EXTENDED RESULTS ON INTEGER VALUES OF GENERATING FUNCTIONS FOR SEQUENCES GIVEN BY PELL'S EQUATIONS

YUJI TSUNO

Abstract. Hong posed the question when rational numbers map to integers for the generating function of Fibonacci numbers. This problem was solved by Pongsriiam and independently by Bulawa and Lee. The key to solving this problem is to consider the Fibonacci sequence and the Lucas sequence as sequences obtained from the integer solutions of Pell's equation $5x^2 - y^2 = \pm 4$. In this study, based on previous research, we change Hong's question and consider the case of the generating functions for the sequences obtained from the integer solutions of Pell's equation $x^2 - dy^2 = \pm 1$ (d is a nonsquare natural number). Similar to previous results, our main results are expressed in the form of ratios of adjacent terms of the sequences obtained from the integer solutions of Pell's equation $x^2 - dy^2 = \pm 1$. Furthermore, the results of Bulawa and Lee pertained to a class of sequences with recurrence relations that were more generalized than those obeyed by the Fibonacci and Lucas sequences. These sequences in our study arise as solutions to the equation $x^2 - dy^2 = \pm 1$, and, as such, obey the type of recurrence relations considered by Bulawa and Lee; however, the initial conditions of these sequences were not considered by those authors. Therefore, our study extends previous research.

1. Previous Results and Main Results

The Fibonacci sequence $\{F_n\}_{n\in\mathbb{N}}$ is defined by $F_0 = 0$, $F_1 = 1$, and

$$F_{n+2} = F_{n+1} + F_n.$$

Its generating function is

$$F(t) = \sum_{n=0}^{\infty} F_n t^n = \frac{t}{1-t-t^2}.$$

The Lucas sequence $\{L_n\}_{n\in\mathbb{N}}$ is defined by $L_0 = 2$, $L_1 = 1$, and

$$L_{n+2} = L_{n+1} + L_n.$$

Its generating function is given by

$$L(t) = \sum_{n=0}^{\infty} L_n t^n = \frac{2-t}{1-t-t^2}.$$

For the generating functions $F(t)$ and $L(t)$, Hong [2] proved that

$$\text{if } t \in \left\{\frac{F_n}{F_{n+1}}\right\}_{n\in\mathbb{N}}, \text{ then } F(t) \in \mathbb{Z};$$
$$\text{if } t \in \left\{\frac{F_n}{F_{n+1}}\right\}_{n\in\mathbb{N}} \text{ or } t \in \left\{\frac{L_n}{L_{n+1}}\right\}_{n\in\mathbb{N}}, \text{ then } L(t) \in \mathbb{Z}.$$

He questioned whether values of generating function for the Fibonacci sequence (respectively the Lucas sequence) would be integers only in these cases. His question is important; without this question, our results would not exist. To answer this question, Pongsriiam [4] provided a necessary and sufficient condition when rational numbers map to integers by the generating function of the Fibonacci numbers (respectively the Lucas numbers). In addition, Bulawa and Lee [1] independently provided a necessary and sufficient condition when rational numbers in the interval of convergence map to integers by the generating function of generalized Fibonacci numbers (respectively the Lucas numbers). Their results are the basis for our research.

Theorem 1.1 (Hong [2], Pongsriiam [4], Bulawa and Lee [1]). *Let t be a rational number. For the generating function $L(t)$, we have $L(t) \in \mathbb{Z}$ if and only if*

$$t \in \left\{ \frac{F_n}{F_{n+1}}, -\frac{L_{n+1}}{L_n}, \frac{L_n}{L_{n+1}} \right\}_{n \in \mathbb{N}} \text{ or } t \in \left\{ -\frac{F_{n+1}}{F_n} \right\}_{n \in \mathbb{N}^+}.$$

Theorem 1.2 (Hong [2], Pongsriiam [4], Bulawa and Lee [1]). *Let t be a rational number. For the generating function $F(t)$, we have $F(t) \in \mathbb{Z}$ if and only if*

$$t \in \left\{ \frac{F_n}{F_{n+1}} \right\}_{n \in \mathbb{N}} \text{ or } t \in \left\{ -\frac{F_{n+1}}{F_n} \right\}_{n \in \mathbb{N}^+}.$$

In the proof of the above theorems, the following well-known identities for Fibonacci and Lucas numbers are employed.

