NONDECREASING DEUTSCH PATHS

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Abstract. A variation of Dyck paths allows for down-steps of arbitrary length, not just one. This is motivated by ideas published by Emeric Deutsch around the turn of the millenium. We are interested in the subclass of them where the sequence of the levels of valleys is nondecreasing. This was studied around 20 years ago in the classic case.

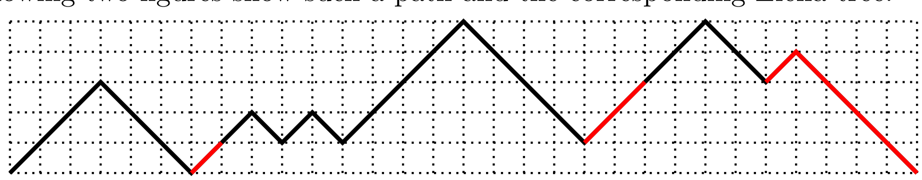
1. Introduction

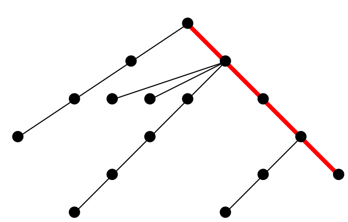
The paper [1] introduced the subfamily of Dyck paths such that the level of the valleys is nondecreasing when scanning the path from left to right (nondecreasing Dyck paths). The generating function

$$\frac{1-2z}{1-3z+z^2}=\sum_{n\ge0}F_{2n-1}z^n,$$

where $2n$ denotes the length of the path, and F_m are Fibonacci numbers, was already given. In [8], the present writer found that the usual translation of Dyck paths into plane trees, when restricted to the subfamily of nondecreasing trees, leads to a simple tree structure, called Elena trees, from which it is easy to find generating functions, bijections, and consider various parameters of them. The paper [4] is also of relevance here. The study of various parameters was picked up again in [6].

The following two figures show such a path and the corresponding Elena tree.





When the Dyck path leaves a certain level and never comes back to it (except for the final home-run), we draw a red edge. In the corresponding tree, this is the backbone, and the rest are just paths, hanging down.

Recently, following ideas of Emeric Deutsch [3], an extension of Dyck paths was studied in [9] (Deutsch paths). Here, instead of just the usual down-step $(1,-1)$, all possible down-steps $(1,-j)$ for $j=1,2,3,\dots$ are allowed.

The present note is dedicated to the description and characterization of nondecreasing Deutsch paths.

2. Enumeration

A sequence of up-steps followed by a sequence of down-steps ending at the same level is easy to enumerate for Dyck paths: it exists only if the length is $2k$, and then there is just one such object.

For Deutsch paths, this is a bit trickier, because of the various down-steps that are available and that can be combined. We compute this number when the number of up-steps equals $j \ge 1$ and the total number of steps is $k > j$:

$$\sum_{1\le j<k} [z^j]\left(\frac{z}{1-z}\right)^{k-j} = \sum_{1\le j<k} [z^{2j-k}]\left(\frac{1}{1-z}\right)^{k-j} = \sum_{1\le j<k} \binom{j-1}{2j-k}$$
$$= \sum_{0\le j<k-1} \binom{k-j-2}{j} = F_{k-1}.$$

The generating function of these numbers is

$$\sum_{k\ge 2} z^k F_{k-1} = \frac{z^2}{1-z-z^2}.$$

A sequence of such objects (or, equivalently, a bundle of such paths) has generating function

$$\sum_{k\ge 0}\left(\frac{z^2}{1-z-z^2}\right)^k = \frac{1}{1-\dfrac{z^2}{1-z-z^2}} = \frac{1-z-z^2}{(1+z)(1-2z)} =: H.$$

This allows us to compute the total number of objects:

$$\sum_{1\le j<k} H^j \binom{j-1}{k-1-j} z^k = \sum_{1\le j} H^j z^{j+1}(1+z)^{j+1} = \frac{z^2(1-z-z^2)}{(1+z)(1-z)(1-2z-z^2)}.$$

We find it convenient to allow the empty path as well, which means that we add 1 and get

$$\frac{1}{4(1-z)} + \frac{1}{4(1+z)} + \frac{1}{2}\frac{1-2z}{1-2z-z^2} = \cfrac{1}{1-\cfrac{z^2}{1-\cfrac{z}{1-\cfrac{z}{1-z^2}}}}.$$

We will explain this formula again in the next section in a more combinatorial fashion. Let $a = 1+\sqrt{2}$, $b = 1-\sqrt{2}$. Because

$$\frac{1}{2}\frac{1-2z}{1-2z-z^2} = \frac{1}{2}\frac{1-2z}{(1-az)(1-bz)} = \left(\frac{1}{4}-\frac{1}{8}\sqrt{2}\right)\frac{1}{1-az} + \left(\frac{1}{4}+\frac{1}{8}\sqrt{2}\right)\frac{1}{1-bz},$$

we found the number of nondecreasing Deutsch paths of length n:

$$\frac{1}{4}(1+(-1)^n) + \frac{1}{4}(a^n+b^n) - \frac{1}{4\sqrt{2}}(a^n-b^n).$$

The numbers

$$\frac{a^n+b^n}{2}$$

are sequence A001333 in [7], and the sequence A000129 (Pell numbers)

$$\frac{a^n-b^n}{2\sqrt{2}}$$

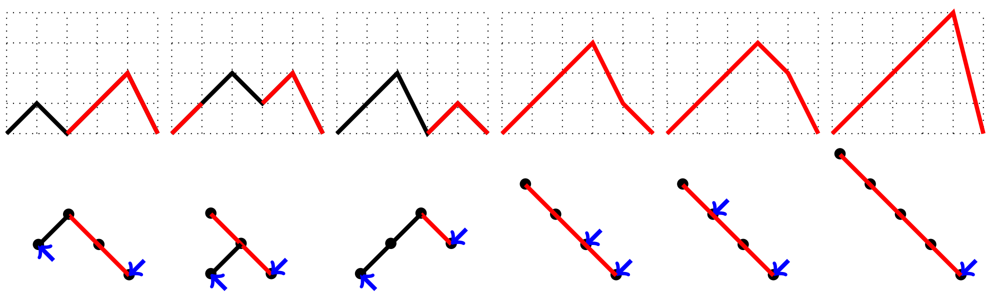
is even more famous.

