SOME SERIES INVOLVING PRODUCTS BETWEEN THE HARMONIC NUMBERS AND THE FIBONACCI NUMBERS

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Abstract. We find various ordinary generating functions for sequences involving products between the harmonic numbers and the Fibonacci numbers. These are then used to establish some classes of series associated with these products. Our approach is based on applying well-known ordinary generating functions for the harmonic numbers.

1. Introduction

The nth harmonic number is defined by

$$H_n = \sum_{k=1}^{n} \frac{1}{k}. \tag{1.1}$$

By convention, $H_0 \equiv 0$. The study of infinite series involving harmonic numbers was initiated by Euler in the mid-18th century, but it was not until the mid-1990s and the work of Bailey, Borwein, and Girgensohn [1] and Borwein, Borwein, and Girgensohn [3] that interest in series of this type was revived. Since this time, infinite series containing harmonic numbers have been well studied, with the literature on such sums now vast (for a sample, see [12]). However, with perhaps two exceptions, the author is not aware of any attempt to establish series that involve the product between the harmonic numbers and the Fibonacci numbers. This is surprising considering the variety of series containing the harmonic numbers that have been investigated to date. Chen, in two papers spaced 10 years apart [4, 5], gave three series for the related problem of a product between the Fibonacci numbers and a variant harmonic number term Λ_n, where

$$\Lambda_n = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} = \sum_{k=1}^{n} \frac{1}{2k-1} = H_{2n} - \frac{1}{2} H_n.$$

As we shall see, numerous interesting series involving the product between the harmonic numbers and the Fibonacci numbers can be found. Here, we apply Lehmer's definition [8] regarding when a particular series can be considered interesting. Lehmer writes a series can be considered *interesting* if its sum can be expressed in closed form in terms of well-known constants. We would add the number of constants appearing in the expression found should not be too long or complicated in appearance.

It is the purpose of the present paper to establish a number of ordinary generating functions for sequences containing products between the harmonic numbers and the Fibonacci numbers. These are found using well-known generating functions involving the harmonic numbers. As a demonstration of the general method, classes of series of the form

$$\sum_{n=1}^{\infty} \frac{(\pm 1)^n H_{a(n)} F_{b(n)}}{2^{b(n)}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(\pm 1)^n H_{a(n)} F_{b(n)}}{2^{b(n)}(b(n)+1)}, \tag{1.2}$$

are given. Here, $a(n)$ and $b(n)$ are equal to n or $2n$, whereas F_n are the Fibonacci numbers defined, as usual, through the recurrence relation $F_n = F_{n-1} + F_{n-2}$, with $F_0 = 0$ and $F_1 = 1$. All sums found are believed to be new with many considered to be interesting.

Throughout this paper, the golden ratio, having the numerical value of $(1+\sqrt{5})/2$, is denoted φ. We shall often have a need for the following algebraic properties of φ.

$$1 - 1/\varphi^2 = 1/\varphi, \tag{1.3a}$$

$$1 + \varphi = \varphi^2, \tag{1.3b}$$

$$\varphi - 1/\varphi = 1, \tag{1.3c}$$

$$\sqrt{5} = 2\varphi - 1, \tag{1.3d}$$

$$5 + \sqrt{5} = 2\varphi\sqrt{5}, \tag{1.3e}$$

$$\text{and} \quad 5 - \sqrt{5} = 2\sqrt{5}/\varphi. \tag{1.3f}$$

2. Some Ordinary Generating Functions

In this section, we establish a number of ordinary generating functions for sequences involving the product between the harmonic numbers and the Fibonacci numbers.

Theorem 2.1. *For $|x| < 1/\varphi$, the ordinary generating function for the sequence $\{H_n F_n\}_{n\geqslant 1}$ is*

$$\sum_{n=1}^{\infty} H_n F_n x^n = \frac{1}{\sqrt{5}} \left[\frac{\varphi}{\varphi + x} \log\left(\frac{\varphi + x}{\varphi}\right) - \frac{1}{1 - \varphi x} \log\left(1 - \varphi x\right) \right]. \tag{2.1}$$

