A FAMILY OF SUMS OF GIBONACCI POLYNOMIAL PRODUCTS OF ORDER 4 REVISITED

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Abstract. We explore the Pell, Jacobsthal, and Vieta implications of the sums of gibonacci polynomial products of order 4 investigated in [4].

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary complex variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 5, 6].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [5].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial* [2, 6]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$ and $j_n(1) = L_n$.

Let $a(x) = x$ and $b(x) = -1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = V_n(x)$, the nth *Vieta polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = v_n(x)$, the nth *Vieta-Lucas polynomial* [3, 6].

Finally, let $a(x) = 2x$ and $b(x) = -1$. When $z_0(x) = 1$ and $z_1(x) = x$, $z_n(x) = T_n(x)$, the nth *Chebyshev polynomial of the first kind*; and when $z_0(x) = 1$ and $z_1(x) = 2x$, $z_n(x) = U_n(x)$, the nth *Chebyshev polynomial of the second kind* [3, 6].

Table 1 shows the close relationships among the Jacobsthal, Vieta, and Chebyshev subfamilies, where $i = \sqrt{-1}$ [3, 6]. They play a significant role in our investigations.

$J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$	$j_n(x) = x^{n/2} l_n(1/\sqrt{x})$
$V_n(x) = i^{n-1} f_n(-ix)$	$v_n(x) = i^n l_n(-ix)$
$V_n(2x) = U_{n-1}(x)$	$v_n(2x) = 2T_n(x)$.

Table 1. Links Among the Subfamilies

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. We also let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $c_n = J_n(x)$ or $j_n(x)$, $d_n = V_n$ or v_n, and $e_n = T_n$ or U_n; and correspondingly, $G_n = F_n$ or L_n, $B_n = P_n$ or Q_n, and $C_n = J_n$ or j_n. We also omit a lot of basic algebra.

A *gibonacci polynomial product of order* m is a product of gibonacci polynomials g_{n+k} of the form $\prod_{k\in\mathcal{Z}} g_{n+k}^{s_j}$, where $\sum_{s_j\ge 1} s_j = m$ [7, 8].

1.1. **Sums of Gibonacci Polynomial Products of Order 4.** In [4], we studied the following sums of gibonacci polynomial products of order 4:

