GRAPH-THEORETIC CONFIRMATIONS OF FOUR SUMS OF GIBONACCI POLYNOMIAL PRODUCTS OF ORDER 4

THOMAS KOSHY

Abstract. Using graph-theoretic techniques, we confirm four identities involving sums of gibonacci polynomial products of order 4, investigated in [2].

1. Introduction

Gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x)+b(x)z_n(x)$, where x is an arbitrary complex variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 3, 4].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [3].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. We let $g_n = f_n$ or $l_n, b_n = p_n$ or q_n, and also omit a lot of basic algebra.

Table 1 lists some well-known fundamental Fibonacci identities. We will employ them in our discourse [3].

$f_{n+1} + f_{n-1}$	$=$	l_n	f_{2n}	$=$	$f_n l_n$
$f_{n+1}^2 + f_n^2$	$=$	f_{2n+1}	$f_{n+2} + f_{n-2}$	$=$	$(x^2+2)f_n$
$f_{n+2} - f_{n-2}$	$=$	xl_n	f_{a+b}	$=$	$f_{a+1}f_b + f_a f_{b-1}$
$f_{n+k}f_{n-k} - f_n^2$	$=$	$(-1)^{n-k+1}f_k^2$			

Table 1: Fundamental Fibonacci Identities

The last two identities are the *Fibonacci addition formula* and the *Cassini-like formula*, respectively.

1.1. **Sums of Gibonacci Polynomial Products of Order 4.** Several sums of gibonacci polynomial products of order 4 are studied in [2]; in the interest of brevity, we focus only on the sums in equations (2.9), (2.6), (2.24), and (2.26) in [2], and they play a major role in our explorations:

$$\begin{aligned} x^3 f_{4n} &= f_{n+2}^3 f_n - 2f_{n+2}^2 f_n^2 - f_{n+2}^2 f_n f_{n-2} + 2(x^2+1) f_{n+2} f_n^3 + f_{n+2} f_n f_{n-2}^2 \\ &\quad -2(x^2+1) f_n^3 f_{n-2} + 2f_n^2 f_{n-2}^2 - f_n f_{n-2}^3; \qquad (1) \\ x^4 f_{4n+1} &= f_{n+2}^4 - 4f_{n+2}^3 f_n + 2(2x^2+3) f_{n+2}^2 f_n^2 - (x^4+6x^2+4) f_{n+2} f_n^3 - 2x^2 f_{n+2} f_n^2 f_{n-2} \\ &\quad + (x^2+1)^2 f_n^4 + (x^4+2x^2) f_n^3 f_{n-2}; \qquad (2) \\ x^4 l_{4n+2} &= (x^2+2) f_{n+2}^4 - 8f_{n+2}^3 f_n + (5x^2+12) f_{n+2}^2 f_n^2 - 2(x^4+6x^2+4) f_{n+2} f_n^3 \\ &\quad - 2x^2 f_{n+2} f_n^2 f_{n-2} + (2x^4+5x^2+2) f_n^4 + 2(x^4+2x^2) f_n^3 f_{n-2} - x^2 f_n^2 f_{n-2}^2; \qquad (3) \\ x^3 l_{4n+3} &= (x^2+3) f_{n+2}^4 - 4f_{n+2}^3 f_n + x^2 f_{n+2}^2 f_n^2 - (x^4+6x^2+4) f_{n+2} f_n^3 \\ &\quad + 4f_{n+2} f_n^2 f_{n-2} + (x^4+3x^2+3) f_n^4 + (x^4+2x^2) f_n^3 f_{n-2} - (x^2+2) f_n^2 f_{n-2}^2. \qquad (4) \end{aligned}$$

The remaining sums can be pursued in a similar manner.

It follows by identities (2.2) and (2.14) in [2] that

$$\begin{aligned} x^3 f_{4n+4} &= (x^2+2) f_{n+2}^4 - 4f_{n+2}^3 f_n + (x^2+3) f_{n+2}^2 f_n^2 - (x^4+6x^2+4) f_{n+2} f_n^3 \\ &\quad + 2f_{n+2} f_n^2 f_{n-2} + (x^4+3x^2+2) f_n^4 + (x^4+2x^2) f_n^3 f_{n-2} - (x^2+1) f_n^2 f_{n-2}^2. \qquad (5) \end{aligned}$$

We will employ this result in Subsection 3.4.