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \ (n \geq 1), \ [3, \text{p. } 86, \text{ Theorem } 5.3]$$
$$L_{n-1}L_{n+1} - L_n^2 = 5(-1)^{n-1} \ (n \geq 1), \ [3, \text{p. } 117, 36]$$
$$L_nF_m = F_{n+m} - (-1)^m F_{n-m} \ (n \geq m), \ [3, \text{p. } 118, 58, 59]$$
$$5F_nF_m = L_{n+m} - (-1)^m L_{n-m} \ (n \geq m), \ [3, \text{p. } 111, 83, 84]$$
$$F_nL_m - L_nF_m = 2(-1)^m F_{n-m} \ (n \geq m), \ [3, \text{p. } 427, 13]$$
$$L_n = F_{n-1} + F_{n+1} \ (n \geq 1), \ [3, \text{p. } 93, \text{ Corollary } 5.5]$$
$$5F_n = L_{n-1} + L_{n+1} \ (n \geq 1), \ [3, \text{p. } 86, \text{ Theorem } 5.3]$$
$$F_{2n} = F_nL_n, \ [3, \text{p. } 86, \text{ Theorem } 5.3]$$

Further, the following theorem for Fibonacci and Lucas numbers is important in the proof of the above theorems.

Theorem 1.3 ([3, Theorem 5.8]; [5]). *Let (x, y) be a pair of nonnegative integers. If (x, y) satisfies Pell's equation*

$$5x^2 - y^2 = \pm 4,$$

there exists a nonnegative integer n such that $x = F_n$ and $y = L_n$. Conversely,

$$5F_n^2 - L_n^2 = 4(-1)^{n+1}$$

for any nonnegative integer n.

Pell's equation

$$5x^2 - y^2 = \pm 4$$

was important in the previous studies discussed. In other words, the results of those studies are related to the integer values of the generating functions of the sequences obtained from the integer solutions of Pell's equation $5x^2 - y^2 = \pm 4$. If this Pell's equation is changed to another type of Pell's equation, how will the results change? In this paper, we consider the Pell's equation

$$x^2 - dy^2 = \pm 1 \ (d \text{ is a nonsquare natural number}).$$

First, let (a, b) be the minimal solution of Pell's equation $x^2 - dy^2 = \pm 1$ (i.e., (a, b) is the positive integer solution of $x^2 - dy^2 = 1$ or $x^2 - dy^2 = -1$ such that (a, b) minimizes the quantity $a + b\sqrt{d}$).

Let

$$X_n = \frac{(a + b\sqrt{d})^n + (a - b\sqrt{d})^n}{2} \tag{1}$$

$$Y_n = \frac{(a + b\sqrt{d})^n - (a - b\sqrt{d})^n}{2\sqrt{d}} \tag{2}$$

for $n \geq 0$.

Then (X_n, Y_n) $(n \geq 0)$ are solutions of Pell's equation $x^2 - dy^2 = \pm 1$. Moreover, all nonnegative integer solutions are given by (X_n, Y_n) $(n \geq 0)$. See, for example, [6, p. 214, Theorem 3.8]. Let

$$\delta = a^2 - db^2.$$

Thus, the generating function of the sequence $\{X_n\}_{n \in \mathbb{N}}$ is given by

$$X(t) = \frac{1 - at}{1 - 2at + \delta t^2},$$

and the generating function of the sequence $\{Y_n\}_{n \in \mathbb{N}}$ is given by

$$Y(t) = \frac{bt}{1 - 2at + \delta t^2}.$$

These functions are obtained from equations (1) and (2) and

$$Y_{n+2} = 2aY_{n+1} - \delta Y_n \tag{3}$$

$$X_{n+2} = 2aX_{n+1} - \delta X_n \tag{4}$$

Furthermore, the convergence radii of these generating functions is

$$\frac{1}{a + b\sqrt{d}}.$$

Here, we describe the main results of this study.