$$\begin{aligned}
x^3 f_{4n} &= f_{n+2}^3 f_n - 2f_{n+2}^2 f_n^2 - f_{n+2}^2 f_n f_{n-2} + 2(x^2+1) f_{n+2} f_n^3 + f_{n+2} f_n f_{n-2}^2 \\
&\quad -2(x^2+1) f_n^3 f_{n-2} + 2f_n^2 f_{n-2}^2 - f_n f_{n-2}^3; &(1.1)\\
x^4 f_{4n-1} &= f_{n+2}^4 - 4(x^2+1) f_{n+2}^3 f_n + (4x^4+13x^2+6) f_{n+2}^2 f_n^2 \\
&\quad - (x^6+7x^4+10x^2+4) f_{n+2} f_n^3 - 2(x^4+2x^2) f_{n+2} f_n^2 f_{n-2} \\
&\quad + (x^6+3x^4+2x^2+1) f_n^4 + (x^6+3x^4+2x^2) f_n^3 f_{n-2} + x^2 f_n^2 f_{n-2}^2; &(1.2)\\
x^4 f_{4n+1} &= f_{n+2}^4 - 4f_{n+2}^3 f_n + 2(2x^2+3) f_{n+2}^2 f_n^2 - (x^4+6x^2+4) f_{n+2} f_n^3 \\
&\quad - 2x^2 f_{n+2} f_n^2 f_{n-2} + (x^2+1)^2 f_n^4 + (x^4+2x^2) f_n^3 f_{n-2}; &(1.3)\\
x^3 f_{4n+2} &= f_{n+2}^4 - 3f_{n+2}^2 f_n^2 + 2f_{n+2} f_n^2 f_{n-2} + f_n^4 - f_n^2 f_{n-2}^2; &(1.4)\\
x^4 f_{4n+3} &= (x^2+1) f_{n+2}^4 - 4f_{n+2}^3 f_n + (x^2+6) f_{n+2}^2 f_n^2 - (x^4+6x^2+4) f_{n+2} f_n^3 \\
&\quad + (x^4+3x^2+1) f_n^4 + (x^4+2x^2) f_n^3 f_{n-2} - x^2 f_n^2 f_{n-2}^2; &(1.5)\\
x^4 l_{4n} &= 2f_{n+2}^4 - 4(x^2+2) f_{n+2}^3 f_n + (4x^4+17x^2+12) f_{n+2}^2 f_n^2 \\
&\quad - (x^6+8x^4+16x^2+8) f_{n+2} f_n^3 - 2(x^4+3x^2) f_{n+2} f_n^2 f_{n-2} \\
&\quad + (x^6+4x^4+4x^2+2) f_n^4 + (x^6+4x^4+4x^2) f_n^3 f_{n-2} + x^2 f_n^2 f_{n-2}^2; &(1.6)\\
x^3 l_{4n-1} &= -f_{n+2}^4 + 4(x^2+3) f_{n+2}^3 f_n - (4x^4+21x^2+24) f_{n+2}^2 f_n^2 \\
&\quad + (x^6+9x^4+22x^2+12) f_{n+2} f_n^3 + 2(x^4+4x^2+2) f_{n+2} f_n^2 f_{n-2} \\
&\quad - (x^6+5x^4+6x^2+1) f_n^4 - (x^6+5x^4+6x^2) f_n^3 f_{n-2} - (x^2+2) f_n^2 f_{n-2}^2; &(1.7)\\
x^3 l_{4n+1} &= f_{n+2}^4 + 4f_{n+2}^3 f_n - 4(x^2+3) f_{n+2}^2 f_n^2 + (x^4+6x^2+4) f_{n+2} f_n^3 \\
&\quad + 2(x^2+2) f_{n+2} f_n^2 f_{n-2} - (x^4+2x^2-1) f_n^4 - (x^4+2x^2) f_n^3 f_{n-2} \\
&\quad -2f_n^2 f_{n-2}^2; &(1.8)\\
x^4 l_{4n+2} &= (x^2+2) f_{n+2}^4 - 8f_{n+2}^3 f_n + (5x^2+12) f_{n+2}^2 f_n^2 - 2(x^4+6x^2+4) f_{n+2} f_n^3 \\
&\quad - 2x^2 f_{n+2} f_n^2 f_{n-2} + (2x^4+5x^2+2) f_n^4 + 2(x^4+2x^2) f_n^3 f_{n-2} \\
&\quad -x^2 f_n^2 f_{n-2}^2; &(1.9)\\
x^3 l_{4n+3} &= (x^2+3) f_{n+2}^4 - 4f_{n+2}^3 f_n + x^2 f_{n+2}^2 f_n^2 - (x^4+6x^2+4) f_{n+2} f_n^3 \\
&\quad + 4f_{n+2} f_n^2 f_{n-2} + (x^4+3x^2+3) f_n^4 + (x^4+2x^2) f_n^3 f_{n-2} \\
&\quad -(x^2+2) f_n^2 f_{n-2}^2. &(1.10)
\end{aligned}$$

They too play a pivotal role in our explorations.

With this background, we now explore the implications of these identities to the Pell, Jacobsthal, and Chebyshev subfamilies.

2. Pell Implications

Because $b_n(x) = g_n(2x)$, the gibonacci identities (1.1) through (1.10) yield the following Pell counterparts, respectively:

$$\begin{aligned}
8x^3p_{4n} &= p_{n+2}^3p_n - 2p_{n+2}^2p_n^2 - p_{n+2}^2p_np_{n-2} + 2(4x^2+1)p_{n+2}p_n^3 + p_{n+2}p_np_{n-2}^2 \\
&\quad - 2(4x^2+1)p_n^3p_{n-2} + 2p_n^2p_{n-2}^2 - p_np_{n-2}^3; \\
16x^4p_{4n-1} &= p_{n+2}^4 - 4(4x^2+1)p_{n+2}^3p_n + 2(32x^4+26x^2+3)p_{n+2}^2p_n^2 \\
&\quad - 4(16x^6+28x^4+10x^2+1)p_{n+2}p_n^3 - 16(2x^4+x^2)p_{n+2}p_n^2p_{n-2} \\
&\quad + (64x^6+48x^4+8x^2+1)p_n^4 + 8(8x^6+6x^4+x^2)p_n^3p_{n-2} + 4x^2p_n^2p_{n-2}^2; \\
16x^4p_{4n+1} &= p_{n+2}^4 - 4p_{n+2}^3p_n + 2(8x^2+3)p_{n+2}^2p_n^2 - 4(4x^4+6x^2+1)p_{n+2}p_n^3 \\
&\quad -8x^2p_{n+2}p_n^2p_{n-2} + (4x^2+1)^2p_n^4 + 8(2x^4+x^2)p_n^3p_{n-2}; \\
8x^3p_{4n+2} &= p_{n+2}^4 - 3p_{n+2}^2p_n^2 + 2p_{n+2}p_n^2p_{n-2} + p_n^4 - p_n^2p_{n-2}^2; \\
16x^4p_{4n+3} &= (4x^2+1)p_{n+2}^4 - 4p_{n+2}^3p_n + 2(2x^2+3)p_{n+2}^2p_n^2 - 4(4x^4+6x^2+1)p_{n+2}p_n^3 \\
&\quad + (16x^4+12x^2+1)p_n^4 + 8(2x^4+x^2)p_n^3p_{n-2} - 4x^2p_n^2p_{n-2}^2;
\end{aligned}$$

