ON GENERALIZED ZECKENDORF DECOMPOSITIONS AND GENERALIZED GOLDEN STRINGS

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Abstract. Zeckendorf proved that every positive integer has a unique representation as a sum of nonconsecutive Fibonacci numbers. A natural generalization of this theorem is to look at the sequence defined as follows: for $n \geq 2$, let $F_{n,1} = F_{n,2} = \cdots = F_{n,n} = 1$ and $F_{n,m+1} = F_{n,m} + F_{n,m+1-n}$ for all $m \geq n$. It is known that every positive integer has a unique representation as a sum of $F_{n,m}$'s, where the indices of summands are at least n apart. We call this the n-decomposition. Griffiths showed an interesting relationship between the Zeckendorf decomposition and the golden string. In this paper, we continue the work to show a relationship between the n-decomposition and the generalized golden string.

1. Introduction

We define the Fibonacci sequence to be $F_1 = 1$, $F_2 = 1$, and $F_m = F_{m-1} + F_{m-2}$ for $m \geq 3$. The Fibonacci numbers have fascinated mathematicians for centuries with many interesting properties. A theorem of Zeckendorf [14] states that every positive integer m can be uniquely written as a sum of nonconsecutive Fibonacci numbers. This gives the so-called Zeckendorf decomposition of m. A more formal statement of Zeckendorf's theorem is as follows.

Theorem 1.1. *For any $m \in \mathbb{N}$, there exists a unique increasing sequence of positive integers $(c_1, c_2, \ldots, c_k)$ such that $c_1 \geq 2$, $c_i \geq c_{i-1} + 2$ for $i = 2, 3, \ldots, k$, and $m = \sum_{i=1}^{k} F_{c_i}$.*

Much work has been done to understand the structure of Zeckendorf decompositions and their applications (see [1, 2, 3, 4, 7, 8, 9, 12]) and to generalize them (see [5, 6, 10, 11, 13]). Before stating our main results, we mention several related results from the literature. Given $n \in \mathbb{N}_{\geq 2}$, we define the sequence $F_{n,1} = \cdots = F_{n,n} = 1$ and $F_{n,m+1} = F_{n,m} + F_{n,m+1-n}$ for all $m \geq n$. The following theorem, which is a generalization of Theorem 1.1, follows immediately from the proof of [5, Theorem 1.3].

Theorem 1.2. *For any $m \in \mathbb{N}$, there exists a unique increasing sequence of positive integers $(c_1, c_2, \ldots, c_k)$ such that $c_1 \geq n$, $c_i \geq c_{i-1} + n$ for $i = 2, 3, \ldots, k$, and $m = \sum_{i=1}^{k} F_{n,c_i}$.*

For conciseness, we call the decomposition in Theorem 1.2 the n-decomposition. In [8], Griffiths made a connection between the golden string and the Zeckendorf decomposition. In particular, the golden string $S_\infty = a_2a_1a_2a_2a_1a_2a_1a_2a_2a_1a_2a_2a_1\ldots$ is defined to be the infinite string of a_1 and a_2 constructed recursively as follows. Let $S_1 = a_1$ and $S_2 = a_2$, and for $m \geq 3$, S_m is defined to be the concatenation of the strings S_{m-1} and S_{m-2}, which we denote by $S_{m-1} \circ S_{m-2}$. Thus,

$$\begin{aligned} S_3 &= S_2 \circ S_1 = a_2 \circ a_1 = a_2a_1, \\ S_4 &= S_3 \circ S_2 = a_2a_1 \circ a_2 = a_2a_1a_2, \end{aligned}$$

and so on. We generalize the golden string in an obvious way. Given $n \in \mathbb{N}_{\geq 2}$, we let

$$\begin{aligned} S_1 &= a_1, \ldots, S_n = a_n, \\ S_m &= S_{m-1} \circ S_{m-n} \text{ for } m \geq n+1. \end{aligned}$$

We call S_∞ obtained from the recursive process the n-string. For example, when $n = 2$, we have the golden string; when $n = 3$, we have the 3-string:

$$a_3 a_1 a_2 a_3 a_3 a_1 a_3 a_1 a_2 a_3 a_1 a_2 a_3 a_3 \ldots.$$

Lemmas 3.1 and 3.2 in [8] show that the Zeckendorf decomposition (2-decomposition) is linked to the golden string (2-string). This might lead us to suspect that in general, the n-decomposition is linked to the n-string. Our next theorems show that the suspicion is indeed well-founded.

