INFINITE SUMS INVOLVING GIBONACCI POLYNOMIAL PRODUCTS REVISITED

THOMAS KOSHY

Abstract. Using graph-theoretic tools, we confirm six sums involving gibonacci polynomial products explored in [3]. The graph-theoretic versions of their Pell counterparts follow from them.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [2].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $\Delta = \sqrt{x^2+4}$, $2\alpha(x) = x + \Delta$, and $2\beta(x) = x - \Delta$, and omit a lot of basic algebra.

1.1. Sums Involving Gibonacci Polynomial Products.

In [3], we investigated the following sums involving gibonacci polynomial products:

$$\sum_{n=0}^{m} \frac{x}{f_{2n}^2 + 1} = \frac{f_{2m+2}}{f_{2m+1}}; \tag{1.1}$$

$$\sum_{n=0}^{m} \frac{x^3 + 2x}{f_{2n+1}^2 + x^2} = \frac{f_{4m+4}}{f_{2m+3}f_{2m+1}}; \tag{1.2}$$

$$\frac{x^2+2}{f_n^4 - (-1)^n(x^2-1)f_n^2 - x^2} = \frac{1}{f_{n-2}f_{n-1}f_nf_{n+1}} + \frac{1}{f_{n-1}f_nf_{n+1}f_{n+2}}; \tag{1.3}$$

$$\sum_{n=0}^{m} \frac{x}{l_{2n}^2 + x^2} = \frac{f_{2m+2}}{\Delta^2 f_{2m+1}}; \tag{1.4}$$

$$\sum_{n=0}^{m} \frac{x^3 + 2x}{l_{2n+1}^2 + (x^2+2)^2} = \frac{f_{4m+4}}{\Delta^2 f_{2m+3}f_{2m+1}}; \tag{1.5}$$

$$\frac{x^2+2}{l_n^4 + (-1)^n(x^2-1)\Delta^2 l_n^2 - \Delta^4 x^2} = \frac{1}{l_{n-2}l_{n-1}l_nl_{n+1}} + \frac{1}{l_{n-1}l_nl_{n+1}l_{n+2}}. \tag{1.6}$$

Our goal is to confirm them using graph-theoretic techniques. To this end, we first develop the needed tools.

2. Graph-theoretic Tools

Consider the *Fibonacci digraph* D in Figure 1 with vertices v_1 and v_2, where a *weight* is assigned to each edge [2, 4].

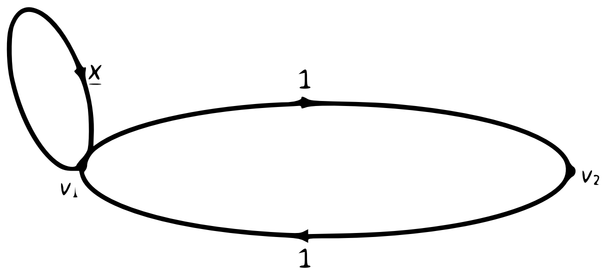

Figure 1. Weighted Fibonacci Digraph D_1

It follows by induction from its *weighted adjacency matrix* $Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ that

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \geq 1$ [2, 4].

A *walk* from vertex v_i to vertex v_j is a sequence v_i-e_i-v_{i+1}-$\cdots$-v_{j-1}-e_{j-1}-v_j of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is *closed* if $v_i = v_j$; otherwise, it is *open*. The *length* of a walk is the number of edges in the walk. The *weight* of a walk is the product of the weights of the edges along the walk.

The sum of the weights of closed walks of length n originating at v_1 in the digraph is f_{n+1} and that of those originating at v_2 is f_{n-1} [2, 4]. Consequently, the sum of the weights of all closed walks of length n in the digraph is $f_{n+1} + f_{n-1} = l_n$. These facts play a pivotal role in the graph-theoretic proofs.

Let A and B denote sets of walks of varying lengths originating at a vertex v. Then the sum of the weights of the elements (a, b) in the product set $A \times B$ is *defined* as the product of the sums of weights from each component. This definition can be extended to any finite number of components [4].

With these tools, we are now ready for the graph-theoretic proofs.