$$\begin{aligned}
8x^4q_{4n} &= p_{n+2}^4 - 4(2x^2+1)p_{n+2}^3p_n + 2(16x^4+17x^2+3)p_{n+2}^2p_n^2 \\
&\quad - 4(8x^6+16x^4+8x^2+1)p_{n+2}p_n^3 - 4(4x^4+3x^2)p_{n+2}p_n^2p_{n-2} \\
&\quad + (32x^6+32x^4+8x^2+1)p_n^4 + 8(4x^6+4x^4+x^2)p_n^3p_{n-2} + 2x^2p_n^2p_{n-2}^2; \\
8x^3q_{4n-1} &= -p_{n+2}^4 + 4(4x^2+3)p_{n+2}^3p_n - 4(16x^4+21x^2+6)p_{n+2}^2p_n^2 \\
&\quad + 4(16x^6+36x^4+22x^2+3)p_{n+2}p_n^3 + 4(8x^4+8x^2+1)p_{n+2}p_n^2p_{n-2} \\
&\quad - (64x^6+80x^4+24x^2+1)p_n^4 - 8(8x^6+10x^4+3x^2)p_n^3p_{n-2} \\
&\quad - 2(2x^2+1)p_n^2p_{n-2}^2; \\
8x^3q_{4n+1} &= p_{n+2}^4 + 4p_{n+2}^3p_n - 4(4x^2+3)p_{n+2}^2p_n^2 + 4(4x^4+6x^2+1)p_{n+2}p_n^3 \\
&\quad + 4(2x^2+1)p_{n+2}p_n^2p_{n-2} - (16x^4+8x^2-1)p_n^4 \\
&\quad -8(2x^4+x^2)p_n^3p_{n-2} - 2p_n^2p_{n-2}^2; \\
8x^4q_{4n+2} &= (2x^2+1)p_{n+2}^4 - 4p_{n+2}^3p_n + 2(5x^2+3)p_{n+2}^2p_n^2 - 4(4x^4+6x^2+1)p_{n+2}p_n^3 \\
&\quad - 4x^2p_{n+2}p_n^2p_{n-2} + (16x^4+10x^2+1)p_n^4 + 8(2x^4+x^2)p_n^3p_{n-2} - 2x^2p_n^2p_{n-2}^2; \\
8x^3q_{4n+3} &= (4x^2+3)p_{n+2}^4 - 4p_{n+2}^3p_n + 4x^2p_{n+2}^2p_n^2 - 4(4x^4+6x^2+1)p_{n+2}p_n^3 \\
&\quad + 4p_{n+2}p_n^2p_{n-2} + (16x^4+12x^2+3)p_n^4 + 8(2x^4+x^2)p_n^3p_{n-2} \\
&\quad -2(2x^2+1)p_n^2p_{n-2}^2,
\end{aligned}$$

where $b_n = b_n(x)$.