Proof. Recalling the ordinary generating function for the sequence $\{H_n\}_{n\geqslant 1}$, namely [6, 1.513(6), p.52]

$$\sum_{n=1}^{\infty} H_n x^n = -\frac{\log(1-x)}{1-x}, \quad |x| < 1, \tag{2.2}$$

enforcing, in turn, substitutions of $x \mapsto \varphi x$ and $x \mapsto -\frac{x}{\varphi}$ into the above ordinary generating function for the sequence of harmonic numbers gives

$$\sum_{n=1}^{\infty} H_n \varphi^n x^n = -\frac{\log(1-\varphi x)}{1-\varphi x}, \tag{2.3}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n x^n}{\varphi^n} = -\frac{\varphi \log\left(\frac{\varphi + x}{\varphi}\right)}{\varphi + x}, \tag{2.4}$$

respectively. From Binet's formula

$$F_n = \frac{1}{\sqrt{5}} \left(\varphi^n - \frac{(-1)^n}{\varphi^n} \right),$$

combining (2.3) and (2.4) yields the desired result. □

Theorem 2.2. *For $|x| < 1/\varphi$, the ordinary generating function for the sequence $\left\{\frac{1+(-1)^n}{2} H_{\frac{n}{2}} F_n\right\}_{n\geqslant 1}$ is*

$$\sum_{n=1}^{\infty} H_n F_{2n} x^{2n} = \frac{1}{\sqrt{5}} \left[\frac{\varphi^2}{\varphi^2 - x^2} \log\left(\frac{\varphi^2 - x^2}{\varphi^2}\right) - \frac{1}{1 - \varphi^2 x^2} \log(1 - \varphi^2 x^2) \right]. \tag{2.5}$$

Proof. On replacing x with x^2 in (2.2), we have

$$\sum_{n=1}^{\infty} H_n x^{2n} = -\frac{\log(1-x^2)}{1-x^2}, \quad |x| < 1.$$

Enforcing, in turn, substitutions of $x \mapsto \varphi x$ and $x \mapsto \frac{x}{\varphi}$ into the above ordinary generating function, after combining the results with Binet's formula in a manner similar to what was done in Theorem 2.1, the desired result follows. □

We next give, as a lemma, the generating function for the sequence $\{H_{2n}\}_{n\geqslant 1}$, as it does not seem to be well known.

Lemma 2.3. *For $|x| < 1$, the ordinary generating function for the sequence $\{H_{2n}\}_{n\geqslant 1}$ is*

$$\sum_{n=1}^{\infty} H_{2n} x^n = \begin{cases} \dfrac{2\sqrt{x}\operatorname{arctanh}(\sqrt{x}) - \log(1-x)}{2(1-x)}, & 0 \leqslant x < 1; \\ \dfrac{-2\sqrt{-x}\arctan(\sqrt{-x}) - \log(1-x)}{2(1-x)}, & -1 < x < 0. \end{cases} \tag{2.6}$$

Proof. Let

$$f(x) = \sum_{n=1}^{\infty} H_{2n} x^{2n} \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} H_{2n-1} x^{2n-1}.$$

Observe for $0 \leqslant x < 1$, the desired ordinary generating function will be $f(\sqrt{x})$, whereas for $-1 < x < 0$, the desired ordinary generating function will be $f(\sqrt{-x})$.

From the recurrence relation for the harmonic numbers $H_{2n} = H_{2n-1} + \frac{1}{2n}$, we can rewrite $f(x)$ as

$$\begin{aligned} f(x) &= x \sum_{n=1}^{\infty} H_{2n} x^{2n-1} \\ &= x \sum_{n=1}^{\infty} \left(H_{2n-1} + \frac{1}{2n} \right) x^{2n-1} \\ &= x \sum_{n=1}^{\infty} H_{2n-1} x^{2n-1} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{x^{2n}}{n} \\ &= x g(x) - \frac{1}{2} \log(1-x^2). \end{aligned} \tag{2.7}$$

Furthermore, from properties for power series, we have

$$f(x) = \frac{1}{2} \sum_{n=1}^{\infty} H_n x^n + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n H_n x^n,$$

and

$$\begin{aligned} g(x) &= \frac{1}{2} \sum_{n=1}^{\infty} H_{n-1} x^{n-1} + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n H_{n-1} x^{n-1} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} H_n x^n - \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n H_n x^n, \end{aligned}$$

after reindexing $n \mapsto n+1$ on both sums and recalling $H_0 = 0$. Adding gives

$$f(x) + g(x) = \sum_{n=1}^{\infty} H_n x^n = -\frac{\log(1-x)}{1-x}. \tag{2.8}$$