Theorem 1.4. *Let t be a rational number. Then, we have $X(t) \in \mathbb{Z}$ if and only if*

$$t \in \left\{ \frac{Y_n}{Y_{n+1}}, \frac{X_n}{X_{n+1}}, \delta \frac{X_{n+1}}{X_n} \right\}_{n \in \mathbb{N}} \quad \text{or } t \in \left\{ \delta \frac{Y_{n+1}}{Y_n} \right\}_{n \in \mathbb{N}^+}.$$

Theorem 1.5. *Let t be a rational number. Then, we have $Y(x) \in \mathbb{Z}$ if and only if*

$$t \in \left\{ \frac{Y_n}{Y_{n+1}} \right\}_{n \in \mathbb{N}} \quad \text{or } x \in \left\{ \delta \frac{Y_{n+1}}{Y_n} \right\}_{n \in \mathbb{N}^+}.$$

It is interesting that our main results have the same form as the theorems given by Hong; Pongsriiam; and Bulawa and Lee. This poses the question, "Will all solutions to other types of Pell's equations have the same form?"

We have the following corollaries from the main results.

Corollary 1.6. *Let t be a rational number. We assume that t is in the interval of convergence of the generating function $X(t)$. If $\delta = 1$, then $X(t) \in \mathbb{Z}$ if and only if*

$$t \in \left\{ \frac{Y_n}{Y_{n+1}} \right\}_{n \in \mathbb{N}}.$$

If $\delta = -1$, then $X(t) \in \mathbb{Z}$ if and only if

$$t \in \left\{ \frac{Y_{2n}}{Y_{2n+1}}, \frac{X_{2n+1}}{X_{2n+2}} \right\}_{n \in \mathbb{N}}.$$

Corollary 1.7. *Let t be a rational number. We assume that t is in the interval of convergence of the generating function $Y(t)$. If $\delta = 1$, then we have $Y(t) \in \mathbb{Z}$ if and only if*

$$t \in \left\{ \frac{Y_n}{Y_{n+1}} \right\}_{n \in \mathbb{N}}.$$

If $\delta = -1$, then we have $Y(t) \in \mathbb{Z}$ if and only if

$$t \in \left\{ \frac{Y_{2n}}{Y_{2n+1}} \right\}_{n \in \mathbb{N}}.$$

These are deduced from (1) and (2) and because $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ are monotonically increasing sequences.

Remark 1.8. *Let s_1 and s_2 be nonzero integers. Also, assume that s_1 is divisible by s_2. Consider the sequence $\{R_n\}_{n \in \mathbb{N}}$ defined by*

$$R_{n+2} = s_1 R_{n+1} + s_2 R_n$$

with some initial values R_0 and R_1. For $R_0 = 0$ and $R_1 = 1$, Bulawa and Lee [1] provided a necessary and sufficient condition when rational values in the interval of convergence of the generating function for the sequence $\{R_n\}_{n \in \mathbb{N}}$ are integers. Let (a, b) be the minimal solution of Pell's equation $x^2 - dy^2 = \pm 1$. Although our research considers a special case ($s_1 = 2a$ and $s_2 = \pm 1$), our approach is more general with respect to initial values (R_0 and R_1).

