) and (2.6), we have

$$\begin{aligned}
x^4f_{4n+3} &= x^2(x^3f_{4n+2})+x^4f_{4n+1}\\
&= x^2\left(f_{n+2}^4-3f_{n+2}^2f_n^2+2f_{n+2}f_n^2f_{n-2}+f_n^4-f_n^2f_{n-2}^2\right)\\
&\quad+[f_{n+2}^4-4f_{n+2}^3f_n+2(2x^2+3)f_{n+2}^2f_n^2-(x^4+6x^2+4)f_{n+2}f_n^3\\
&\quad-2x^2f_{n+2}f_n^2f_{n-2}+(x^2+1)^2f_n^4+(x^4+2x^2)f_n^3f_{n-2}]\\
&= (x^2+1)f_{n+2}^4-4f_{n+2}^3f_n+(x^2+6)f_{n+2}^2f_n^2-(x^4+6x^2+4)f_{n+2}f_n^3\\
&\quad+(x^4+3x^2+1)f_n^4+(x^4+2x^2)f_n^3f_{n-2}-x^2f_n^2f_{n-2}^2. \qquad (2.14)
\end{aligned}$$

Consequently,

$$F_{4n+3} = 2F_{n+2}^4 - 4F_{n+2}^3F_n + 7F_{n+2}^2F_n^2 - 11F_{n+2}F_n^3 + 5F_n^4 + 3F_n^3F_{n-2} - F_n^2F_{n-2}^2. \quad (2.15)$$

We now explore a gibonacci sum for x^4f_{4n-1}.

2.5. **A Gibonacci Sum for x^4f_{4n-1}.** Using the gibonacci recurrence, and identities (2.6) and (2.8), we have

$$\begin{aligned} x^4f_{4n-1} &= [f_{n+2}^4 - 4f_{n+2}^3f_n + 2(2x^2+3)f_{n+2}^2f_n^2 - (x^4+6x^2+4)f_{n+2}f_n^3 \\ &\quad - 2x^2f_{n+2}f_n^2f_{n-2} + (x^2+1)^2f_n^4 + (x^4+2x^2)f_n^3f_{n-2}] \\ &\quad - x^2[4f_{n+2}^3f_n - (4x^2+9)f_{n+2}^2f_n^2 + (x^4+6x^2+4)f_{n+2}f_n^3 \\ &\quad + 2(x^2+1)f_{n+2}f_n^2f_{n-2} - (x^4+2x^2)f_n^4 - (x^6+2x^4)f_n^3f_{n-2} - f_n^2f_{n-2}^2] \\ &= f_{n+2}^4 - 4(x^2+1)f_{n+2}^3f_n + (4x^4+13x^2+6)f_{n+2}^2f_n^2 \\ &\quad - (x^6+7x^4+10x^2+4)f_{n+2}f_n^3 - 2(x^4+2x^2)f_{n+2}f_n^2f_{n-2} \\ &\quad + (x^6+3x^4+2x^2+1)f_n^4 + (x^6+3x^4+2x^2)f_n^3f_{n-2} + x^2f_n^2f_{n-2}^2. \quad (2.16) \end{aligned}$$

In particular,

$$\begin{aligned} F_{4n-1} &= F_{n+2}^4 - 8F_{n+2}^3F_n + 23F_{n+2}^2F_n^2 - 22F_{n+2}F_n^3 - 6F_{n+2}F_n^2F_{n-2} \\ &\quad + 7F_n^4 + 6F_n^3F_{n-2} + F_n^2F_{n-2}^2. \quad (2.17) \end{aligned}$$

Next, we find a gibonacci sum for x^4l_{4n}.

2.6. **A Gibonacci Sum for x^4l_{4n}.** Using the identity $f_{n+1} + f_{n-1} = l_n$, and identities (2.6) and (2.16), we have

$$\begin{aligned} x^4l_{4n} &= x^4f_{4n+1} + x^4f_{4n-1} \\ &= [f_{n+2}^4 - 4f_{n+2}^3f_n + 2(2x^2+3)f_{n+2}^2f_n^2 - (x^4+6x^2+4)f_{n+2}f_n^3 \\ &\quad - 2x^2f_{n+2}f_n^2f_{n-2} + (x^2+1)^2f_n^4 + (x^4+2x^2)f_n^3f_{n-2}] \\ &\quad + [f_{n+2}^4 - 4(x^2+1)f_{n+2}^3f_n + (4x^4+13x^2+6)f_{n+2}^2f_n^2 \\ &\quad - (x^6+7x^4+10x^2+4)f_{n+2}f_n^3 - 2(x^4+2x^2)f_{n+2}f_n^2f_{n-2} \\ &\quad + (x^6+3x^4+2x^2+1)f_n^4 + (x^6+3x^4+2x^2)f_n^3f_{n-2} + x^2f_n^2f_{n-2}^2] \\ &= 2f_{n+2}^4 - 4(x^2+2)f_{n+2}^3f_n + (4x^4+17x^2+12)f_{n+2}^2f_n^2 \\ &\quad - (x^6+8x^4+16x^2+8)f_{n+2}f_n^3 - 2(x^4+3x^2)f_{n+2}f_n^2f_{n-2} \\ &\quad + (x^6+4x^4+4x^2+2)f_n^4 + (x^6+4x^4+4x^2)f_n^3f_{n-2} + x^2f_n^2f_{n-2}^2. \quad (2.18) \end{aligned}$$