Eliminating $g(x)$ from (2.7) and (2.8) yields

$$f(x) = \frac{2x \operatorname{arctanh}(x) - \log(1-x^2)}{2(1-x^2)}. \tag{2.9}$$

For $0 \leqslant x < 1$, enforcing a substitution of $x \mapsto \sqrt{x}$ in (2.9) yields the desired result. For $-1 < x < 0$, enforcing a substitution of $x \mapsto i\sqrt{-x}$, where i is the imaginary unit in (2.9), results in the term

$$i\sqrt{-x} \operatorname{arctanh}(i\sqrt{-x}) = -\sqrt{-x} \arctan(\sqrt{-x}),$$

from which the desired result then follows. □

Theorem 2.4. *If $|x| < 1/\varphi$, the ordinary generating function for the sequence $\{H_{2n}F_n\}_{n\geqslant 1}$ is*

$$\sum_{n=1}^{\infty} H_{2n}F_n x^n = \begin{cases} \frac{\varphi}{2\sqrt{5}(\varphi+x)}\left[2\sqrt{\frac{x}{\varphi}} \arctan\left(\sqrt{\frac{x}{\varphi}}\right) + \log\left(\frac{\varphi+x}{\varphi}\right)\right] \\ \quad + \frac{1}{2\sqrt{5}(1-\varphi x)}\left[2\sqrt{\varphi x} \operatorname{arctanh}(\sqrt{\varphi x}) + \log(1-\varphi x)\right], & 0 \leqslant x < \frac{1}{\varphi}; \\ -\frac{1}{2\sqrt{5}(1-\varphi x)}\left[2\sqrt{-\varphi x} \arctan(\sqrt{-\varphi x}) + \log(1-\varphi x)\right] \\ \quad - \frac{\varphi}{2\sqrt{5}(\varphi+x)}\left[2\sqrt{\frac{-x}{\varphi}} \operatorname{arctanh}\left(\sqrt{\frac{-x}{\varphi}}\right) + \log\left(\frac{\varphi+x}{\varphi}\right)\right], & -\frac{1}{\varphi} < x < 0. \end{cases} \tag{2.10}$$

Proof. The proof proceeds as in Theorem 2.1, applied to the ordinary generating function given in Lemma 2.3. □

Theorem 2.5. *If $|x| \leqslant 1/\varphi$ and $x \neq 1/\varphi$, the ordinary generating function for the sequence $\{H_nF_n/(n+1)\}_{n\geqslant 1}$ is*

$$\sum_{n=1}^{\infty} \frac{H_nF_n}{n+1} x^n = \frac{1}{2\sqrt{5}x}\left[\frac{1}{\varphi}\log^2(1-\varphi x) + \varphi \log^2\left(\frac{\varphi+x}{\varphi}\right)\right]. \tag{2.11}$$

Proof. Integrating the ordinary generating function given by (2.2) from 0 to x immediately yields

$$\sum_{n=1}^{\infty} \frac{H_n x^n}{n+1} = \frac{\log^2(1-x)}{2x}, \quad |x| \leqslant 1, x \neq 1. \tag{2.12}$$

We now proceed as in Theorem 2.1, applied to the ordinary generating function given above, thus completing the proof. □

Lemma 2.6. *If* $|x| \leqslant 1$ *and* $x \neq 1$*, the ordinary generating function for the sequence* $\{H_{2n}/(n+1)\}_{n\geqslant 1}$ *is*

$$\sum_{n=1}^{\infty} \frac{H_{2n}x^n}{n+1} = \begin{cases} \dfrac{\operatorname{arctanh}^2(\sqrt{x}) - 2\sqrt{x}\operatorname{arctanh}(\sqrt{x}) - \log(1-x) + \frac{1}{4}\log^2(1-x)}{x}, & 0 \leqslant x < 1; \\ \dfrac{-\arctan^2(\sqrt{-x}) + 2\sqrt{-x}\arctan(\sqrt{-x}) - \log(1-x) + \frac{1}{4}\log^2(1-x)}{x}, & -1 \leqslant x < 0. \end{cases} \tag{2.13}$$