Theorem 1.3. *Let $n \geq 3$ and $m \geq 1$. The following items hold.*

(1) *S_m contains $F_{n,m}$ letters, of which*

$$\begin{aligned} F_{n,m-(n-1)} &\text{ are } a_n\text{'s}, \\ F_{n,m-n} &\text{ are } a_1\text{'s}, \\ F_{n,m-(n+1)} &\text{ are } a_2\text{'s}, \\ &\vdots \\ F_{n,m-(2n-2)} &\text{ are } a_{n-1}\text{'s}. \end{aligned}$$

(2) *For any $m \in \mathbb{N}_{\geq 1}$, the concatenation*

$$S_{n+(n-1)m} \circ \cdots \circ S_{n+(n-1)} \circ S_n$$

gives the first $F_{n,n} + F_{n,2n-1} + \cdots + F_{n,(m+1)n-m}$ letters of S_∞.

Theorem 1.4. *Let $n \geq 3$ and $m \geq 1$. Let $F_{n,c_1} + F_{n,c_2} + \cdots + F_{n,c_k}$ be the n-decomposition of $m \in \mathbb{N}$. Then, $S_{c_k} \circ S_{c_{k-1}} \circ \cdots \circ S_{c_1}$ gives the first m letters of S_∞.*

Remark 1.5. So that Theorem 1.3 (1) makes sense, we need to extend the sequence $F_{n,m}$ to the left while following the recursive relation. It is straightforward that for $n \geq 3$, we have $F_{n,0} = \cdots = F_{n,2-n} = 0$, $F_{n,1-n} = 1$, and $F_{-n} = \cdots = F_{3-2n} = 0$.

For each $m \geq 1$, let $N_{a_i}(m)$ denote the number of a_i in the string S_∞ up to m.

Theorem 1.6. *Let $n \geq 3$ and $m \geq 1$. If $m = F_{n,c_1} + F_{n,c_2} + \cdots + F_{n,c_k}$ is an n-decomposition of m, then*

$$N_{a_i}(m) = \begin{cases} F_{n,c_1-(n+i-1)} + F_{n,c_2-(n+i-1)} + \cdots + F_{n,c_k-(n+i-1)}, & \text{if } 1 \leq i \leq n-1; \\ F_{n,c_1-(n-1)} + F_{n,c_2-(n-1)} + \cdots + F_{n,c_k-(n-1)}, & \text{if } i = n. \end{cases}$$

Our final result extends [7, Theorem 3.4], which describes the set of all positive integers having the summand F_k in their Zeckendorf decomposition. The following theorem sheds another light on the relationship between the n-string and the n-decomposition.

Theorem 1.7. *For $k \geq n \geq 3$, the set of all positive integers having the summand $F_{n,k}$ in their n-decomposition is given by*

$$Z_n(k) = \left\{ j + F_{n,k} + \sum_{i=1}^{n} F_{n,k+i} \cdot N_{a_i}(m) : 0 \leq j \leq F_{n,k-(n-1)} - 1 \text{ and } m \geq 0 \right\}.$$

Remark 1.8. In [7, Theorem 3.4], $Z_2(k)$ has a closed form thanks to [8, Theorem 3.3], which provides a formula for $N_{a_i}(m)$ in the case of the golden string. The formula was deduced using Binet's formula for the Fibonacci numbers. However, the author of the present paper is unable to find such a closed form for $Z_n(k)$, when $n \geq 3$. Thus, Theorem 1.7 only gives another (not quicker) way to find $Z_n(k)$ and shows a relationship between the n-string and the n-decomposition.

As we proceed to the proof of the main theorems, a number of immediate results are encountered.

2. Relationship Between the n-Decomposition and the n-String

The following lemma will be used in due course.