2. Some Graph-theoretic Tools

Our goal is to confirm the polynomial identities (1) through (4) using graph-theoretic techniques. To this end, consider the *Fibonacci digraph* D in Figure 1 with vertices v_1 and v_2, where a *weight* is assigned to each edge [3, 5].

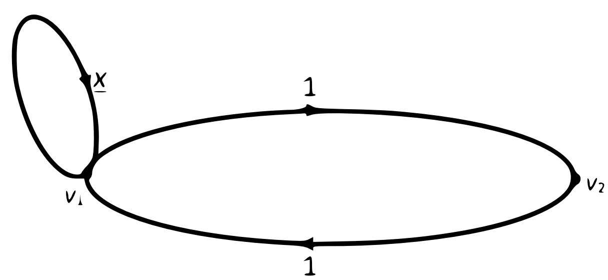

Figure 1. Weighted Fibonacci Digraph D_1

It follows from its *weighted adjacency matrix* $Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ that

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \geq 1$ [3, 5].

A *walk* from vertex v_i to vertex v_j is a sequence v_i-e_i-v_{i+1}-$\cdots$-v_{j-1}-e_{j-1}-v_j of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is *closed* if $v_i = v_j$; otherwise, it is *open*. The *length* of a walk is the number of edges in the walk. The *weight* of a walk is the product of the weights of the edges along the walk.

The ijth entry of Q^n gives the sum of the weights of all walks of length n from v_i to v_j in the weighted digraph D, where $1 \leq i, j \leq n$ [3, 5]. Consequently, the sum of the weights of closed walks of length n originating at v_1 in the digraph is f_{n+1} and that of those originating at v_2 is f_{n-1}. So, the sum of the weights of all closed walks of length n in the digraph is $f_{n+1}+f_{n-1}=l_n$. These facts play a pivotal role in the graph-theoretic proofs.

Let A, B, C, and D denote the sets of walks of varying lengths originating at a vertex v, respectively. Then, the sum of the weights of the elements (a,b,c,d) in the product set $A\times B\times C\times D$ is *defined* as the product of the sums of weights from each component [5].

With these tools at our finger tips, we are now ready for the graph-theoretic proofs.

3. Graph-theoretic Proofs

3.1. **Confirmation of Identity** (1).

Proof. Let S denote the sum of the weights of closed walks of length $4n-1$ in the digraph D originating (and ending) at v_1. Then $S=f_{4n}$, and hence $x^3S=x^3f_{4n}$.

We will now compute the sum x^3S in a different way. To this end, let w be an arbitrary closed walk of length $4n-1$ from v_1 to v_1. It can land at v_1 or v_2 at the nth, $2n$th, and $3n$th steps:

$$w=\underbrace{v_1-\ \cdots\ -v}_{\text{subwalk of length } n}\ \underbrace{v-\ \cdots\ -v}_{\text{subwalk of length } n}\ \underbrace{v-\ \cdots\ -v}_{\text{subwalk of length } n}\ \underbrace{v-\ \cdots\ -v_1}_{\text{subwalk of length } n-1},$$

where $v=v_1$ or v_2.

Table 2 shows the various possible cases and the respective sums of weights of closed walks originating at v_1 of length $4n-1$.

w lands at v_1 at the nth step?	w lands at v_1 at the $2n$th step?	w lands at v_1 at the $3n$th step?	w lands at v_1 at the $(4n-1)$st step?	sum of the weights of walks w
yes	yes	yes	yes	$f_{n+1}^3f_n$
yes	yes	no	yes	$f_{n+1}^2f_nf_{n-1}$
yes	no	yes	yes	$f_{n+1}f_n^3$
yes	no	no	yes	$f_{n+1}f_nf_{n-1}^2$
no	yes	yes	yes	$f_{n+1}f_n^3$
no	yes	no	yes	$f_n^3f_{n-1}$
no	no	yes	yes	$f_n^3f_{n-1}$
no	no	no	yes	$f_nf_{n-1}^3$

Table 2: Sums of the Weights of Closed Walks Originating at v_1 of Length $4n-1$.