Proof. For $0 \leqslant x < 1$, replacing x with t in (2.6), before integrating from 0 to x, gives

$$\sum_{n=1}^{\infty} \frac{H_{2n}x^{n+1}}{n+1} = \int_0^x \frac{\sqrt{t}\operatorname{arctanh}(\sqrt{t})}{1-t}\,dt - \frac{1}{2}\int_0^x \frac{\log(1-t)}{1-t}\,dt.$$

The second of the integrals appearing to the right of the equality is elementary. Here,

$$\int_0^x \frac{\log(1-t)}{1-t}\,dt = -\frac{1}{2}\log^2(1-x).$$

In the first of the integrals, enforcing a substitution of $t \mapsto t^2$, followed by a partial fraction decomposition, gives

$$\begin{aligned} \int_0^x \frac{\sqrt{t}\operatorname{arctanh}(\sqrt{t})}{1-t}\,dt &= 2\int_0^{\sqrt{x}} \frac{t^2\operatorname{arctanh}(t)}{1-t^2}\,dt \\ &= 2\int_0^{\sqrt{x}} \frac{\operatorname{arctanh}(t)}{1-t^2}\,dt - 2\int_0^{\sqrt{x}} \operatorname{arctanh}(t)\,dt \\ &= \operatorname{arctanh}^2(\sqrt{x}) - 2\sqrt{x}\operatorname{arctanh}(\sqrt{x}) - \log(1-x). \end{aligned}$$

Combining the results found for the two integrals completes the proof. For $-1 \leqslant x < 0$, replacing $\sqrt{x}$ with $i\sqrt{-x}$, where i is the imaginary unit, results in the terms

$$\operatorname{arctanh}^2(i\sqrt{-x}) = -\arctan^2(\sqrt{-x}),$$

and

$$i\sqrt{-x}\operatorname{arctanh}(i\sqrt{-x}) = -\sqrt{-x}\arctan(\sqrt{-x}),$$

and gives the desired result. □

Theorem 2.7. *If $|x| \leqslant 1/\varphi$ and $x \neq 1/\varphi$, the ordinary generating function for the sequence $\{H_{2n}F_n/(n+1)\}_{n\geqslant 1}$ is*

$$\sum_{n=1}^{\infty} \frac{H_{2n}F_n x^n}{n+1}$$
$$= \begin{cases} \frac{1}{\varphi\sqrt{5}x}\left[\operatorname{arctanh}^2(\sqrt{\varphi x}) - 2\sqrt{\varphi x}\operatorname{arctanh}(\sqrt{\varphi x}) - \log(1-\varphi x) + \frac{1}{4}\log^2(1-\varphi x)\right] \\ +\frac{\varphi}{\sqrt{5}x}\left[-\arctan^2\left(\frac{x}{\varphi}\right) + 2\sqrt{\frac{x}{\varphi}}\arctan\left(\sqrt{\frac{x}{\varphi}}\right) - \log\left(\frac{\varphi+x}{\varphi}\right) + \frac{1}{4}\log^2\left(\frac{\varphi+x}{\varphi}\right)\right], \\ 0 \leqslant x < \frac{1}{\varphi}; \\ \\ \frac{1}{\varphi\sqrt{5}x}\left[-\arctan^2(\sqrt{-\varphi x}) + 2\sqrt{-\varphi x}\arctan(\sqrt{-\varphi x}) - \log(1-\varphi x) + \frac{1}{4}\log^2(1-\varphi x)\right] \\ +\frac{\varphi}{\sqrt{5}x}\left[\operatorname{arctanh}^2\left(\frac{-x}{\varphi}\right) - 2\sqrt{\frac{-x}{\varphi}}\operatorname{arctanh}\left(\sqrt{\frac{-x}{\varphi}}\right) - \log\left(\frac{\varphi+x}{\varphi}\right) + \frac{1}{4}\log^2\left(\frac{\varphi+x}{\varphi}\right)\right], \\ -\frac{1}{\varphi} \leqslant x < 0. \end{cases} \tag{2.14}$$

Proof. The proof proceeds as in Theorem 2.1, applied to the ordinary generating function given in Lemma 2.6. ☐

3. Two Classes of Series Leading to Some Interesting Sums

We now present two classes of series, these being those given in (1.2). We give eight series in the first class and six in the second. Some of the series we find can be described as interesting according to Lehmer's criterion. These are found by substituting particular values into the ordinary generating functions given in Section 2, together with series manipulations using classic results for convergent series where needed.