Lemma 2.1. *Let $n \geq 3$ and S_∞ be the n-string. The following items hold.*

1. *Fix i and j such that $n \leq i \leq j-(n-1)$. Then, $S_j \circ S_i$ gives the first $F_{n,i} + F_{n,j}$ letters of S_∞.*
2. *Fix $j \geq i \geq n$. There exists S^* (possibly empty), a concatenation of some S_i's, such that $S_i \circ S^* = S_j$.*
3. *For $m \geq 2$, there exists S^* (possibly empty), a concatenation of some S_i's, such that*
$$(S_{n+(n-1)m} \circ \cdots \circ S_{n+(n-1)} \circ S_n) \circ S^* \;=\; S_{n+(n-1)m+1} \circ S_i$$
for some $n \leq i \leq n+(n-1)(m-1)+1$.
4. *For $m \geq 2$, there exists S^* (possibly empty), a concatenation of some S_i's, such that*
$$(S_{n+(n-1)m} \circ \cdots \circ S_{n+(n-1)} \circ S_n) \circ S^* \;=\; S_{n+(n-1)m+2}.$$

Proof. We first prove (1). By construction, $S_{j+1} = S_j \circ S_{j-(n-1)}$ and S_{j+1} gives the first $F_{n,j+1}$ letters of S_∞. Hence, $S_j \circ S_{j-(n-1)}$ gives the first $F_{n,j+1}$ letters of S_∞. Because $n \leq i \leq j-(n-1)$, S_i gives the first $F_{n,i}$ letters of $S_{j-(n-1)}$. Therefore, $S_j \circ S_i$ gives the first $F_{n,i} + F_{n,j}$ letters of S_∞, as desired.

To prove (2), it suffices to show that there exists S^* such that $S_i \circ S^* = S_{i+1}$. By construction and because $i \geq n$, we can let $S^* = S_{i-(n-1)}$.

Next, we prove (3). We proceed by induction on m.
<u>Base cases</u>. For $m = 2$,
$$S_{n+2(n-1)} \circ S_{n+(n-1)} \circ S_n \;=\; S_{n+2(n-1)+1} \circ S_n.$$
Thus, letting S^* be the empty string, our claim holds. For $m = 3$, we have
$$S_{n+3(n-1)} \circ S_{n+2(n-1)} \circ S_{n+(n-1)} \circ S_n \;=\; S_{n+3(n-1)+1} \circ S_{n+(n-1)+1}.$$
Again, letting S^* be the empty string, our claim holds.
<u>Inductive hypothesis</u>. Suppose that our claim holds for all $2 \leq m \leq \ell$ for some $\ell \geq 3$. We have
$$\begin{aligned} &S_{n+(n-1)(\ell+1)} \circ S_{n+(n-1)\ell} \circ S_{n+(n-1)(\ell-1)} \circ \cdots \circ S_{n+(n-1)} \circ S_n \\ &\;=\; S_{n+(n-1)(\ell+1)+1} \circ (S_{n+(n-1)(\ell-1)} \circ \cdots \circ S_{n+(n-1)} \circ S_n) \\ &\;=\; S_{n+(n-1)(\ell+1)+1} \circ S_{n+(n-1)(\ell-1)+1} \circ S_i \text{ for some } n \leq i \leq n+(n-1)(\ell-2)+1. \end{aligned}$$

The last equality is due to the inductive hypothesis. By (2), there exists some S', a concatenation of some S_i's, such that $S_i \circ S' = S_{n+(n-1)(\ell-2)+1}$. Hence,

$$\begin{aligned}
&(S_{n+(n-1)(\ell+1)} \circ S_{n+(n-1)\ell} \circ S_{n+(n-1)(\ell-1)} \circ \cdots \circ S_{n+(n-1)} \circ S_n) \circ S' \\
&= S_{n+(n-1)(\ell+1)+1} \circ S_{n+(n-1)(\ell-1)+1} \circ S_i \circ S' \\
&= S_{n+(n-1)(\ell+1)+1} \circ S_{n+(n-1)(\ell-1)+1} \circ S_{n+(n-1)(\ell-2)+1} \\
&= S_{n+(n-1)(\ell+1)+1} \circ S_{n+(n-1)(\ell-1)+2}.
\end{aligned}$$

Therefore, our claim holds for $m = \ell + 1$. This completes our proof of (3).