In particular, we then have

$$\begin{aligned}
8P_{4n} &= P_{n+2}^3P_n - 2P_{n+2}^2P_n^2 - P_{n+2}^2P_nP_{n-2} + 10P_{n+2}P_n^3 + P_{n+2}P_nP_{n-2}^2 - 10P_n^3P_{n-2} \\
&\quad + 2P_n^2P_{n-2}^2 - P_nP_{n-2}^3; \\
16P_{4n-1} &= P_{n+2}^4 - 20P_{n+2}^3P_n + 122P_{n+2}^2P_n^2 - 220P_{n+2}P_n^3 - 48P_{n+2}P_n^2P_{n-2} \\
&\quad + 121P_n^4 + 120P_n^3P_{n-2} + 4P_n^2P_{n-2}^2; \\
16P_{4n+1} &= P_{n+2}^4 - 4P_{n+2}^3P_n + 22P_{n+2}^2P_n^2 - 44P_{n+2}P_n^3 - 8P_{n+2}P_n^2P_{n-2} \\
&\quad + 25P_n^4 + 24P_n^3P_{n-2}; \\
8P_{4n+2} &= P_{n+2}^4 - 3P_{n+2}^2P_n^2 + 2P_{n+2}P_n^2P_{n-2} + P_n^4 - P_n^2P_{n-2}^2;
\end{aligned}$$

$$\begin{aligned}
16P_{4n+3} &= 5P_{n+2}^4 - 4P_{n+2}^3P_n + 10P_{n+2}^2P_n^2 - 44P_{n+2}P_n^3 + 29P_n^4 \\
&\quad + 24P_n^3P_{n-2} - 4P_n^2P_{n-2}^2; \\
16Q_{4n} &= P_{n+2}^4 - 12P_{n+2}^3P_n + 72P_{n+2}^2P_n^2 - 132P_{n+2}P_n^3 - 28P_{n+2}P_n^2P_{n-2} \\
&\quad + 73P_n^4 + 72P_n^3P_{n-2} + 2P_n^2P_{n-2}^2; \\
16Q_{4n-1} &= -P_{n+2}^4 + 28P_{n+2}^3P_n - 172P_{n+2}^2P_n^2 + 308P_{n+2}P_n^3 + 68P_{n+2}P_n^2P_{n-2} \\
&\quad - 169P_n^4 - 168P_n^3P_{n-2} - 6P_n^2P_{n-2}^2; \\
16Q_{4n+1} &= P_{n+2}^4 + 4P_{n+2}^3P_n - 28P_{n+2}^2P_n^2 + 44P_{n+2}P_n^3 + 12P_{n+2}P_n^2P_{n-2} \\
&\quad - 23P_n^4 - 24P_n^3P_{n-2} - 2P_n^2P_{n-2}^2; \\
16Q_{4n+2} &= 3P_{n+2}^4 - 4P_{n+2}^3P_n + 16P_{n+2}^2P_n^2 - 44P_{n+2}P_n^3 - 4P_{n+2}P_n^2P_{n-2} \\
&\quad + 27P_n^4 + 24P_n^3P_{n-2} - 2P_n^2P_{n-2}^2; \\
16Q_{4n+3} &= 7P_{n+2}^4 - 4P_{n+2}^3P_n + 4P_{n+2}^2P_n^2 - 44P_{n+2}P_n^3 + 4P_{n+2}P_n^2P_{n-2} \\
&\quad + 31P_n^4 + 24P_n^3P_{n-2} - 6P_n^2P_{n-2}^2.
\end{aligned}$$

Next, we explore the Jacobasthal implications of the gibonacci sums.

3. Sums of Jacobsthal Polynomial Products of Order 4

Using the gibonacci-Jacobsthal relationships in Table 1, we can extract the Jacobsthal counterparts of the gibonacci identities (1.1) through (1.10):

$$\begin{aligned}
J_{4n} &= J_{n+2}^3J_n - 2xJ_{n+2}^2J_n^2 - x^2J_{n+2}^2J_nJ_{n-2} + 2(x^2+x)J_{n+2}J_n^3 \\
&\quad + x^4J_{n+2}J_nJ_{n-2}^2 - 2(x^4+x^3)J_n^3J_{n-2} + 2x^5J_n^2J_{n-2}^2 - x^6J_nJ_{n-2}^3; \qquad (3.1)\\
xJ_{4n-1} &= J_{n+2}^4 - 4(x+1)J_{n+2}^3J_n + (6x^2+13x+4)J_{n+2}^2J_n^2 \\
&\quad - (4x^3+10x^2+7x+1)J_{n+2}J_n^3 - 2(2x^3+x^2)J_{n+2}J_n^2J_{n-2} \\
&\quad + (x^4+2x^3+3x^2+x)J_n^4 + (2x^4+3x^3+x^2)J_n^3J_{n-2} + x^5J_n^2J_{n-2}^2; \qquad (3.2)\\
J_{4n+1} &= J_{n+2}^4 - 4xJ_{n+2}^3J_n + 2(3x^2+2x)J_{n+2}^2J_n^2 - (4x^3+6x^2+x)J_{n+2}J_n^3 \\
&\quad - 2x^3J_{n+2}J_n^2J_{n-2} + (x^2+x)^2J_n^4 + (2x^4+x^3)J_n^3J_{n-2}; \qquad (3.3)\\
J_{4n+2} &= J_{n+2}^4 - 3x^2J_{n+2}^2J_n^2 + 2x^4J_{n+2}J_n^2J_{n-2} + x^4J_n^4 - x^6J_n^2J_{n-2}^2; \qquad (3.4)\\
J_{4n+3} &= (x+1)J_{n+2}^4 - 4x^2J_{n+2}^3J_n + (6x^3+x^2)J_{n+2}^2J_n^2 - (4x^4+6x^3+x^2)J_{n+2}J_n^3 \\
&\quad + (x^5+3x^4+x^3)J_n^4 + (2x^5+x^4)J_n^3J_{n-2} - x^6J_n^2J_{n-2}^2; \qquad (3.5)
\end{aligned}$$