It follows from the table that the sum S of the weights of all walks w is given by

$$\begin{aligned} S &= f_{n+1}^3f_n+f_{n+1}^2f_nf_{n-1}+2f_{n+1}f_n^3+f_{n+1}f_nf_{n-1}^2+2f_n^3f_{n-1}+f_nf_{n-1}^3;\\ x^3S &= A+B+C+D+E+F, \end{aligned}$$

where

$$\begin{aligned}
A &= x^3 f_{n+1}^3 f_n \\
&= (f_{n+2} - f_n)^3 f_n \\
&= f_{n+2}^3 f_n - 3f_{n+2}^2 f_n^2 + 3f_{n+2} f_n^3 - f_n^4; \\
B &= x^3 f_{n+1}^2 f_n f_{n-1} \\
&= (f_{n+2} - f_n)^2 f_n (f_n - f_{n-2}) \\
&= f_{n+2}^2 f_n^2 - f_{n+2}^2 f_n f_{n-2} - 2f_{n+2} f_n^3 + 2f_{n+2} f_n^2 f_{n-2} + f_n^4 - f_n^3 f_{n-2}; \\
C &= 2x^3 f_{n+1} f_n^3 \\
&= 2x^2 f_n^3 (f_{n+2} - f_n) \\
&= 2x^2 f_{n+2} f_n^3 - 2x^4 f_n^4; \\
D &= x^3 f_{n+1} f_n f_{n-1}^2 \\
&= (f_{n+2} - f_n) f_n (f_n - f_{n-2})^2 \\
&= f_{n+2} f_n^3 - 2f_{n+2} f_n^2 f_{n-2} + f_{n+2} f_n f_{n-2}^2 - f_n^4 + 2f_n^3 f_{n-2} - f_n^2 f_{n-2}^2; \\
E &= 2x^3 f_n^3 f_{n-1} \\
&= 2x^2 f_n^3 (f_n - f_{n-2}) \\
&= 2x^2 f_n^4 - 2x^2 f_n^3 f_{n-2}; \\
F &= x^3 f_n f_{n-1}^3 \\
&= f_n (f_n - f_{n-2})^3 \\
&= f_n^4 - 3f_n^3 f_{n-2} + 3f_n^2 f_{n-2}^2 - f_n f_{n-2}^3.
\end{aligned}$$

Consequently,

$$\begin{aligned}
x^3 S &= f_{n+2}^3 f_n - 2f_{n+2}^2 f_n^2 - f_{n+2}^2 f_n f_{n-2} + 2(x^2+1) f_{n+2} f_n^3 + f_{n+2} f_n f_{n-2}^2 \\
&\quad - 2(x^2+1) f_n^3 f_{n-2} + 2f_n^2 f_{n-2}^2 - f_n f_{n-2}^3.
\end{aligned}$$

Equating this value of $x^3 S$ with its earlier value yields identity (1), as desired. □

3.2. **Confirmation of Identity** (2).

Proof. Let S' denote the sum of the weights of closed walks of length $4n$ originating at v_1 in the digraph. Then $S' = f_{4n+1}$, and hence $x^4 S' = x^4 f_{4n+1}$.

We will now compute $x^4 S'$ in a different way, and then equate the two values. To achieve this, we let w be an arbitrary closed walk of length $4n$ originating at v_1. It can land at v_1 or v_2 at the nth, $2n$th, and $3n$th steps:

$$w = \underbrace{v_1 - \cdots - v}_{\text{subwalk of length } n} \; \underbrace{v - \cdots - v}_{\text{subwalk of length } n} \; \underbrace{v - \cdots - v}_{\text{subwalk of length } n} \; \underbrace{v - \cdots - v_1}_{\text{subwalk of length } n},$$

where $v = v_1$ or v_2.

Table 3 summarizes the possible cases and the sums of the weights of the closed walks originating at v_1 of length $4n$.

w lands at v_1 at the nth step?	w lands at v_1 at the $2n$th step?	w lands at v_1 at the $3n$th step?	w lands at v_1 at the $4n$th step?	sum of the weights of walks w
yes	yes	yes	yes	f_{n+1}^4
yes	yes	no	yes	$f_{n+1}^2 f_n^2$
yes	no	yes	yes	$f_{n+1}^2 f_n^2$
yes	no	no	yes	$f_{n+1} f_n^2 f_{n-1}$
no	yes	yes	yes	$f_{n+1}^2 f_n^2$
no	yes	no	yes	f_n^4
no	no	yes	yes	$f_{n+1} f_n^2 f_{n-1}$
no	no	no	yes	$f_n^2 f_{n-1}^2$

Table 3: Sums of the Weights of Closed Walks Originating at v_1 of Length $4n$.