2. Proofs of the Main Results

Before we prove the main results, consider the following identities.

$$Y_{n-1}Y_{n+1} - (Y_n)^2 = -\delta^{n-1}b^2 \quad (n \geq 1) \tag{5}$$

$$2X_{n-1}X_{n+1} = X_{2n} + \delta^{n-1}X_2 \quad (n \geq 1) \tag{6}$$

$$Y_n X_m - X_n Y_m = \delta^m Y_{n-m} \quad (n \geq m) \tag{7}$$

$$X_n Y_m = \frac{Y_{n+m} - \delta^m Y_{n-m}}{2} \quad (n \geq m) \tag{8}$$

$$Y_n Y_m = \frac{X_{n+m} - \delta^m X_{n-m}}{2d} \quad (n \geq m) \tag{9}$$

$$X_{n+1} = aX_n + dbY_n \quad (n \geq 0) \tag{10}$$

$$Y_{n+1} = aY_n + bX_n \quad (n \geq 0) \tag{11}$$

$$2(X_n)^2 = X_{2n} + \delta^n \quad (n \geq 0) \tag{12}$$

$$2X_n X_{n+1} = X_{2n+1} + \delta^n a \quad (n \geq 0) \tag{13}$$

These identities are obtained from (1) and (2).

2.1. **Proof of Theorem 1.4.** First, we show that

$$X\left(\frac{Y_n}{Y_{n+1}}\right), X\left(\frac{X_n}{X_{n+1}}\right), X\left(\delta\frac{X_{n+1}}{X_n}\right)\ (n \geq 0),$$

and

$$X\left(\delta\frac{Y_{n+1}}{Y_n}\right)\ (n \geq 1).$$

are integers.

If $n = 0$, the result is clear. If $n \geq 1$, using (3) and (5), we obtain

$$\begin{aligned} X\left(\frac{Y_n}{Y_{n+1}}\right) &= \frac{Y_{n+1}(Y_{n+1} - aY_n)}{Y_{n+1}(Y_{n+1} - 2aY_n) + \delta(Y_n)^2} \\ &\overset{(3)}{=} \frac{Y_{n+1}(Y_{n+1} - aY_n)}{-\delta Y_{n+1}Y_{n-1} + \delta(Y_n)^2} \\ &\overset{(5)}{=} \frac{Y_{n+1}(Y_{n+1} - aY_n)}{\delta^n b^2}. \end{aligned}$$

Moreover, Y_n $(n \geq 0)$ is divisible by b because we have $Y_0 = 0$, $Y_1 = b$, and $Y_{n+2} = 2aY_{n+1} - \delta Y_n$. Therefore, $X(\frac{Y_n}{Y_{n+1}}) \in \mathbb{Z}$. In the same manner, we have

$$\begin{aligned} X\left(\frac{X_n}{X_{n+1}}\right) &= \frac{X_{n+1}(X_{n+1} - aX_n)}{X_{n+1}(X_{n+1} - 2aX_n) + \delta X_n^2} \overset{(4)}{=} \frac{X_{n+1}(X_{n+1} - aX_n)}{-\delta X_{n+1}X_{n-1} + \delta X_n^2} \\ &\overset{(6)(12)}{=} \frac{X_{n+1}(X_{n+1} - aX_n)}{-d\delta^n b^2} \overset{(10)}{=} \frac{-X_{n+1}Y_n}{\delta^n b} \end{aligned}$$

by using (4), (6), (10), (12), and $a^2 - db^2 = \delta$. Therefore, $X(\frac{X_n}{X_{n+1}}) \in \mathbb{Z}$.

Similarly, we have

$$X\left(\delta\frac{Y_{n+1}}{Y_n}\right) = \frac{Y_n(Y_n - \delta aY_{n+1})}{\delta Y_{n+1}(Y_{n+1} - 2aY_n) + Y_n^2} \overset{(3)}{=} \frac{Y_n(Y_n - \delta aY_{n+1})}{(Y_n)^2 - Y_{n+1}Y_{n-1}} \overset{(5)}{=} \frac{Y_n(Y_n - \delta aY_{n+1})}{\delta^{n-1}b^2}$$

by using (3) and (5). Therefore, $X(\delta\frac{Y_{n+1}}{Y_n}) \in \mathbb{Z}$.