This implies

$$\begin{aligned} L_{4n} &= 2F_{n+2}^4 - 12F_{n+2}^3F_n + 33F_{n+2}^2F_n^2 - 33F_{n+2}F_n^3 - 8F_{n+2}F_n^2F_{n-2} \\ &\quad + 11F_n^4 + 9F_n^3F_{n-2} + F_n^2F_{n-2}^2. \quad (2.19) \end{aligned}$$

2.7. **A Gibonacci Sum for x^3l_{4n+1}.** Using the identity $f_{n+1}+f_{n-1}=l_n$, and identities (2.2) and (2.8), we have

$$\begin{aligned} x^3l_{4n+1} &= x^3f_{4n+2}+x^3f_{4n} \\ &= (f_{n+2}^4-3f_{n+2}^2f_n^2+2f_{n+2}f_n^2f_{n-2}+f_n^4-f_n^2f_{n-2}^2) \\ &+ [4f_{n+2}^3f_n-(4x^2+9)f_{n+2}^2f_n^2+(x^4+6x^2+4)f_{n+2}f_n^3+2(x^2+1)f_{n+2}f_n^2f_{n-2} \\ &- (x^4+2x^2)f_n^4-(x^4+2x^2)f_n^3f_{n-2}-f_n^2f_{n-2}^2] \\ &= f_{n+2}^4+4f_{n+2}^3f_n-4(x^2+3)f_{n+2}^2f_n^2+(x^4+6x^2+4)f_{n+2}f_n^3 \\ &+ 2(x^2+2)f_{n+2}f_n^2f_{n-2}-(x^4+2x^2-1)f_n^4-(x^4+2x^2)f_n^3f_{n-2}-2f_n^2f_{n-2}^2. \end{aligned} \tag{2.20}$$

It then follows that

$$\begin{aligned} L_{4n+1} &= F_{n+2}^4+4F_{n+2}^3F_n-16F_{n+2}^2F_n^2+11F_{n+2}F_n^3+6F_{n+2}F_n^2F_{n-2}-2F_n^4 \\ &- 3F_n^3F_{n-2}-2F_n^2F_{n-2}^2. \end{aligned} \tag{2.21}$$

2.8. **A Gibonacci Sum for x^3l_{4n-1}.** Using gibonacci recurrence, and identities (2.18) and (2.20), we have

$$\begin{aligned} x^3l_{4n-1} &= x^3l_{4n+1}-x^4l_{4n} \\ &= [f_{n+2}^4+4f_{n+2}^3f_n-4(x^2+3)f_{n+2}^2f_n^2+(x^4+6x^2+4)f_{n+2}f_n^3 \\ &+ 2(x^2+2)f_{n+2}f_n^2f_{n-2}-(x^4+2x^2-1)f_n^4-(x^4+2x^2)f_n^3f_{n-2}-2f_n^2f_{n-2}^2] \\ &- [2f_{n+2}^4-4(x^2+2)f_{n+2}^3f_n+(4x^4+17x^2+12)f_{n+2}^2f_n^2 \\ &- (x^6+8x^4+16x^2+8)f_{n+2}f_n^3-2(x^4+3x^2)f_{n+2}f_n^2f_{n-2} \\ &+ (x^6+4x^4+4x^2+2)f_n^4+(x^6+4x^4+4x^2)f_n^3f_{n-2}+x^2f_n^2f_{n-2}^2] \\ &= -f_{n+2}^4+4(x^2+3)f_{n+2}^3f_n-(4x^4+21x^2+24)f_{n+2}^2f_n^2 \\ &+ (x^6+9x^4+22x^2+12)f_{n+2}f_n^3+2(x^4+4x^2+2)f_{n+2}f_n^2f_{n-2} \\ &- (x^6+5x^4+6x^2+1)f_n^4-(x^6+5x^4+6x^2)f_n^3f_{n-2}-(x^2+2)f_n^2f_{n-2}^2. \end{aligned} \tag{2.22}$$

This implies

$$\begin{aligned} L_{4n-1} &= -F_{n+2}^4+16F_{n+2}^3F_n-49F_{n+2}^2F_n^2+44F_{n+2}F_n^3+14F_{n+2}F_n^2F_{n-2} \\ &- 13F_n^4-12F_n^3F_{n-2}-3F_n^2F_{n-2}^2. \end{aligned} \tag{2.23}$$

Next, we express x^4l_{4n+2} as a gibonacci sum.