It follows from the table that

$$\begin{aligned} S' &= f_{n+1}^4 + 3f_{n+1}^2 f_n^2 + 2f_{n+1} f_n^2 f_{n-1} + f_n^4 + f_n^2 f_{n-1}^2; \\ x^4 S' &= G + H + I + J + K, \end{aligned}$$

where

$$\begin{aligned} G &= x^4 f_{n+1}^4 \\ &= (f_{n+2} - f_n)^4 \\ &= f_{n+2}^4 - 4f_{n+2}^3 f_n^2 + 6f_{n+2}^2 f_n^2 - 4f_{n+2} f_n^3 + f_n^4; \\ H &= 3x^4 f_{n+1}^2 f_n^2 \\ &= 3x^2 f_n^2 (f_{n+2} - f_n)^2 \\ &= 3x^2 f_{n+2}^2 f_n^2 - 6x^2 f_{n+2} f_n^3 + 3x^2 f_n^4; \\ I &= 2x^4 f_{n+1} f_n^2 f_{n-1} \\ &= 2x^2 f_n^2 (f_{n+2} - f_n)(f_n - f_{n-2}) \\ &= 2x^2 f_{n+2} f_n^3 - 2x^2 f_{n+2} f_n^2 f_{n-2} - 2x^2 f_n^4 + 2x^2 f_n^3 f_{n-2}; \\ J &= x^4 f_n^4; \\ K &= x^4 f_n^2 f_{n-1}^2 \\ &= x^2 f_n^2 (f_n - f_{n-2})^2 \\ &= x^2 f_n^4 - 2x^2 f_n^3 f_{n-2} + x^2 f_n^2 f_{n-2}^2. \end{aligned}$$

Thus,

$$\begin{aligned} x^4 S' = \; & f_{n+2}^4 - 4f_{n+2}^3 f_n + 3(x^2+2) f_{n+2}^2 f_n^2 - 4(x^2+1) f_{n+2} f_n^3 - 2x^2 f_{n+2} f_n^2 f_{n-2} \\ & + (x^2+1)^2 f_n^4 + x^2 f_n^2 f_{n-2}^2. \end{aligned} \tag{6}$$

Now let

$$L = x^2 f_{n+2}^2 f_n^2 - (x^4 + 2x^2) f_{n+2} f_n^3 + (x^4 + 2x^2) f_n^3 f_{n-2} - x^2 f_n^2 f_{n-2}^2. \tag{7}$$

Using the identity $f_{n+2} + f_{n-2} = (x^2+2) f_n$, we have

$$\begin{aligned} L &= x^2 f_{n+2} f_n^2 \left[f_{n+2} - (x^2+2) f_n\right] + x^2 f_n^2 f_{n-2} \left[(x^2+2) f_n - f_{n-2}\right] \\ &= 0. \end{aligned}$$

Consequently, adding the value of L in equation (7) to that of x^4S' in equation (6), we get

$$\begin{aligned} x^4S' &= f_{n+2}^4 - 4f_{n+2}^3 f_n + 2(2x^2+3)f_{n+2}^2 f_n^2 - (x^4+6x^2+4)f_{n+2}f_n^3 - 2x^2 f_{n+2}f_n^2 f_{n-2} \\ &\quad + (x^2+1)^2 f_n^4 + (x^4+2x^2)f_n^2 f_{n-2}^2. \end{aligned}$$

This value of x^4S', coupled with its original value, yields identity (2), as expected. □

3.3. **Confirmation of Identity** (3).

Proof. Let S^* denote the sum of the weights of all closed walks of length $4n+2$ in the digraph. Then $S^* = l_{4n+2}$, and hence $x^4S^* = x^4 l_{4n+2}$.

We will now compute x^4S^* in a different way, and then equate the two values. Let w be an arbitrary closed walk of length $4n+2$.

Case 1. Suppose w originates (and ends) at v_1. It can land at v_1 or v_2 at the $(n+1)$st, $(2n+2)$nd, and $(3n+2)$nd steps:

$$w = \underbrace{v_1 - \cdots - v}_{\text{subwalk of length } n+1}\ \underbrace{v - \cdots - v}_{\text{subwalk of length } n+1}\ \underbrace{v - \cdots - v}_{\text{subwalk of length } n}\ \underbrace{v - \cdots - v_1}_{\text{subwalk of length } n},$$

where $v = v_1$ or v_2.