In the same manner, we have

$$\begin{aligned} X\left(\delta\frac{X_{n+1}}{X_n}\right) &= \frac{X_n(X_n - a\delta X_{n+1})}{\delta X_{n+1}(X_{n+1} - 2aX_n) + (X_n)^2} \overset{(4)}{=} \frac{X_n(X_n - a\delta X_{n+1})}{-X_{n+1}X_{n-1} + (X_n)^2} \\ &\overset{(6)(12)}{=} \frac{X_n(X_n - a\delta X_{n+1})}{-\delta^{n-1}db^2} \overset{(10)}{=} \frac{X_n(bX_n + aY_n)}{\delta^n b} \end{aligned}$$

by (4), (6), (10), (12), and $a^2 - db^2 = \delta$. Hence, $X(\delta\frac{X_{n+1}}{X_n}) \in \mathbb{Z}$.

Next, if $X(t) = k$ (k is an integer) for some rational number t, we show that

$$t \in \left\{\frac{Y_n}{Y_{n+1}}, \frac{X_n}{X_{n+1}}, \delta\frac{X_{n+1}}{X_n}\right\}_{n\in\mathbb{N}} \text{ or } t \in \left\{\delta\frac{Y_{n+1}}{Y_n}\right\}_{n\in\mathbb{N}^+}.$$

If $k = 0$, then

$$\frac{1 - at}{1 - 2at + \delta t^2} = 0.$$

Hence,

$$t = \frac{1}{a} = \frac{X_0}{X_1}.$$

If $k \neq 0$, then

$$\frac{1-at}{1-2at+\delta t^2} = k.$$

Therefore,

$$\delta k t^2 + a(1-2k)t + k - 1 = 0.$$

Hence,

$$t = \frac{-a(1-2k) \pm \sqrt{a^2(1-2k)^2 - 4\delta k(k-1)}}{2\delta k}.$$

Here, there exists a nonnegative integer M such that

$$a^2(1-2k)^2 + 4\delta k(k-1) = M^2,$$

because t is a rational number. Moreover,

$$M^2 - db^2(2k-1)^2 = \delta,$$

because $a^2 - db^2 = \delta$. Using (8), we have $Y_{2N} = 2X_N Y_N$ for any nonnegative integer N. Because Y_N is divisible by b,

$$M^2 - d(Y_{2N})^2 \neq \delta.$$

Therefore, there exists a nonnegative integer n such that $M = X_{2n+1}$. Moreover, we obtain $b(2k-1) = Y_{2n+1}$ $(n \geq 0)$ or $b(2k-1) = -Y_{2n+1}$ $(n \geq 1)$. Hence,

$$k = \frac{Y_{2n+1}+b}{2b} (n \geq 0)$$

or

$$k = \frac{-Y_{2n+1}+b}{2b} (n \geq 1).$$

From the above, we have

$$t = \frac{aY_{2n+1} + bX_{2n+1}}{\delta(Y_{2n+1}+b)} \ (n \geq 0), \tag{A}$$

$$t = \frac{aY_{2n+1} - bX_{2n+1}}{\delta(Y_{2n+1}+b)} \ (n \geq 0), \tag{B}$$

$$t = \frac{-aY_{2n+1} + bX_{2n+1}}{\delta(-Y_{2n+1}+b)} \ (n \geq 1), \tag{C}$$

$$\text{or} \quad t = \frac{-aY_{2n+1} - bX_{2n+1}}{\delta(-Y_{2n+1}+b)} \ (n \geq 1) \tag{D}$$

For (A) to (D), using equations (5) through (13), we obtain

$$t \in \left\{ \frac{Y_n}{Y_{n+1}}, \frac{X_n}{X_{n+1}}, \delta\frac{X_{n+1}}{X_n} \right\}_{n\in\mathbb{N}} \text{ or } t \in \left\{ \delta\frac{Y_{n+1}}{Y_n} \right\}_{n\in\mathbb{N}^+}.$$

To prove this claim, we first assume that n is even or $\delta = 1$.