3. Combinatorial Considerations

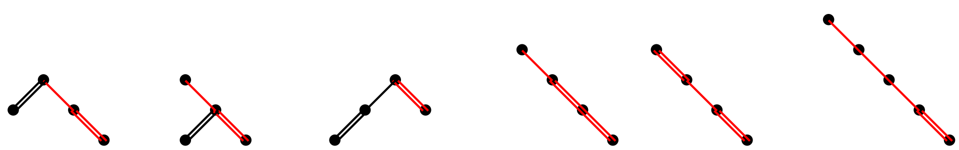
For increasing Dyck paths, it is enough to count the number of up-steps, because the number of down-steps is the same. This is no longer the case in the Deutsch model. So, we have to count the total number of steps, which is the length of the (nondecreasing) Deutsch path.

When we form the associated tree, we must somehow indicate which down-step was used. One way of doing this is to use arrows to indicate where a new down-step start.

To explain the concepts, we prepared a list of all nondecreasing Deutsch paths of length five and the corresponding trees.



We notice that the total number of edges plus the number of blue arrows is always five, because each blue arrow indicates where a new down-step starts, and it must be counted just once. Alternatively, we can work with double edges, so these edges need to be counted twice. In the original Dyck model, every edge would be counted twice, leading to the length of the Dyck path.



We recall tilings with squares and dominoes (length 2) from [2] and that length n tilings are enumerated by the Fibonacci number F_{n+1}.

Note that $\frac{z}{1-z^2}$ enumerates sequences of type $122\dots2$. Hence,

$$\frac{1}{1-\dfrac{z}{1-z^2}} = 1 + \sum_{n\ge1} [\text{tilings of length } n, \text{ first tile is a square}] z^n.$$

Gluing one square to this starting square, the tilings of length $n+1$ start with a domino. If we delete the domino, it is a general tiling of length $n-1$, enumerated by the Fibonacci number F_n.

So,

$$\frac{1}{1-\dfrac{z}{1-z^2}} = 1 + \sum_{n\ge1} F_n z^n \quad \text{and} \quad \frac{z}{1-\dfrac{z}{1-z^2}} = z + \sum_{n\ge2} F_{n-1} z^n.$$

That links the formula

$$\frac{z^2}{1-z-z^2} = \sum_{n\ge2} F_{n-1} z^n$$

from before to a tiling type argument.

Using the description as a tree with single or double edges, we derive another form of the total generating function, which looks a little different:

$$1+\cfrac{z^2}{1-\cfrac{z+z^2}{1-\cfrac{z^2}{1-z-z^2}}}\cdot\cfrac{1}{1-\cfrac{z^2}{1-z-z^2}}.$$

Now, we explain the continued fraction

$$\cfrac{1}{1-\cfrac{z^2}{1-\cfrac{z}{1-\cfrac{z}{1-z^2}}}}$$

in a combinatorial way. We use tools that are common in combinatorics for words, in particular, the symbolic equation $(a+b)^*=b^*(ab^*)^*$, valid for any two nonempty formal languages or, power series that start with 0. We write p for a path and e for a single edge and E for a double edge. As we discussed earlier, the generating function

$$\cfrac{z}{1-\cfrac{z}{1-z^2}}$$

corresponds to $e+p$. We split the backbone of the tree (the edges depicted in red) as $[(e\dots e)E]\dots[(e\dots e)E]$. What hangs on the first node is enumerated by b^*. Assume that the first edge of the backbone is e. Then, the generating function according to this edge and the bundle of paths hanging down from node number 2 is enumerated by ep^*. Should the next edge also be e, it is enumerated via another ep^*, and so on. If the first group of single edges in the backbone has l such edges, it is enumerated via $p^*(ep^*)^l$. The number l can be any integer ≥ 0. However, the group ends with an E, and this one contributes z^2. This explains the term

$$\Gamma=\cfrac{z^2}{1-\cfrac{z}{1-\cfrac{z}{1-z^2}}}.$$

But, there is another '$*$' coming in, because the backbone consists of an arbitrary number of such groups. That explains the final formula

$$\frac{1}{1-\Gamma}.$$

We switched freely between symbolic expressions (words/languages) and generating functions, where the variable z counts an edge. The book [5] describes how this works.

The decomposition of trees, as just described, is by no means obvious, and without seeing the nice continued fraction, might stay undetected.

4. Conclusion

We defined nondecreasing Deutsch paths and found the corresponding generating function. This was eased by translating it into a tree model using simple and double edges. The explicit enumeration formula involves Pell numbers and (as an intermediate step) Fibonacci numbers.

Parameters like the (average of the) number of double edges, degree of the root, length of the occurring paths, etc., can be computed easily using a second variable in the generating function. The manipulations related to the rational functions that one gets in this way are best done by a computer. Because this is a routine procedure, we refrain from doing this here.

References

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MSC2020: 05A15, 05A19, 11B39

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