The first class of series are of the form

$$\sum_{n=1}^{\infty} \frac{(\pm 1)^n H_{a(n)}F_{b(n)}}{2^{b(n)}},$$

where $a(n)$ and $b(n)$ are equal to n or $2n$. Setting $x = \pm\frac{1}{2}$ in (2.1) gives

$$\sum_{n=1}^{\infty} \frac{H_n F_n}{2^n} = \log(4) + \frac{12}{\sqrt{5}}\log(\varphi), \tag{3.1}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n F_n}{2^n} = \frac{1}{5}\log\left(\frac{5}{4}\right) - \frac{2}{\sqrt{5}}\log(\varphi). \tag{3.2}$$

For absolutely convergent series, because

$$\sum_{n=1}^{\infty} a_{2n} = \frac{1}{2}\sum_{n=1}^{\infty} a_n + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n a_n, \tag{3.3}$$

on applying this result, we immediately see that

$$\sum_{n=1}^{\infty} \frac{H_{2n}F_{2n}}{4^n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n F_n}{2^n} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n H_n F_n}{2^n}.$$

Combining with the results for the sums found in (3.1) and (3.2), one obtains

$$\sum_{n=1}^{\infty} \frac{H_{2n}F_{2n}}{4^n} = \frac{1}{10}\log(1280) + \sqrt{5}\log(\varphi). \tag{3.4}$$

The series found in (3.1), (3.2), and (3.4) are interesting. The alternating case corresponding to the sum given in (3.4) can also be found by applying the following classic result for convergent series. If $a_n > 0$ and all series converge, then

$$\sum_{n=1}^{\infty}(-1)^n a_{2n} = \operatorname{Re}\sum_{n=1}^{\infty} i^n a_n. \tag{3.5}$$

Here, i is the imaginary unit, whereas Re denotes the real part. Applying this result to our series of interest gives

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}F_{2n}}{4^n} = \operatorname{Re}\sum_{n=1}^{\infty} i^n \frac{H_n F_n}{2^n}.$$

Setting $x = \frac{i}{2}$ and $\frac{i}{2}$ in (2.1), noting that

$$\log\left(\frac{2\varphi + i}{2\varphi}\right) = \frac{1}{2}\log\left(1 + \frac{1}{4\varphi^2}\right) + i\arctan\left(\frac{1}{2\varphi}\right),$$

and

$$\log\left(\frac{2 - i\varphi}{2}\right) = \frac{1}{2}\log\left(1 + \frac{\varphi^2}{4}\right) - i\arctan\left(\frac{\varphi}{2}\right),$$

we find

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}F_{2n}}{4^n} &= \frac{4}{25 + 3\sqrt{5}}\left[\varphi\log\left(1 + \frac{1}{4\varphi^2}\right) + \arctan\left(\frac{1}{2\varphi}\right)\right] \\ &\quad - \frac{4}{5 + 11\sqrt{5}}\left[\log\left(1 + \frac{\varphi^2}{4}\right) + \varphi\arctan\left(\frac{\varphi}{2}\right)\right],\end{aligned} \tag{3.6}$$

a not particularly interesting series due to its rather long, complicated looking expression.

Setting $x = \frac{i}{2}$ in (2.5) gives the two interesting series

$$\sum_{n=1}^{\infty} \frac{H_n F_{2n}}{4^n} = \frac{1}{5}\log\left(\frac{256}{25}\right) + \frac{4}{\sqrt{5}}\log(\varphi), \tag{3.7}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n F_{2n}}{4^n} = \frac{2}{29}\log\left(\frac{29}{16}\right) - \frac{44}{29\sqrt{5}}\operatorname{arccoth}\left(\frac{11}{\sqrt{5}}\right). \tag{3.8}$$

For the final two series in the first class of series, setting $x = \pm\frac{1}{2}$ in (2.10) gives