For (A), we have

$$t = \frac{aY_{2n+1} + bX_{2n+1}}{\delta(Y_{2n+1}+b)} \overset{(8)}{=} \frac{2aX_{n+1}Y_n + ab + bX_{2n+1}}{2\delta(X_{n+1}Y_n + b)} \overset{(7)(13)}{=} \frac{2aX_{n+1}Y_n + 2bX_nX_{n+1}}{2\delta Y_{n+1}X_n} \overset{(11)}{=} \delta\frac{X_{n+1}}{X_n}.$$

For (B), we have

$$t = \frac{aY_{2n+1} - bX_{2n+1}}{\delta(Y_{2n+1}+b)} \overset{(8)(13)}{=} \frac{2aX_{n+1}Y_n + 2ab - 2bX_nX_{n+1}}{2\delta(X_{n+1}Y_n + b)} \overset{(7)}{=} \frac{aY_{n+1} - bX_{n+1}}{\delta Y_{n+1}} \overset{(10)(11)}{=} \frac{Y_n}{Y_{n+1}}.$$

For (C), we have

$$t = \frac{-aY_{2n+1} + bX_{2n+1}}{\delta(-Y_{2n+1} + b)} \overset{(8)(13)}{=} \frac{-2aX_{n+1}Y_n - 2ab + 2bX_nX_{n+1}}{-2\delta X_{n+1}Y_n}$$
$$\overset{(7)}{=} \frac{-2aX_nY_{n+1} + 2bX_nX_{n+1}}{-2\delta Y_nX_{n+1}} \overset{(10)(11)}{=} \frac{X_n}{X_{n+1}}.$$

For (D), we have

$$t = \frac{-aY_{2n+1} - bX_{2n+1}}{\delta(-Y_{2n+1} + b)} \overset{(8)(13)}{=} \frac{-2aX_{n+1}Y_n - 2bX_nX_{n+1}}{-2\delta X_{n+1}Y_n} \overset{(11)}{=} \delta\frac{Y_{n+1}}{Y_n}.$$

Next, we consider that n is odd and $\delta = -1$.

For (A), we have

$$t = \frac{aY_{2n+1} + bX_{2n+1}}{-Y_{2n+1} - b} \overset{(8)}{=} \frac{2aX_{n+1}Y_n + (-1)^nab + bX_{2n+1}}{-2X_{n+1}Y_n - (-1)^nb - b} \overset{(13)}{=} -\frac{aY_n + bX_n}{Y_n} \overset{(11)}{=} -\frac{Y_{n+1}}{Y_n}.$$

For (B), we have

$$t = \frac{aY_{2n+1} - bX_{2n+1}}{-Y_{2n+1} - b} \overset{(8)(13)}{=} \frac{2aX_{n+1}Y_n - 2bX_nX_{n+1} - 2ab}{-2X_{n+1}Y_n}$$
$$\overset{(7)}{=} \frac{X_n(aY_{n+1} - bX_{n+1})}{-X_{n+1}Y_n} \overset{(10)(11)}{=} \frac{X_n}{X_{n+1}}.$$

For (C), we have

$$t = \frac{-aY_{2n+1} + bX_{2n+1}}{Y_{2n+1} - b} \overset{(8)(13)}{=} \frac{-2aX_{n+1}Y_n + 2bX_nX_{n+1} - 2(-1)^nab}{2X_{n+1}Y_n - 2b}$$
$$\overset{(7)}{=} -\frac{aY_{n+1} - bX_{n+1}}{Y_{n+1}} \overset{(10)(11)}{=} \frac{Y_n}{Y_{n+1}}.$$

For (D), we have

$$t = \frac{-aY_{2n+1} - bX_{2n+1}}{Y_{2n+1} - b} \overset{(8)(13)}{=} \frac{-2aX_{n+1}Y_n - 2bX_nX_{n+1}}{2X_{n+1}Y_n - 2b} \overset{(7)}{=} -\frac{X_{n+1}(aY_n + bX_n)}{Y_{n+1}X_n} \overset{(11)}{=} -\frac{X_{n+1}}{X_n}.$$

This completes the proof.