Using Tables 1 and 4, it follows that the sum S_1^* of the weights of all such walks w is given by

$$\begin{aligned} S_1^* &= f_{n+2}^2 f_{n+1}^2 + f_{n+2}^2 f_n^2 + f_{n+2}f_{n+1}^2 f_n + f_{n+2}f_{n+1}f_n f_{n-1} + f_{n+1}^4 + 2f_{n+1}^2 f_n^2 + f_{n+1}f_n^2 f_{n-1} \\ &= \left(f_{n+2}^2 + f_{n+1}^2\right)\left(f_{n+1}^2 + f_n^2\right) + f_{n+1}f_n(f_{n+2}+f_n)(f_{n+1}+f_{n-1}) \\ &= f_{2n+3}f_{2n+1} + f_{2n+2}f_{2n} \\ &= f_{4n+3}. \end{aligned}$$

w lands at v_1 at the $(n+1)$st step?	w lands at v_1 at the $(2n+2)$nd step?	w lands at v_1 at the $(3n+2)$nd step?	w lands at v_1 at the $(4n+2)$nd step?	sum of the weights of walks w
yes	yes	yes	yes	$f_{n+2}^2 f_{n+1}^2$
yes	yes	no	yes	$f_{n+2}^2 f_n^2$
yes	no	yes	yes	$f_{n+2}f_{n+1}^2 f_n$
yes	no	no	yes	$f_{n+2}f_{n+1}f_n f_{n-1}$
no	yes	yes	yes	f_{n+1}^4
no	yes	no	yes	$f_{n+1}^2 f_n^2$
no	no	yes	yes	$f_{n+1}^2 f_n^2$
no	no	no	yes	$f_{n+1}f_n^2 f_{n-1}^2$

Table 4: Sums of the Weights of Closed Walks Originating at v_1 of Length $4n+2$

Case 2. Suppose w originates at v_2. It also can land at v_1 or v_2 at the $(n+1)$st, $(2n+2)$nd, and $(3n+2)$nd steps:

$$w = \underbrace{v_2 - \cdots - v}_{\text{subwalk of length } n+1}\ \underbrace{v - \cdots - v}_{\text{subwalk of length } n+1}\ \underbrace{v - \cdots - v}_{\text{subwalk of length } n}\ \underbrace{v - \cdots - v_2}_{\text{subwalk of length } n},$$

where $v = v_1$ or v_2.

Using Tables 1 and 5, it follows that the sum S_2^* of the weights of all such walks w is given by

$$\begin{aligned} S_2^* &= f_{n+2}f_{n+1}^2f_n + f_{n+2}f_{n+1}f_nf_{n-1} + 2f_{n+1}^2f_n^2 + f_{n+1}^2f_{n-1}^2 + f_{n+1}f_n^2f_{n-1} + f_n^4 + f_n^2f_{n-1}^2 \\ &= f_{n+1}f_n(f_{n+2}+f_n)(f_{n+1}+f_{n-1}) + \left(f_{n+1}^2+f_n^2\right)\left(f_n^2+f_{n-1}^2\right) \\ &= f_{2n+2}f_{2n} + f_{2n+1}f_{2n-1} \\ &= f_{4n+1}. \end{aligned}$$

w lands at v_1 at the $(n+1)$st step?	w lands at v_1 at the $(2n+2)$nd step?	w lands at v_1 at the $(3n+2)$nd step?	w lands at v_2 at the $(4n+2)$nd step?	sum of the weights of walks w
yes	yes	yes	yes	$f_{n+2}f_{n+1}^2f_n$
yes	yes	no	yes	$f_{n+2}f_{n+1}f_nf_{n-1}$
yes	no	yes	yes	$f_{n+1}^2f_n^2$
yes	no	no	yes	$f_{n+1}^2f_{n-1}^2$
no	yes	yes	yes	$f_{n+1}^2f_n^2$
no	yes	no	yes	$f_{n+1}f_n^2f_{n-1}$
no	no	yes	yes	f_n^4
no	no	no	yes	$f_n^2f_{n-1}^2$