Finally, (4) follows immediately from items (2) and (3). □

We are ready to prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. First, we prove (1). The first part is clear by construction. We prove the second part by induction on m.

Base cases. If $1 \leq m \leq n-1$, then $S_m = a_m$ by construction. All numbers in the set $\{m-(n-1), m-n, \ldots, m-(2n-2)\}$ lie between $3-2n$ and 0 inclusive. By Remark 1.5, for all $3-2n \leq i \leq 0$ except $i = 1-n$, we have $F_{n,i} = 0$, whereas $F_{n,1-n} = 1$. Write $F_{n,1-n} = F_{n,m-(m+(n-1))}$ to see that our claim holds. For $m = n$, it is easy to check that our claim also holds.

Inductive hypothesis. Suppose that our claim holds for all $1 \leq m \leq \ell$ for some $\ell \geq n$. Because $S_{\ell+1} = S_\ell \circ S_{\ell+1-n}$, the number of a_n's in $S_{\ell+1}$ is

$$\begin{aligned}
&F_{n,\ell-(n-1)} + F_{n,\ell+1-n-(n-1)} \text{ by the inductive hypothesis,} \\
&= F_{n,\ell-n+2} = F_{n,(\ell+1)-(n-1)}
\end{aligned}$$

Similarly, the formulas for the number of a_i's in $S_{\ell+1}$ are all correct. This completes our proof of (1). □

Next, we prove (2). For $m = 1$, we have $S_{n+(n-1)} \circ S_n = S_{2n}$, which gives the first $F_{n,2n} = F_{n,n} + F_{n,2n-1}$ letters of S_∞. For $m \geq 2$, (2) follows from the first part of (1) and Lemma 2.1 (4).

Proof of Theorem 1.4. The proof is by induction on k, the number of terms in the n-decomposition.

Base case. If $k = 1$, the statement of the theorem is certainly true.

Inductive hypothesis. Assume that the statement is true for some $k = \ell \geq 1$. Consider the n-decomposition

$$m = F_{n,c_1} + F_{n,c_2} + \cdots + F_{n,c_\ell} + F_{n,c_{\ell+1}}.$$

By the inductive hypothesis, we know that

$$S_{c_\ell} \circ S_{c_{\ell-1}} \circ \cdots \circ S_{c_1}$$

gives the first $F_{n,c_1} + F_{n,c_2} + \cdots + F_{n,c_\ell}$ letters of S_∞, and therefore, by Theorem 1.3 (2), of

$$S_{c_{\ell+1}-1} \circ S_{c_{\ell+1}-2} \circ \cdots \circ S_{c_{\ell+1}-\ell}.$$

Hence, we know that

$$S_{c_{\ell+1}} \circ S_{c_\ell} \circ S_{c_{\ell-1}} \circ \cdots \circ S_{c_1}$$

gives the first m letters of

$$S_{\ell+1} \circ S_{c_{\ell+1}-1} \circ S_{c_{\ell+1}-2} \circ \cdots \circ S_{c_{\ell+1}-\ell},$$

and hence, by Theorem 1.3 (2), of S_∞. □

3. Fixed Term in the n-Decomposition

In this section, we fix $k \geq n \geq 3$. Our goal is to characterize the set $Z_n(k)$, the set of all positive integers having the summand $F_{n,k}$ in their n-decomposition. The following lemma generalizes [7, Lemma 2.5].

Lemma 3.1. *Let $m \geq 1$. For $v \geq 1$ and $1 \leq u \leq n$, we have*

$$F_{n,m+vn+u} - (F_{n,m+vn+(u-1)} + \cdots + F_{n,m+n+(u-1)}) = F_{n,m+u}. \tag{3.1}$$