3. Graph-theoretic Proof

3.1. Confirmation of Identity (1.1).

Proof. Let A_n denote the sum of the weights of walks in the set A of closed walks of length $2n-1$ in the digraph originating at v_1, where $1 \leq n \leq m$. Then, the sum of the weights of the elements in the product set $A \times A$ is given by A_n^2. Let $S_n = A_n^2 + 1$, and

$$S_m = \sum_{n=1}^{m} \frac{x}{S_n} = \sum_{n=1}^{m} \frac{x}{A_n^2 + 1}.$$

We will now compute S_m in a different way. To this end, let w be an arbitrary walk in A. It can land at v_1 or v_2 at the $(n-1)$st step:

$$w = \underbrace{v_1 - \cdots - v}_{\text{subwalk of length } n-1} \quad \underbrace{v - \cdots - v_1}_{\text{subwalk of length } n}, \text{ where } v = v_1 \text{ or } v_2.$$

Table 1 shows the possible cases and the sum of the weights in each case. It follows from the table that the sum A_n of the weights of walks in A is given by $A_n = f_{n+1}f_n + f_nf_{n-1} = f_{2n}$. So $S_n = A_n^2 + 1 = f_{2n}^2 + 1$, and hence

$$S_m = \sum_{n=1}^{m} \frac{x}{f_{2n}^2 + 1}.$$

w lands at v_1 at the $(n-1)$st step?	w lands at v_1 at the $(2n-1)$st step?	sum of the weights of walks w
yes	yes	f_nf_{n+1}
no	yes	$f_{n-1}f_n$

Table 1: Sums of the Weights of Closed Walks Originating at v_1

Based on the initial values

$$\begin{aligned} S_1 &= \frac{x}{x^2+1} = \frac{f_2}{f_3}; \\ S_2 &= \frac{x^3+2x}{x^4+3x^2+1} = \frac{f_4}{f_5}; \text{ and} \\ S_3 &= \frac{x^5+4x^3+3x}{x^6+5x^4+6x^2+1} = \frac{f_6}{f_7}, \end{aligned}$$

of S_m, we conjecture that

$$\sum_{n=1}^{m} \frac{x}{f_{2n}^2 + 1} = \frac{f_{2m}}{f_{2m+1}}. \tag{3.1}$$

This can be confirmed using induction or recursion [3]. For example, let C_m and D_m denote the left side and right side of (3.1). By the addition formula $f_{a-b} = (-1)^b(f_af_{b-1} - f_{a-1}f_b)$ and the Cassini-like identity $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ [2], we have

$$\begin{aligned} D_m - D_{m-1} &= \frac{f_{2m}}{f_{2m+1}} - \frac{f_{2m-2}}{f_{2m-1}} \\ &= \frac{(-1)^{2m-1}(f_{2m+1}f_{2m-2} - f_{2m}f_{2m-1})}{f_{2m+1}f_{2m-1}} \\ &= \frac{f_{(2m+1)-(2m-1)}}{f_{2m+1}f_{2m-1}} \\ &= \frac{x}{f_{2m}^2 + 1} \\ &= C_m - C_{m-1}. \end{aligned}$$

So $C_m - D_m = C_{m-1} - D_{m-1} = \cdots = C_1 - D_1 = 0$, and hence $C_m = D_m$, as expected. Thus, the conjecture is true.

It follows from equation (3.1) that

$$\sum_{n=0}^{m} \frac{x}{f_{2n}^2 + 1} = \frac{f_{2m+2}}{f_{2m+1}},$$

as desired. □

It particular, we have

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n}^2 + 1} = \frac{1 + \sqrt{5}}{2},$$

as in [3, 5, 7].

3.2. **Confirmation of Identity** (1.2)**.**

Proof. Let A_n denote the sum of the weights of elements in the set A of closed walks of length $2n$ in the digraph originating at v_1, where we define $A_0 = 1$ and $0 \leq n \leq m$. Then, the sum of the weights of the elements in the product set $A \times A$ is given by A_n^2. Let $S_n = A_n^2 +$ (weight of the loop)$^2 = A_n^2 + x^2$, and

$$S_m = \sum_{n=0}^{m} \frac{x^3 + 2x}{S_n} = \sum_{n=0}^{m} \frac{x^3 + 2x}{A_n^2 + x^2}.$$