2.2. **Proof of Theorem 1.5.** First, we show that

$$Y\left(\frac{Y_n}{Y_{n+1}}\right) \ (n \geq 0)$$

and

$$Y\left(\delta\frac{Y_{n+1}}{Y_n}\right) \ (n \geq 1)$$

are integers. If $n \geq 0$, the result is clear. If $n \geq 1$, using (3) and (5), we obtain

$$Y\left(\frac{Y_n}{Y_{n+1}}\right) = \frac{bY_nY_{n+1}}{Y_{n+1}(Y_{n+1} - 2aY_n) + \delta(Y_n)^2} \overset{(3)}{=} \frac{bY_nY_{n+1}}{-\delta Y_{n+1}Y_{n-1} + \delta(Y_n)^2} \overset{(5)}{=} \frac{Y_nY_{n+1}}{\delta^nb}.$$

In the same manner, using (3) and (5), we obtain

$$Y\left(\delta\frac{Y_{n+1}}{Y_n}\right) = \frac{bY_nY_{n+1}}{\delta Y_{n+1}(Y_{n+1} - 2aY_n) + (Y_n)^2} \overset{(3)}{=} \frac{bY_nY_{n+1}}{-Y_{n+1}Y_{n-1} + (Y_n)^2} \overset{(5)}{=} \frac{Y_nY_{n+1}}{\delta^{n-1}b}.$$

Because Y_n $(n \geq 0)$ is divisible by b, $Y(\frac{Y_n}{Y_{n+1}})$ and $Y(\delta\frac{Y_{n+1}}{Y_n})$ are integers.

Next, if $Y(t) = k$ (k is an integer) for some rational number t, we show that

$$t \in \left\{ \frac{Y_n}{Y_{n+1}} \right\}_{n \in \mathbb{N}} \text{ or } t \in \left\{ \delta \frac{Y_{n+1}}{Y_n} \right\}_{n \in \mathbb{N}^+}.$$

If $k = 0$, then

$$\frac{bt}{1 - 2at + \delta t^2} = 0.$$

Hence,

$$t = 0 = \frac{Y_0}{Y_1}.$$

If $k \neq 0$, then

$$\frac{bt}{1 - 2at + \delta t^2} = k.$$

Hence,

$$k\delta t^2 + (-2ak - b)t + k = 0.$$

Therefore,

$$t = \frac{2ak + b \pm \sqrt{(2ak + b)^2 - 4\delta k^2}}{2\delta k}.$$

Here, because t is a rational number, there exists a nonnegative integer M such that

$$(2ak + b)^2 - 4\delta k^2 = M^2.$$

Moreover, using $a^2 - db^2 = \delta$, we obtain

$$(2kbd + a)^2 - dM^2 = \delta.$$

Hence, there exists a nonnegative integer n such that

$$X_{2n+1} = \delta^n (2kbd + a), Y_{2n+1} = M (n \geq 0).$$

If $\delta = 1$, $a \pm 1$ is not divisible by db. If $a \pm 1$ is divisible by db, there exists a positive integer l such that

$$a = dbl \pm 1.$$

However,

$$(dbl \pm 1)^2 - db^2 > 1$$

This is a contradiction because $a^2 - db^2 = 1$. Moreover, using (12), for any nonnegative integer N, $X_{2N} - 1$ is divisible by db. Hence, $X_{2N} \pm a$ is not divisible by db. Therefore, there exists a nonnegative integer n such that

$$X_{2n+1} = \pm(2kbd + a).$$

Furthermore, if $X_{2n+1} = -(2kbd + a)$, using (10) and (13), $2a$ is divisible by db because $X_{2n+1} - a$ is divisible by db. Hence, 4 is divisible by db. But, this is also a contradiction because 4 is not divisible by db. Therefore,