Proof. If $v = 1$, the identity holds because of the linear recurrence relation of $\{F_{n,i}\}$. Assume that $v \geq 2$. We have

$$\begin{aligned}
&(F_{n,m+vn+(u-1)} + \cdots + F_{n,m+n+(u-1)}) + F_{n,m+u} \\
&= (F_{n,m+vn+(u-1)} + \cdots + F_{n,m+2n+(u-1)}) + (F_{n,m+n+(u-1)} + F_{n,m+u}) \\
&= (F_{n,m+vn+(u-1)} + \cdots + F_{n,m+2n+(u-1)}) + F_{n,m+n+u} \\
&\vdots \\
&= F_{n,m+vn+(u-1)} + F_{n,m+(v-1)n+u} = F_{n,m+vn+u}.
\end{aligned}$$

This completes our proof. □

Lemma 3.2. *Let i and $j \in \mathbb{N}$. If $F_{n,u}$ and $F_{n,v}$ are the largest summands in the n-decompositions of i and j, respectively, then $u > v$ implies that $i > j$.*

The statement of the lemma is obvious because the n-decomposition of a number can be found by the greedy algorithm.

Lemma 3.3. *For $j \geq 1$, the $(F_{n,nj})$th character of S_∞ is a_n; the $(F_{n,nj+1})$th character of S_∞ is a_1; …; $(F_{n,nj+(n-1)})$th character of S_∞ is a_{n-1}.*

Proof. The proof is by induction on j.
Base case. By construction, we know that the statement is true for $j = 1$.
Inductive hypothesis. Suppose the statement is true for $j = m$ for some $m \geq 1$. The $(F_{n,n(m+1)})$th of S_∞ is the last letter of $S_{n(m+1)} = S_{n(m+1)-1} \circ S_{nm}$. By the inductive hypothesis, the last letter of S_{nm} is a_n and so, the $(F_{n,n(m+1)})$th letter of S_∞ is a_n. The proof is completed by similar arguments for the $(F_{n,n(m+1)+u})$th letter, where $1 \leq u \leq n-1$. □

Let $\mathcal{X}_{n,k}$ denote the set of all positive integers whose n-decompositions have $F_{n,k}$ as the smallest summand. Next, let $\mathcal{Q}_{n,k} = \{q(j)\}_{j\geq 1}$ be the strictly increasing infinite sequence that results from arranging the elements of $\mathcal{X}_{n,k}$ into ascending numeric order. Table 1 gives the ordered list of summands for each $q(j)$, where the rth row corresponds to the rth smallest element from $\mathcal{X}_{n,k}$. (Lemma 3.2 helps explain the ordering of rows.)

Row						
1	$F_{n,k}$					
2	$F_{n,k}$	$F_{n,k+n}$				
3	$F_{n,k}$		$F_{n,k+n+1}$			
		$\vdots$				
$(n+1)$	$F_{n,k}$			$F_{n,k+2n-1}$		
$(n+2)$	$F_{n,k}$				$F_{n,k+2n}$	
$(n+3)$	$F_{n,k}$	$F_{n,k+n}$			$F_{n,k+2n}$	
$(n+4)$	$F_{n,k}$					$F_{n,k+2n+1}$
$(n+5)$	$F_{n,k}$	$F_{n,k+n}$				$F_{n,k+2n+1}$
$(n+6)$	$F_{n,k}$		$F_{n,k+n+1}$			$F_{n,k+2n+1}$
		$\vdots$				

Table 1. The n-decompositions, in numeric order, of the positive integers having $F_{n,k}$ as their smallest summand.

Lemma 3.4. *For $j \geq n$, the rows of Table 1 for which $F_{n,k+j}$ is the largest summand are those numbered from $F_{n,j}+1$ to $F_{n,j+1}$ inclusive.*

Proof. The proof is by induction on j.
Base cases. For $n \leq j \leq 2n-1$, there is exactly one row with $F_{n,k+j}$ being the largest summand. In particular, the row with $F_{n,k+j}$ being the largest summand is the $(j-(n-2))th$. Hence, the base cases are done if we prove the two following claims.

(1) $F_{n,j+1} - F_{n,j} = 1$;
(2) $F_{n,j} + 1 = j - (n-2)$.