Table 5: Sums of the Weights of Closed Walks Originating at v_2 of Length $4n+2$

Using the identities (2.6) and (2.14)

$$\begin{aligned} x^4f_{4n+1} &= f_{n+2}^4 - 4f_{n+2}^3f_n + 2(2x^2+3)f_{n+2}^2f_n^2 - (x^4+6x^2+4)f_{n+2}f_n^3 - 2x^2f_{n+2}f_n^2f_{n-2} \\ &\quad + (x^2+1)^2f_n^4 + (x^4+2x^2)f_n^3f_{n-2}; \\ x^4f_{4n+3} &= (x^2+1)f_{n+2}^4 - 4f_{n+2}^3f_n + (x^2+6)f_{n+2}^2f_n^2 - (x^4+6x^2+4)f_{n+2}f_n^3 \\ &\quad + (x^4+3x^2+1)f_n^4 + (x^4+2x^2)f_n^3f_{n-2} - x^2f_n^2f_{n-2}^2, \end{aligned}$$

in [2], we get

$$\begin{aligned} x^4S^* &= x^4S_1^* + x^4S_2^* \\ &= (x^2+2)f_{n+2}^4 - 8f_{n+2}^3f_n + (5x^2+12)f_{n+2}^2f_n^2 - 2(x^4+6x^2+4)f_{n+2}f_n^3 \\ &\quad - 2x^2f_{n+2}f_n^2f_{n-2} + (2x^4+5x^2+2)f_n^4 + 2(x^4+2x^2)f_n^3f_{n-2} - x^2f_n^2f_{n-2}^2. \end{aligned}$$

Equating the two values of x^4S^* yields identity (3), as desired. □

Finally, we turn to the graph-theoretic confirmation of identity (4).

3.4. **Confirmation of Identity** (4).

Proof. Let S denote the sum of the weights of all closed walks of length $4n+3$ in the digraph. Then $S = l_{4n+3}$; so $x^3S = x^3l_{4n+3}$.

We will now compute x^3S in a different way. To this end, let w be an arbitrary walk of length $4n+3$.

Case 1. Suppose w originates (and ends) at v_1. It can land at v_1 or v_2 at the $(n+1)$st, $(2n+2)$nd, and $(3n+3)$rd steps:

$$w = \underbrace{v_1 - \cdots - v}_{\text{subwalk of length } n+1}\ \underbrace{v - \cdots - v}_{\text{subwalk of length } n+1}\ \underbrace{v - \cdots - v}_{\text{subwalk of length } n+1}\ \underbrace{v - \cdots - v_1}_{\text{subwalk of length } n},$$

where $v = v_1$ or v_2.

It follows by Tables 1 and 6 that the sum S_1 of the weights of all such walks w is given by

$$\begin{aligned} S_1 &= f_{n+2}^3 f_{n+1} + f_{n+2}^2 f_{n+1} f_n + 2 f_{n+2} f_{n+1}^3 + f_{n+2} f_{n+1} f_n^2 + 2 f_{n+1}^3 f_n + f_{n+1} f_n^3 \\ &= f_{n+1}\left(f_{n+2}^2 + 2f_{n+1}^2 + f_n^2\right)(f_{n+2} + f_n) \\ &= f_{2n+2}\left(f_{n+2}^2 + 2f_{n+1}^2 + f_n^2\right) \\ &= f_{2n+2}(f_{2n+3} + f_{2n+1}) \\ &= f_{4n+4}. \end{aligned}$$

w lands at v_1 at the $(n+1)$st step?	w lands at v_1 at the $(2n+2)$nd step?	w lands at v_1 at the $(3n+3)$rd step?	w lands at v_1 at the $(4n+3)$rd step?	sum of the weights of walks w
yes	yes	yes	yes	$f_{n+2}^3 f_{n+1}$
yes	yes	no	yes	$f_{n+2}^2 f_{n+1} f_n$
yes	no	yes	yes	$f_{n+2} f_{n+1}^3$
yes	no	no	yes	$f_{n+2} f_{n+1} f_n^2$
no	yes	yes	yes	$f_{n+2} f_{n+1}^3$
no	yes	no	yes	$f_{n+1}^3 f_n$
no	no	yes	yes	$f_{n+1}^3 f_n$
no	no	no	yes	$f_{n+1} f_n^3$

Table 6: Sums of the Weights of Closed Walks Originating at v_1 of Length $4n + 3$.

Case 2. Suppose w originates at v_2. It also can land at v_1 or v_2 at the $(n+1)$st, $(2n+2)$nd, and $(3n+3)$rd steps:

$$w = \underbrace{v_2 - \cdots - v}_{\text{subwalk of length } n+1} \quad \underbrace{v - \cdots - v}_{\text{subwalk of length } n+1} \quad \underbrace{v - \cdots - v}_{\text{subwalk of length } n+1} \quad \underbrace{v - \cdots - v_2}_{\text{subwalk of length } n},$$

where $v = v_1$ or v_2.