$$X_{2n+1} = 2kbd + a.$$

If $\delta = -1$, for any nonnegative integer N, $(X_{2N})^2 - d(Y_{2N})^2 \neq -1$. Hence, there exists a nonnegative integer n such that

$$X_{2n+1} = \pm(2kbd + a).$$

Moreover, using (10) and (13), $X_{2n+1} - (-1)^n a$ is divisible by $2bd$. Therefore, we have

$$X_{2n+1} = (-1)^n (2kbd + a).$$

Hence, we obtain

$$t = \frac{a\delta^n X_{2n+1} - \delta + bdY_{2n+1}}{\delta^{n+1} X_{2n+1} - \delta a} (n \geq 0) \tag{E}$$

$$\text{or} \quad t = \frac{a\delta^n X_{2n+1} - \delta - bdY_{2n+1}}{\delta^{n+1} X_{2n+1} - \delta a} (n \geq 0). \tag{F}$$

To finish the proof, we first assume $\delta = 1$.

For (E), we have

$$t = \frac{aX_{2n+1} - 1 + bdY_{2n+1}}{X_{2n+1} - a} \overset{(10)}{=} \frac{X_{2n+2} - 1}{X_{2n+1} - a} \overset{(9)}{=} \frac{Y_{n+1}}{Y_n}.$$

For (F), we have

$$t = \frac{aX_{2n+1} - 1 - bdY_{2n+1}}{X_{2n+1} - a} \overset{(10)(11)}{=} \frac{X_{2n} - 1}{X_{2n+1} - a} \overset{(9)}{=} \frac{Y_n}{Y_{n+1}}.$$

Next, we assume $\delta = -1$.

If n is even, for (E),

$$t = \frac{aX_{2n+1} + bdY_{2n+1} + 1}{-X_{2n+1} + a} \overset{(10)}{=} \frac{X_{2n+2} + 1}{-X_{2n+1} + a} \overset{(9)}{=} -\frac{Y_{n+1}}{Y_n}.$$

If n is odd, for (E),

$$t = \frac{-aX_{2n+1} + bdY_{2n+1} + 1}{X_{2n+1} + a} \overset{(10)(11)}{=} \frac{X_{2n} + 1}{X_{2n+1} + a} \overset{(9)}{=} \frac{Y_n}{Y_{n+1}}.$$

If n is even, for (F),

$$t = \frac{aX_{2n+1} - bdY_{2n+1} + 1}{-X_{2n+1} + a} \overset{(10)(11)}{=} \frac{-X_{2n} + 1}{-X_{2n+1} + a} \overset{(9)}{=} \frac{Y_n}{Y_{n+1}}.$$

If n is odd, for (F),

$$t = \frac{-aX_{2n+1} - bdY_{2n+1} + 1}{X_{2n+1} + a} \overset{(10)}{=} \frac{-X_{2n+2} + 1}{X_{2n+1} + a} \overset{(9)}{=} -\frac{Y_{n+1}}{Y_n}.$$

This completes the proof.

3. Acknowledgements

I thank the referees for their useful remarks and Editage for English language editing.

References

[1] A. Bulawa and W. K. Lee, *Integral values of the generating functions for the Fibonacci and related sequences*, The Fibonacci Quarterly, **55.1** (2017), 74–81.
[2] D. S. Hong, *When is the generating function of the Fibonacci numbers an integer?*, College Mathematics Journal, **46** (2015), 110–112.
[3] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, New York, 2001.
[4] P. Pongsriiam, *Integral values of the generating functions of Fibonacci and Lucas numbers*, College Mathematics Journal, **48** (2017), 97–101.
[5] P. Schub, *A minor Fibonacci curiosity*, Scripta Mathematica, **16** (1950), 214.
[6] T. Takagi, *Shotou Seisuuron Kougi*, 2nd ed., Kyoritsu Shuppan, Tokyo, 1971.

MSC2020: 11B39

Iiyama high school, Nagano 389-2253, Japan
Email address: tsuno.yuji@gmail.com