For the first claim, we have $F_{n,j+1} - F_{n,j} = F_{n,j-(n-1)} = 1$ for all $n \leq j \leq 2n-1$. This is due to the construction of $\{F_{n,j}\}$. To prove the second claim, observe that as j goes from n to $2n-1$, we have $F_{n,j}$ increase from 1 to n.
Inductive hypothesis. Suppose that the statement of the lemma holds for $n \leq j \leq m$ for some $m \geq 2n-1$. We want to show that the rows with $F_{n,k+m+1}$ being the largest summand is from $F_{n,m+1}+1$ to $F_{n,m+2}$. By the inductive hypothesis, the number of rows with the largest summand not larger than $F_{n,k+m+1-n}$ is

$$1 + \sum_{j=n}^{m+1-n} (F_{n,j+1} - F_{n,j}) \;=\; 1 + F_{n,m+2-n} - F_{n,n} \;=\; F_{n,m+2-n},$$

which is also the number of rows with $F_{n,k+m+1}$ being the largest summand.

By the inductive hypothesis, the rows with F_{k+m} being the largest is from $F_{n,m}+1$ to $F_{n,m+1}$. Therefore, the rows with $F_{n,k+m+1}$ being the largest is from $F_{n,m+1}+1$ to $F_{n,m+1}+F_{n,m+2-n} = F_{n,m+2}$. □

Lemma 3.5. *For $j \geq 1$, we have*

$$q(j+1) - q(j) \;=\; \begin{cases} F_{n,k+1}, & \textit{if the } j\textit{th character of } S_\infty \textit{ is } a_1; \\ F_{n,k+2}, & \textit{if the } j\textit{th character of } S_\infty \textit{ is } a_2; \\ \vdots & \\ F_{n,k+n}, & \textit{if the } j\textit{th character of } S_\infty \textit{ is } a_n. \end{cases} \tag{3.2}$$

Proof. The proof is by induction on j.

Base cases. Consider $1 \le j \le F_{n,2n} - 1 = n$. From Table 1, we know that

$$q(j+1) - q(j) = \begin{cases} F_{n,k+n}, & \text{if } j = 1; \\ F_{n,k+(j-1)}, & \text{if } 2 \le j \le n. \end{cases} \quad (3.3)$$

Because $S_\infty = a_n a_1 a_2 \ldots a_{n-1} \ldots$ and (3.3), we have proved the base cases for $1 \le j \le F_{n,2n} - 1$.
Inductive hypothesis. Assume that the statement is true for $1 \le j \le F_{n,m} - 1$ for some $m \ge 2n$. By Lemma 3.4, the first $F_{n,m-n+1}$ rows of Table 1 are those for which the largest summand is no greater than $F_{n,k+m-n}$, and the rows for which $F_{n,k+m}$ is the largest summand are from $F_{n,m} + 1$ to $F_{n,m+1}$.

Due to the ordering of rows in Table 1, we have $q(i) + F_{n,k+m} = q(i + F_{n,m})$ for $1 \le i \le F_{n,m-n+1}$. Hence, for $1 \le i \le F_{n,m-n+1} - 1$, we have

$$\begin{aligned} q(i+1+F_{n,m}) - q(i+F_{n,m}) &= (q(i+1) + F_{n,k+m}) - (q(i) + F_{n,k+m}) \\ &= q(i+1) - q(i). \end{aligned}$$

By the construction of S_∞, the string of the first $F_{n,m-n+1}$ characters is identical to the string of characters from the $(F_{n,m}+1)$th to the $F_{n,m+1}$th, inclusive. Therefore, we know that (3.2) is true for $1 + F_{n,m} \le j \le F_{n,m+1} - 1$. It remains to show that (3.2) is true for $j = F_{n,m}$. We have

$$q(F_{n,m}+1) - q(F_{n,m}) = F_{n,k+m} - (F_{n,k+m-1} + F_{n,k+m-1-n} + \cdots + F_{n,k+m-1-\ell n}),$$

where ℓ satisfies $n \le m - 1 - \ell n < 2n$. Write $m = vn + u$ for some $1 \le u \le n$. It follows that $1 - (u-1)/n \le v - \ell < 2 - (u-1)/n$, so $v - \ell = 1$. Hence,

$$\begin{aligned} & q(F_{n,m}+1) - q(F_{n,m}) \\ &= F_{n,k+vn+u} - (F_{n,k+vn+u-1} + F_{n,k+(v-1)n+u-1} + \cdots + F_{n,k+n+u-1}) = F_{n,k+u} \end{aligned}$$

due to Lemma 3.1. Using Lemma 3.3, we conclude that (3.2) is true for $j = F_{n,m}$. This completes our proof. □

Finally, we prove Theorem 1.7.