We will now compute A_n and hence S_m in a different way. Let w be an arbitrary walk in A. It can land at v_1 or v_2 at the nth step:

$$w = \underbrace{v_1 - \;\cdots\; - v}_{\text{subwalk of length } n} \; \underbrace{v - \;\cdots\; - v_1}_{\text{subwalk of length } n}, \text{ where } v = v_1 \text{ or } v_2.$$

It follows from Table 2 that the sum A_n of the weights of walks w in A is given by $A_n = f_{n+1}^2 + f_n^2 = f_{2n+1}$. So, $S_n = A_n^2 + x^2 = f_{2n+1}^2 + x^2$. Then,

$$S_m = \sum_{n=0}^{m} \frac{x^3 + 2x}{f_{2n+1}^2 + x^2}.$$

w lands at v_1 at the nth step?	w lands at v_1 at the $(2n)$th step?	sum of the weights of walks w
yes	yes	$f_{n+1}f_{n+1}$
no	yes	f_nf_n

Table 2: Sums of the Weights of Closed Walks Originating at v_1

With the initial values

$$\begin{aligned}
S_0 &= \frac{x^3 + 2x}{x^2 + 1} = \frac{f_4}{f_3 f_1}; \\
S_1 &= \frac{(x^3 + 2x)(x^4 + 4x^2 + 2)}{(x^4 + 3x^2 + 1)(x^2 + 1)} = \frac{f_8}{f_5 f_3}; \text{ and} \\
S_2 &= \frac{(x^5 + 4x^3 + 3x)(x^6 + 6x^4 + 9x^2 + 2)}{(x^6 + 5x^4 + 6x^2 + 1)(x^4 + 3x^2 + 1)} = \frac{f_{12}}{f_7 f_5},
\end{aligned}$$

of S_m, we conjecture that

$$\sum_{n=0}^{m} \frac{x^3+2x}{f_{2n+1}^2+x^2} = \frac{f_{4m+4}}{f_{2m+3}f_{2m+1}}.$$

This can be confirmed by recursion [3], as before; and gives the desired result. □

This result implies that

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}^2+1} = \frac{\sqrt{5}}{3},$$

as in [7].

Next, we pursue the sum in equation (1.3).

3.3. **Confirmation of Identity** (1.3)**.**

Proof. Let A_n denote the sum of the weights of closed walks of length n originating at v_1. Let $S_1 = A_{n-3}A_{n-2}A_{n-1}A_n$, $S_2 = A_{n-2}A_{n-1}A_nA_{n+1}$, and $S = \dfrac{1}{S_1} + \dfrac{1}{S_2}$. Because $A_n = f_{n+1}$, we then have

$$\begin{aligned} S &= \frac{1}{A_{n-3}A_{n-2}A_{n-1}A_n} + \frac{1}{A_{n-2}A_{n-1}A_nA_{n+1}} \\ &= \frac{1}{f_{n-3}f_{n-2}f_{n-1}f_n} + \frac{1}{f_{n-2}f_{n-1}f_nf_{n+1}}. \end{aligned}$$

Now, let $T_n = A_{n-3}A_{n-2}A_{n-1}A_nA_{n+1}$. Using the identities $f_{n+2} + f_{n-2} = (x^2+2)f_n$ and $f_{n+k}f_{n-k} - f_n^2 = (-1)^{n+k+1}f_k^2$ [2], we then have

$$\begin{aligned} S &= \frac{A_{n-3}}{A_{n-3}A_{n-2}A_{n-1}A_nA_{n+1}} + \frac{A_{n+1}}{A_{n-3}A_{n-2}A_{n-1}A_nA_{n+1}} \\ &= \frac{A_{n-3}}{T_n} + \frac{A_{n+1}}{T_n} \\ &= \frac{f_{n+2}+f_{n-2}}{f_{n-2}f_{n-1}f_nf_{n+1}f_{n+2}} \\ &= \frac{(x^2+2)f_n}{f_{n-2}f_{n-1}f_nf_{n+1}f_{n+2}} \\ &= \frac{x^2+2}{f_{n-2}f_{n-1}f_{n+1}f_{n+2}} \\ &= \frac{x^2+2}{(f_{n+2}f_{n-2})(f_{n+1}f_{n-1})} \\ &= \frac{x^2+2}{[f_n^2-(-1)^nx^2][f_n^2+(-1)^n]} \\ &= \frac{x^2+2}{f_n^4-(-1)^n(x^2-1)f_n^2-x^2}. \end{aligned}$$

Equating the two values of S yields the desired result. □

It follows from this result that

$$\sum_{n=3}^{\infty} \frac{1}{F_n^4-1} = \frac{35}{18} - \frac{5\sqrt{5}}{6},$$

as in [3, 5, 7].