$$\begin{aligned}
j_{4n} &= 2J_{n+2}^4 - 4(2x+1)J_{n+2}^3J_n + (12x^2+17x+4)J_{n+2}^2J_n^2 \\
&\quad - (8x^3+16x^2+8x+1)J_{n+2}J_n^3 - 2(3x^3+x^2)J_{n+2}J_n^2J_{n-2} \\
&\quad + (2x^4+4x^3+4x^2+x)J_n^4 + (4x^4+4x^3+x^2)J_n^3J_{n-2} + x^5J_n^2J_{n-2}^2; \qquad (3.6)\\
xj_{4n-1} &= -J_{n+2}^4 + 4(3x+1)J_{n+2}^3J_n - (24x^2+21x+4)J_{n+2}^2J_n^2 \\
&\quad + (12x^3+22x^2+9x+1)J_{n+2}J_n^3 + 2(2x^4+4x^3+x^2)J_{n+2}J_n^2J_{n-2} \\
&\quad - (x^4+6x^3+5x^2+x)J_n^4 - (6x^4+5x^3+x^2)J_n^3J_{n-2} \\
&\quad - (2x^6+x^5)J_n^2J_{n-2}^2; \qquad (3.7)
\end{aligned}$$

$$\begin{aligned}
j_{4n+1} &= J_{n+2}^4 + 4xJ_{n+2}^3J_n - 4(3x^2+x)J_{n+2}^2J_n^2 + (4x^3+6x^2+x)J_{n+2}J_n^3 \\
&\quad + 2(2x^4+x^3)J_{n+2}J_n^2J_{n-2} + (x^4-2x^3-x^2)J_n^4 - (2x^4+x^3)J_n^3J_{n-2} \\
&\quad - 2x^6J_n^2J_{n-2}^2; \qquad (3.8)\\
j_{4n+2} &= (2x+1)J_{n+2}^4 - 8x^2J_{n+2}^3J_n + (12x^3+5x^2)J_{n+2}^2J_n^2 - 2(4x^4+6x^3+x^2)J_{n+2}J_n^3 \\
&\quad - 2x^4J_{n+2}J_n^2J_{n-2} + (2x^5+5x^4+2x^3)J_n^4 + 2(2x^5+x^4)J_n^3J_{n-2} \\
&\quad -x^6J_n^2J_{n-2}^2; \qquad (3.9)\\
j_{4n+3} &= (3x+1)J_{n+2}^4 - 4x^2J_{n+2}^3J_n + x^2J_{n+2}^2J_n^2 - (4x^4+6x^3+x^2)J_{n+2}J_n^3 \\
&\quad + 4x^5J_{n+2}J_n^2J_{n-2} + (3x^5+3x^4+x^3)J_n^4 + (2x^5+x^4)J_n^3J_{n-2} \\
&\quad - (2x^7+x^6)J_n^2J_{n-2}^2, \qquad (3.10)
\end{aligned}$$

where $c_n = c_n(x)$.