Proof of Theorem 1.7. By Lemma 3.5, we can write

$$\mathcal{X}_{n,k} = \left\{ F_{n,k} + \sum_{i=1}^{n} F_{n,k+i} \cdot N_{a_i}(m) : m \ge 0 \right\}.$$

The numbers in $\{F_{n,n}, F_{n,n+1}, \ldots, F_{n,k-n}\}$ are used to obtain the n-decompositions of all integers for which the largest summand is no greater than $F_{n,k-n}$. In particular, such n-decompositions generate all integers from 1 to $F_{n,k-n+1} - 1$, inclusive. Furthermore, such decompositions can be appended to any n-decomposition having $F_{n,k}$ as its smallest summand to produce another n-decomposition. Therefore, we know that

$$Z_n(k) = \left\{ j + F_{n,k} + \sum_{i=1}^{n} F_{n,k+i} \cdot N_{a_i}(m) : 0 \le j \le F_{n,k-(n-1)} - 1 \text{ and } m \ge 0 \right\},$$

as desired. □

References

[1] A. Best, P. Dynes, X. Edelsbrunner, B. McDonald, S. Miller, C. Turnage-Butterbaugh, and M. Weinstein, *Benford behavior of Zeckendorf's decompositions*, The Fibonacci Quarterly, **52.1** (2014), 35–46.

[2] A. Best, P. Dynes, X. Edelsbrunner, B. McDonald, S. Miller, C. Turnage-Butterbaugh, and M. Weinstein, *Gaussian behavior of the number of summands in Zeckendorf's decompositions in small intervals*, The Fibonacci Quarterly, **52.1** (2014), 47–53.

[3] J. Brown, Jr., *Zeckendorf's theorem and some applications*, The Fibonacci Quarterly, **2.3** (1964), 163–168.

[4] L. Carlitz, R. Scoville, and V. Hoggatt, Jr., *Fibonacci representations*, The Fibonacci Quarterly, **10.1** (1972), 1–28.

[5] P. Demontigny, T. Do, A. Kulkarni, S. Miller, D. Moon, and U. Varma, *Generalizing Zeckendorf's theorem to f-decompositions*, J. Number Theory, **141** (2014), 136–158.

[6] R. Dorward, P. Ford, E. Fourakis, P. Harris, S. Miller, E. Palsson, and H. Paugh, *Individual gap measures from generalized Zeckendorf decompositions*, Unif. Distrib. Theory, **12** (2017), 27–36.

[7] M. Griffiths, *Fixed-term Zeckendorf representations*, The Fibonacci Quarterly, **52.4** (2014), 331–335.

[8] M. Griffiths, *The golden string, Zeckendorf representations, and the sum of a series*, American Mathematical Monthly, **118** (2011), 497–507.

[9] E. Hart and L. Sanchis, *On the occurrence of F_n in the Zeckendorf decomposition of nF_n*, The Fibonacci Quarterly, **37.1** (1999), 21–33.

[10] V. Hoggatt, Jr., *Generalized Zeckendorf theorem*, The Fibonacci Quarterly, **10.1** (1972), 89–93.

[11] T. Keller, *Generalizations of Zeckendorf's theorem*, The Fibonacci Quarterly, **10.1** (1972), 95–102.

[12] C. G. Lekkerkerker, *Voorstelling van natuurlyke getallen door een som van getallen van Fibonacci*, Bull. Belg. Math. Soc. Simon Stevin, **29** (1951–1952), 190–195.

[13] S. Miller and R. Li, *Central limit theorems for gaps of generalized Zeckendorf's decompositions*, The Fibonacci Quarterly, **57.3** (2019), 213–230.

[14] E. Zeckendorf, *Representation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. Roy. Sci. Liege, **41** (1972), 179–182.

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