Next, we confirm the Lucas sums in equations (1.4) through (1.6).

3.4. **Confirmation of Identity** (1.4).

Proof. Let B_n denote the sum of the weights of the elements in the set B of closed walks of length $2n$ in the digraph, where we define $B_0 = 2$ and $0 \leq n \leq m$. Then the sum of the weights of the elements in the product set $B \times B$ is B_n^2. Let $S_n = B_n^2 +$ (weight of the loop)$^2 = B_n^2 + x^2$, and

$$S_m = \sum_{n=0}^{m} \frac{x}{S_n} = \sum_{n=0}^{m} \frac{x}{B_n^2 + x^2}.$$

We will now compute B_n and hence S_m in a different way. To this end, let w be an arbitrary element in B.

Case 1. Suppose w originates at v_1. It can land at v_1 or v_2 at the nth step:

$$w = \underbrace{v_1 - \cdots - v}_{\text{subwalk of length } n} \ \underbrace{v - \cdots - v_1}_{\text{subwalk of length } n}, \text{ where } v = v_1 \text{ or } v_2.$$

It follows from Table 3 that the sum of the weights of such walks w is $f_{n+1}^2 + f_n^2 = f_{2n+1}$.

w lands at v_1 at the nth step?	w lands at v_1 at the $(2n)$th step?	sum of the weights of walks w
yes	yes	$f_{n+1}f_{n+1}$
no	yes	f_nf_n

Table 3: Sums of the Weights of Closed Walks Originating at v_1

Case 2. Suppose w originates at v_2. It can land at v_1 or v_2 at the nth step:

$$w = \underbrace{v_2 - \cdots - v}_{\text{subwalk of length } n} \ \underbrace{v - \cdots - v_2}_{\text{subwalk of length } n}, \text{ where } v = v_1 \text{ or } v_2.$$

It follows from Table 4 that the sum of the weights of such walks w is $f_n^2 + f_{n-1}^2 = f_{2n-1}$.

w lands at v_1 at the nth step?	w lands at v_2 at the $(2n)$th step?	sum of the weights of walks w
yes	yes	f_nf_n
no	yes	$f_{n-1}f_{n-1}$

Table 4: Sums of the Weights of Closed Walks Originating at v_2

Thus, the sum B_n of the weights of all closed walks w is given by $B_n = f_{2n+1} + f_{2n-1} = l_{2n}$. Consequently,

$$S_m = \sum_{n=0}^{m} \frac{x}{l_{2n}^2 + x^2}.$$

Using the initial values

$$\begin{aligned} S_0 &= \frac{x}{\Delta^2} = \frac{f_2}{\Delta^2 f_1}; \\ S_1 &= \frac{x^3+2x}{\Delta^2(x^2+1)} = \frac{f_4}{\Delta^2 f_3}; \text{ and} \\ S_2 &= \frac{x^5+4x^3+3x}{\Delta^2(x^4+3x^2+1)} = \frac{f_6}{\Delta^2 f_5}, \end{aligned}$$

of S_m, we conjecture that

$$\sum_{n=0}^{m} \frac{x}{l_{2n}^2 + x^2} = \frac{f_{2m+2}}{\Delta^2 f_{2m+1}}.$$

We can establish its validity by recursion [3]. Let C_m and D_m denote the left side and right side of this equation, respectively. Using the addition formula $f_{a-b} = (-1)^b(f_a f_{b-1} - f_{a-1} f_b)$ and the identity $l_n^2 - \Delta^2 f_n^2 = 4(-1)^n$ [2], we then have

$$\begin{aligned} D_m - D_{m-1} &= \frac{f_{2m+2}}{\Delta^2 f_{2m+1}} - \frac{f_{2m}}{\Delta^2 f_{2m-1}} \\ &= \frac{f_{2m+2} f_{2m-1} - f_{2m+1} f_{2m}}{\Delta^2 f_{2m+1} f_{2m-1}} \\ &= \frac{f_{(2m+2)-2m}}{\Delta^2 \left(f_{2m}^2 + 1\right)} \\ &= \frac{x}{l_{2m}^2 + x^2} \\ &= C_m - C_{m-1}. \end{aligned}$$

Then, $C_m - D_m = C_{m-1} - D_{m-1} = \cdots = C_0 - D_0 = 0$. So $C_m = D_m$, as expected.