2.9. **A Gibonacci Sum for $x^4 l_{4n+2}$.** The gibonacci recurrence, coupled with identities (2.18) and (2.20), yields

$$\begin{aligned}
x^4 l_{4n+2} &= x^5 l_{4n+1} + x^4 l_{4n}\\
&= x^2[f_{n+2}^4 + 4f_{n+2}^3 f_n - 4(x^2+3)f_{n+2}^2 f_n^2 + (x^4+6x^2+4)f_{n+2}f_n^3\\
&+ 2(x^2+2)f_{n+2}f_n^2 f_{n-2} - (x^4+2x^2-1)f_n^4 - (x^4+2x^2)f_n^3 f_{n-2} - 2f_n^2 f_{n-2}^2]\\
&+ [2f_{n+2}^4 - 4(x^2+2)f_{n+2}^3 f_n + (4x^4+17x^2+12)f_{n+2}^2 f_n^2\\
&- (x^6+8x^4+16x^2+8)f_{n+2}f_n^3 - 2(x^4+3x^2)f_{n+2}f_n^2 f_{n-2}\\
&+ (x^6+4x^4+4x^2+2)f_n^4 + (x^6+4x^4+4x^2)f_n^3 f_{n-2} + x^2 f_n^2 f_{n-2}^2]\\
&= (x^2+2)f_{n+2}^4 - 8f_{n+2}^3 f_n + (5x^2+12)f_{n+2}^2 f_n^2 - 2(x^4+6x^2+4)f_{n+2}f_n^3\\
&- 2x^2 f_{n+2}f_n^2 f_{n-2} + (2x^4+5x^2+2)f_n^4 + 2(x^4+2x^2)f_n^3 f_{n-2} - x^2 f_n^2 f_{n-2}^2. \qquad (2.24)
\end{aligned}$$

This yields

$$\begin{aligned}
L_{4n+2} &= 3F_{n+2}^4 - 8F_{n+2}^3 F_n + 17F_{n+2}^2 F_n^2 - 22F_{n+2}F_n^3 - 2F_{n+2}F_n^2 F_{n-2}\\
&+ 9F_n^4 + 6F_n^3 F_{n-2} - F_n^2 F_{n-2}^2. \qquad (2.25)
\end{aligned}$$

Finally, we express $x^3 l_{4n+3}$ as a gibonacci sum.

2.10. **A Gibonacci Sum for $x^3 l_{4n+3}$.** Using gibonacci recurrence, and identities (2.20) and (2.24), we have

$$\begin{aligned}
x^3 l_{4n+3} &= x^4 l_{4n+2} + x^3 l_{4n+1}\\
&= [f_{n+2}^4 + 4f_{n+2}^3 f_n - 4(x^2+3)f_{n+2}^2 f_n^2 + (x^4+6x^2+4)f_{n+2}f_n^3\\
&+ 2(x^2+2)f_{n+2}f_n^2 f_{n-2} - (x^4+2x^2-1)f_n^4 - (x^4+2x^2)f_n^3 f_{n-2} - 2f_n^2 f_{n-2}^2]\\
&+ [(x^2+2)f_{n+2}^4 - 8f_{n+2}^3 f_n + (5x^2+12)f_{n+2}^2 f_n^2 - 2(x^4+6x^2+4)f_{n+2}f_n^3\\
&- 2x^2 f_{n+2}f_n^2 f_{n-2} + (2x^4+5x^2+2)f_n^4 + 2(x^4+2x^2)f_n^3 f_{n-2} - x^2 f_n^2 f_{n-2}^2]\\
&= (x^2+3)f_{n+2}^4 - 4f_{n+2}^3 f_n + x^2 f_{n+2}^2 f_n^2 - (x^4+6x^2+4)f_{n+2}f_n^3\\
&+ 4f_{n+2}f_n^2 f_{n-2} + (x^4+3x^2+3)f_n^4 + (x^4+2x^2)f_n^3 f_{n-2}\\
&- (x^2+2)f_n^2 f_{n-2}^2. \qquad (2.26)
\end{aligned}$$

In particular, we then have

$$\begin{aligned}
L_{4n+3} &= 4F_{n+2}^4 - 4F_{n+2}^3 F_n + F_{n+2}^2 F_n^2 - 11F_{n+2}F_n^3 + 4F_{n+2}F_n^2 F_{n-2}\\
&+ 7F_n^4 + 3F_n^3 F_{n-2} - 3F_n^2 F_{n-2}^2. \qquad (2.27)
\end{aligned}$$

3. Conclusion

Because $b_n(x) = g_n(2x)$, the Pell versions, polynomial and numeric, follow from the gibonacci identities. In the interest of brevity, we omit them.

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