These Jacobsthal equations yield the following numeric identities, respectively:

$$\begin{aligned}
J_{4n} &= J_{n+2}^3J_n - 4J_{n+2}^2J_n^2 - 4J_{n+2}^2J_nJ_{n-2} + 12J_{n+2}J_n^3 + 16J_{n+2}J_nJ_{n-2}^2 \\
&\quad - 48J_n^3J_{n-2} + 64J_n^2J_{n-2}^2 - 64J_nJ_{n-2}^3; \\
2J_{4n-1} &= J_{n+2}^4 - 12J_{n+2}^3J_n + 54J_{n+2}^2J_n^2 - 87J_{n+2}J_n^3 - 40J_{n+2}J_n^2J_{n-2} + 46J_n^4 \\
&\quad + 60J_n^3J_{n-2} + 32J_n^2J_{n-2}^2; \\
J_{4n+1} &= J_{n+2}^4 - 8J_{n+2}^3J_n + 32J_{n+2}^2J_n^2 - 58J_{n+2}J_n^3 - 16J_{n+2}J_n^2J_{n-2} + 36J_n^4 \\
&\quad + 40J_n^3J_{n-2}; \\
J_{4n+2} &= J_{n+2}^4 - 12J_{n+2}^2J_n^2 + 32J_{n+2}J_n^2J_{n-2} + 16J_n^4 - 64J_n^2J_{n-2}^2; \\
J_{4n+3} &= 3J_{n+2}^4 - 16J_{n+2}^3J_n + 52J_{n+2}^2J_n^2 - 116J_{n+2}J_n^3 + 88J_n^4 \\
&\quad + 80J_n^3J_{n-2} - 64J_n^2J_{n-2}^2; \\
j_{4n} &= 2J_{n+2}^4 - 20J_{n+2}^3J_n + 86J_{n+2}^2J_n^2 - 145J_{n+2}J_n^3 - 56J_{n+2}J_n^2J_{n-2} \\
&\quad + 82J_n^4 + 100J_n^3J_{n-2} + 32J_n^2J_{n-2}^2; \\
2j_{4n-1} &= -J_{n+2}^4 + 28J_{n+2}^3J_n - 142J_{n+2}^2J_n^2 + 203J_{n+2}J_n^3 + 136J_{n+2}J_n^2J_{n-2} \\
&\quad - 86J_n^4 - 140J_n^3J_{n-2} - 160J_n^2J_{n-2}^2; \\
j_{4n+1} &= J_{n+2}^4 + 8J_{n+2}^3J_n - 56J_{n+2}^2J_n^2 + 58J_{n+2}J_n^3 + 80J_{n+2}J_n^2J_{n-2} - 4J_n^4 \\
&\quad - 40J_n^3J_{n-2} - 128J_n^2J_{n-2}^2; \\
j_{4n+2} &= 5J_{n+2}^4 - 32J_{n+2}^3J_n + 116J_{n+2}^2J_n^2 - 232J_{n+2}J_n^3 - 32J_{n+2}J_n^2J_{n-2} \\
&\quad + 160J_n^4 + 160J_n^3J_{n-2} - 64J_n^2J_{n-2}^2; \\
j_{4n+3} &= 7J_{n+2}^4 - 16J_{n+2}^3J_n + 4J_{n+2}^2J_n^2 - 116J_{n+2}J_n^3 + 128J_{n+2}J_n^2J_{n-2} + 152J_n^4 \\
&\quad + 80J_n^3J_{n-2} - 320J_n^2J_{n-2}^2,
\end{aligned}$$

where $C_n = J_n$ or j_n.

4. Vieta Implications

Using the gibonacci-Vieta relationships in Table 1, we can find the Vieta versions of identities (1.1) through (1.10):