Thus, the conjecture is true, as expected. □

In particular, we have

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n}^2 + 1} = \frac{1+\sqrt{5}}{10},$$

as in [3].

We now confirm equation (1.5).

3.5. **Confirmation of Identity** (1.5)**.**

Proof. Let C_n denote the sum of the weights of elements in the set C of all closed walks of length $2n+1$ in the digraph, where $0 \le n \le m$. Then, the sum of the weights of the elements in the product set $C \times C$ is C_n^2. Let $S_n = C_n^2 + (x^2+2)^2$ and

$$S_m = \sum_{n=0}^{m} \frac{x^3+2x}{S_n} = \sum_{n=0}^{m} \frac{x^3+2x}{C_n^2 + (x^2+2)^2}.$$

To compute S_m in a different way, we let w be an arbitrary walk in C.

Case 1. Suppose w originates at v_1. It can land at v_1 or v_2 at the nth step:

$$w = \underbrace{v_1 - \ \cdots \ - v}_{\text{subwalk of length } n} \ \underbrace{v - \ \cdots \ - v_1}_{\text{subwalk of length } n+1}, \text{ where } v = v_1 \text{ or } v_2.$$

Table 5 implies that the sum of the weights of such walks w is given by $f_{n+2}f_{n+1}+f_{n+1}f_n = f_{2n+2}$.

w lands at v_1 at the nth step?	w lands at v_1 at the $(2n+1)$st step?	sum of the weights of walks w
yes	yes	$f_{n+1}f_{n+2}$
no	yes	f_nf_{n+1}

Table 5: Sums of the Weights of Closed Walks Originating at v_1

Case 2. Suppose w originates at v_2. It can land at v_1 or v_2 at the nth step:

$$w = \underbrace{v_2 - \ \cdots \ - v}_{\text{subwalk of length } n} \quad \underbrace{v - \ \cdots \ - v_2}_{\text{subwalk of length } n+1}, \text{ where } v = v_1 \text{ or } v_2.$$

It follows from Table 6 that the sum of the weights of such walks is $f_{n+1}f_n + f_nf_{n-1} = f_{2n}$.

w lands at v_1 at the nth step?	w lands at v_2 at the $(2n+1)$st step?	sum of the weights of walks w
yes	yes	f_nf_{n+1}
no	yes	$f_{n-1}f_n$

Table 6: Sums of the Weights of Closed Walks Originating at v_1

Thus, the sum C_n of the weights of all walks in C is given by $C_n = f_{2n+2} + f_{2n} = l_{2n+1}$. Consequently,

$$S_m = \sum_{n=0}^{m} \frac{x^3+2x}{l_{2n+1}^2 + (x^2+2)^2}.$$

It then follows that

$$\begin{aligned}
S_0 &= \frac{x^3+2x}{\Delta^2(x^2+1)} = \frac{f_4}{\Delta^2 f_3 f_1};\\
S_1 &= \frac{(x^3+2x)(x^4+4x^2+2)}{\Delta^2(x^4+3x^2+1)(x^2+1)} = \frac{f_8}{\Delta^2 f_5 f_3}; \text{ and}\\
S_2 &= \frac{(x^5+4x^3+3x)(x^6+6x^4+9x^2+2)}{\Delta^2(x^4+3x^2+1)(x^6+5x^4+6x^2+1)} = \frac{f_{12}}{\Delta^2 f_7 f_5}.
\end{aligned}$$

Based on these initial values of S_n, we conjecture that

$$S_m = \frac{f_{4m+4}}{\Delta^2 f_{2m+3} f_{2m+1}}.$$

We can confirm this using recursion, as in [3].