It follows by Tables 1 and 7 that the sum S_2 of the weights of all such walks w is given by

$$\begin{aligned} S_2 &= f_{n+2}^2 f_{n+1} f_n + f_{n+2} f_{n+1}^2 f_{n-1} + f_{n+2} f_{n+1} f_n^2 + f_{n+1}^3 f_n + 2 f_{n+1}^2 f_n f_{n-1} + f_{n+1} f_n^3 + f_n^3 f_{n-1} \\ &= f_{n+1}(f_{n+2} f_n + f_{n+1} f_{n-1})(f_{n+2} + f_n) + \left(f_{n+1}^2 + f_n^2\right) f_n (f_{n+1} + f_{n-1}) \\ &= f_{2n+1}(f_{2n+2} + f_{2n}) \\ &= f_{4n+2}. \end{aligned}$$

w lands at v_1 at the $(n+1)$st step?	w lands at v_1 at the $(2n+2)$nd step?	w lands at v_1 at the $(3n+3)$rd step?	w lands at v_2 at the $(4n+3)$rd step?	sum of the weights of walks w
yes	yes	yes	yes	$f_{n+2}^2 f_{n+1} f_n$
yes	yes	no	yes	$f_{n+2} f_{n+1}^2 f_{n-1}$
yes	no	yes	yes	$f_{n+1}^3 f_n$
yes	no	no	yes	$f_{n+1}^2 f_n f_{n-1}$
no	yes	yes	yes	$f_{n+2} f_{n+1} f_n^2$
no	yes	no	yes	$f_{n+1}^2 f_n f_{n-1}$
no	no	yes	yes	$f_{n+1} f_n^3$
no	no	no	yes	$f_n^3 f_{n-1}$

Table 7: Sums of the Weights of Closed Walks Originating at v_2 of Length $4n + 3$.

Using the result (2.2)

$$x^3 f_{4n+2} = f_{n+2}^4 - 3 f_{n+2}^2 f_n^2 + 2 f_{n+2} f_n^2 f_{n-2} + f_n^4 - f_n^2 f_{n-2}^2,$$

in [2] and identity (5), we then have

$$\begin{aligned} x^3S &= x^3S_1 + x^3S_2 \\ &= x^3 f_{4n+4} + x^3 f_{4n+2} \\ &= (x^2+3)f_{n+2}^4 - 4f_{n+2}^3 f_n + x^2 f_{n+2}^2 f_n^2 - (x^4+6x^2+4)f_{n+2}f_n^3 \\ &\quad + 4f_{n+2}f_n^2 f_{n-2} + (x^4+3x^2+3)f_n^4 + (x^4+2x^2)f_n^3 f_{n-2} - (x^2+2)f_n^2 f_{n-2}^2. \end{aligned}$$

This value of x^3S, coupled with its earlier version, yields the desired result, as expected. □

4. Conclusion

Because $g_n(1) = F_n$ or L_n, the graph-theoretic confirmations of the numeric versions of the gibonacci identities (1) through (4) follow from the above arguments; and so do their Pell counterparts because $b_n(x) = g_n(2x)$.

5. Acknowledgment

The author thanks the reviewer for a careful reading of the article and for encouraging words.

References

[1] M. Bicknell, *A primer for the Fibonacci numbers: Part VII*, The Fibonacci Quarterly, **8.4** (1970), 407–420.

[2] T. Koshy, *A family of sums of gibonacci polynomial products of order 4*, The Fibonacci Quarterly, **59.2** (2021), 98–107.

[3] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Volume II, Wiley, Hoboken, New Jersey, 2019.

[4] T. Koshy, *Polynomial extensions of the Lucas and Ginsburg identities revisited*, The Fibonacci Quarterly, **55.2** (2017), 147–151.

[5] T. Koshy, *A recurrence for gibonacci cubes with graph-theoretic confirmations*, The Fibonacci Quarterly, **57.2** (2019), 139–147.

MSC2020: Primary 05C20, 11B37, 11B39, 11B38, 11C08.

Department of Mathematics, Framingham State University, Framingham, MA 01701, USA
Email address: tkoshy@emeriti.framingham.edu