$$\begin{aligned}
x^3V_{4n} &= V_{n+2}^3V_n + 2V_{n+2}^2V_n^2 - V_{n+2}^2V_nV_{n-2} - 2(x^2-1)V_{n+2}V_n^3\\
&\quad + V_{n+2}V_nV_{n-2}^2 + 2(x^2-1)V_n^3V_{n-2} - 2V_n^2V_{n-2}^2 - V_nV_{n-2}^3;\\
x^4V_{4n-1} &= -V_{n+2}^4 + 4(x^2-1)V_{n+2}^3V_n - (4x^4-13x^2+6)V_{n+2}^2V_n^2\\
&\quad + (x^6-7x^4+10x^2-4)V_{n+2}V_n^3 + 2(x^4-2x^2)V_{n+2}V_n^2V_{n-2}\\
&\quad + (x^6-3x^4+2x^2-1)V_n^4 - (x^6-3x^4+2x^2)V_n^3V_{n-2} + x^2V_n^2V_{n-2}^2;\\
x^4V_{4n+1} &= V_{n+2}^4 + 4V_{n+2}^3V_n - 2(2x^2-3)V_{n+2}^2V_n^2 + (x^4-6x^2+4)V_{n+2}V_n^3\\
&\quad + \ + 2x^2V_{n+2}V_n^2V_{n-2} + (x^2-1)^2V_n^4 - (x^4-2x^2)V_n^3V_{n-2};\\
x^3V_{4n+2} &= V_{n+2}^4 - 3V_{n+2}^2V_n^2 + 2V_{n+2}V_n^2V_{n-2} + V_n^4 - V_n^2V_{n-2}^2;\\
x^4V_{4n+3} &= (x^2-1)V_{n+2}^4 - 4V_{n+2}^3V_n + (x^2-6)V_{n+2}^2V_n^2 - (x^4-6x^2+4)V_{n+2}V_n^3\\
&\quad - (x^4-3x^2+1)V_n^4 + (x^4-2x^2)V_n^3V_{n-2} - x^2V_n^2V_{n-2}^2; \qquad (4.1)\\
x^4v_{4n} &= 2V_{n+2}^4 - 4(x^2-2)V_{n+2}^3V_n + (4x^4-17x^2+12)V_{n+2}^2V_n^2\\
&\quad - (x^6-8x^4+16x^2-8)V_{n+2}V_n^3 - 2(x^4-3x^2)V_{n+2}V_n^2V_{n-2}\\
&\quad - (x^6-4x^4+4x^2-2)V_n^4 + (x^6-4x^4+4x^2)V_n^3V_{n-2} - x^2V_n^2V_{n-2}^2;\\
x^3v_{4n-1} &= V_{n+2}^4 - 4(x^2-3)V_{n+2}^3V_n + (4x^4-21x^2+24)V_{n+2}^2V_n^2\\
&\quad - (x^6-9x^4+22x^2-12)V_{n+2}V_n^3 - 2(x^4-4x^2+2)V_{n+2}V_n^2V_{n-2}\\
&\quad - (x^6-5x^4+6x^2-1)V_n^4 + (x^6-5x^4+6x^2)V_n^3V_{n-2} - (x^2-2)V_n^2V_{n-2}^2;\\
x^3v_{4n+1} &= V_{n+2}^4 - 4V_{n+2}^3V_n + 4(x^2-3)V_{n+2}^2V_n^2 - (x^4-6x^2+4)V_{n+2}V_n^3\\
&\quad - 2(x^2-2)V_{n+2}V_n^2V_{n-2} - (x^4-2x^2-1)V_n^4 + (x^4-2x^2)V_n^3V_{n-2} - 2V_n^2V_{n-2}^2;\\
x^4v_{4n+2} &= (x^2-2)V_{n+2}^4 - 8V_{n+2}^3V_n + (5x^2-12)V_{n+2}^2V_n^2 - 2(x^4-6x^2+4)V_{n+2}V_n^3\\
&\quad - 2x^2V_{n+2}V_n^2V_{n-2} - (2x^4-5x^2+2)V_n^4 + 2(x^4-2x^2)V_n^3V_{n-2}\\
&\quad -x^2V_n^2V_{n-2}^2; \qquad (4.2)\\
x^3v_{4n+3} &= -(x^2-3)V_{n+2}^4 + 4V_{n+2}^3V_n - x^2V_{n+2}^2V_n^2 + (x^4-6x^2+4)V_{n+2}V_n^3\\
&\quad + 4V_{n+2}V_n^2V_{n-2} + (x^4-3x^2+3)V_n^4 - (x^4-2x^2)V_n^3V_{n-2} + (x^2-2)V_n^2V_{n-2}^2,
\end{aligned}$$

where $d_n = d_n(x)$.

Their proofs, although simple and straightforward, involve a lot of basic algebra. So, for conciseness and convenience, we give the essence of the proofs of just two of them, namely, relationships (4.1) and (4.2). To confirm identity (4.1), replace x with $-ix$ in equation (1.5) and multiply the resulting equation with i^{4n+4}; this yields equation (4.1). To establish identity (4.2), replace x with $-ix$ in equation (1.9) and multiply the resulting equation with i^{4n+2}; this yields the desired result.

5. Chebyshev Implications

Using the Vieta-Chebyshev relationships in Table 1, we can also find the Chebyshev counterparts of identities (1.1) through (1.10). Again, in the interest of brevity, we omit them.

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