Equating the two values of S_m yields the desired result. □

This result implies that

$$\sum_{k=0}^{\infty} \frac{1}{L_{2n+1}^2 + 9} = \frac{\sqrt{5}}{15},$$

as in [3].

Finally, we confirm equation (1.6).

3.6. **Confirmation of Identity** (1.6).

Proof. Let B_n denote the sum of the weights of all closed walks of length n in the digraph. We also let $S_1^* = B_{n-2}B_{n-1}B_nB_{n+1}$, $S_2^* = B_{n-1}B_nB_{n+1}B_{n+2}$, and $S = \frac{1}{S_1^*} + \frac{1}{S_2^*}$. Because $B_n = l_n$, we have

$$\begin{aligned} S &= \frac{1}{B_{n-2}B_{n-1}B_nB_{n+1}} + \frac{1}{B_{n-1}B_nB_{n+1}B_{n+2}} \\ &= \frac{1}{l_{n-2}l_{n-1}l_nl_{n+1}} + \frac{1}{l_{n-1}l_nl_{n+1}l_{n+2}}. \end{aligned}$$

We now compute S in a different way. Let $T_n = B_{n-2}B_{n-1}B_nB_{n+1}B_{n+2}$. Using the identities $l_{n+2} + l_{n-2} = (x^2+2)l_n$ and $l_{n+k}l_{n-k} - l_n^2 = (-1)^{n+k}\Delta^2 f_k^2$ [2], we then have

$$\begin{aligned} S &= \frac{B_{n+2}}{B_{n-2}B_{n-1}B_nB_{n+1}B_{n+2}} + \frac{B_{n-2}}{B_{n-2}B_{n-1}B_nB_{n+1}B_{n+2}} \\ &= \frac{B_{n+2}+B_{n-2}}{T_n} \\ &= \frac{l_{n+2}+l_{n-2}}{l_{n-2}l_{n-1}l_nl_{n+1}l_{n+2}} \\ &= \frac{(x^2+2)l_n}{l_{n-2}l_{n-1}l_nl_{n+1}l_{n+2}} \\ &= \frac{x^2+2}{(l_{n+2}l_{n-2})(l_{n+1}l_{n-1})} \\ &= \frac{x^2+2}{[l_n^2+(-1)^n\Delta^2x^2][l_n^2-(-1)^n\Delta^2]} \\ &= \frac{x^2+2}{l_n^4+(-1)^n(x^2-1)\Delta^2l_n^2-\Delta^4x^2}. \end{aligned}$$

This value of S, coupled with the earlier one, gives the desired result. □

It then follows that

$$\sum_{n=3}^{\infty} \frac{1}{L^4-25} = \frac{5}{63} - \frac{\sqrt{5}}{30},$$

as in [3, 6, 8].

4. Conclusion

Because $b_n(x) = g_n(2x)$, the graph-theoretic confirmations of the Pell versions of the summation formulas in equations (1.1) through (1.6) follow from the above proofs. In the interest of brevity, we omit them.

5. Acknowledgment

The author thanks the reviewer for a careful reading of the article, and for constructive suggestions and encouraging words.

References

[1] M. Bicknell, *A primer for the Fibonacci numbers: Part VII*, The Fibonacci Quarterly, **8.4** (1970), 407–420.
[2] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Volume II, Wiley, Hoboken, New Jersey, 2019.
[3] T. Koshy, *Infinite sums involving gibonacci polynomial products*, The Fibonacci Quarterly, **59.3** (2021), 237–245.
[4] T. Koshy, *A recurrence for gibonacci cubes with graph-theoretic confirmations*, The Fibonacci Quarterly, **57.2** (2019), 139–147.
[5] H. Ohtsuka, *Problem H-783*, The Fibonacci Quarterly, **54.1** (2016), 87.
[6] À. Plaza, *Problem H-810*, The Fibonacci Quarterly, **55.3** (2017), 282.
[7] À. Plaza, *Solution to Problem H-783*, The Fibonacci Quarterly, **56.1** (2018), 90–91.
[8] À. Plaza, *Solution to Problem H-810*, The Fibonacci Quarterly, **57.3** (2019), 284.

MSC2020: Primary 05C20, 05C22, 11B39, 11B83, 11C08

Department of Mathematics, Framingham State University, Framingham, MA 01701
Email address: tkoshy@emeriti.framingham.edu