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_{2n}F_n}{2^n} &= \frac{\varphi^2}{\sqrt{5}}\left[\sqrt{2\varphi}\operatorname{arctanh}\left(\sqrt{\frac{\varphi}{2}}\right) + \log(2\varphi^2)\right] \\ &\quad + \frac{1}{\varphi^2\sqrt{5}}\left[\sqrt{\frac{2}{\varphi}}\arctan\left(\frac{1}{\sqrt{2\varphi}}\right) + \log\left(\frac{\varphi^2}{2}\right)\right],\end{aligned} \tag{3.9}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n} F_n}{2^n} = -\frac{1}{5\varphi}\left[\sqrt{2\varphi}\arctan\left(\sqrt{\frac{\varphi}{2}}\right) + \log\left(\frac{\varphi\sqrt{5}}{2}\right)\right] - \frac{\varphi}{5}\left[\sqrt{\frac{2}{\varphi}}\operatorname{arctanh}\left(\frac{1}{\sqrt{2\varphi}}\right) + \log\left(\frac{2\varphi}{\sqrt{5}}\right)\right], \tag{3.10}$$

two series that again are not particularly interesting due to their long and complicated form.

The second class of series are those of the form

$$\sum_{n=1}^{\infty} \frac{(\pm 1)^n H_{a(n)} F_{b(n)}}{2^{b(n)}(b(n)+1)},$$

where $a(n)$ and $b(n)$ are equal to either n or $2n$. Setting $x = \pm\frac{1}{2}$ in (2.11) gives

$$\sum_{n=1}^{\infty} \frac{H_n F_n}{2^n(n+1)} = \log^2(2) + 4\log^2(\varphi) - \frac{4}{\sqrt{5}}\log(2)\log(\varphi), \tag{3.11}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n F_n}{2^n(n+1)} = \frac{1}{\sqrt{5}}\log\left(\frac{5}{4}\right)\log(\varphi) - \log^2\left(\frac{\sqrt{5}}{2}\right) - \log^2(\varphi). \tag{3.12}$$

Making use of (3.3), when combined with (3.11) and (3.12), yields

$$\sum_{n=1}^{\infty} \frac{H_{2n} F_{2n}}{4^n(2n+1)} = \frac{3}{2}\log^2(\varphi) - \frac{1}{8}\log^2(5) + \frac{1}{2}\log(2)\log(5) - \frac{1}{2\sqrt{5}}\log\left(\frac{64}{5}\right)\log(\varphi). \tag{3.13}$$

Containing three or four terms, series (3.11), (3.12), and (3.13) are borderline interesting. The corresponding alternating case of (3.13) is found on applying (3.5) to our series of interest. Doing so yields

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n} F_{2n}}{4^n(2n+1)} = \operatorname{Re}\sum_{n=1}^{\infty} i^n \frac{H_n F_n}{2^n(n+1)}.$$

Setting $x = \frac{i}{2}$ in (2.11), we find

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n} F_{2n}}{4^n(2n+1)} = \frac{1}{\sqrt{5}}\left[\varphi\log\left(1+\frac{1}{4\varphi^2}\right)\arctan\left(\frac{1}{2\varphi}\right) - \frac{1}{\varphi}\log\left(1+\frac{\varphi^2}{4}\right)\arctan\left(\frac{\varphi}{2}\right)\right], \tag{3.14}$$

a series that could not be said to be interesting.

Setting $x = \pm\frac{1}{2}$ in (2.14) gives

$$\sum_{n=1}^{\infty} \frac{H_{2n} F_n}{2^n(n+1)} = \frac{2}{\varphi\sqrt{5}}\left[\operatorname{acrtanh}^2\left(\sqrt{\frac{\varphi}{2}}\right) - \sqrt{2\varphi}\operatorname{arctanh}\left(\sqrt{\frac{\varphi}{2}}\right) + \log(2\varphi^2) + \frac{1}{4}\log^2(2\varphi^2)\right] + \frac{2\varphi}{\sqrt{5}}\left[-\arctan^2\left(\frac{1}{\sqrt{2\varphi}}\right) + \sqrt{\frac{2}{\varphi}}\arctan\left(\frac{1}{\sqrt{2\varphi}}\right) - \log\left(\frac{\varphi^2}{2}\right) + \frac{1}{4}\log^2\left(\frac{\varphi^2}{2}\right)\right], \tag{3.15}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n} F_n}{2^n(n+1)}$$
$$= \frac{2}{\varphi\sqrt{5}} \left[\arctan^2\left(\sqrt{\frac{\varphi}{2}}\right) - \sqrt{2\varphi}\arctan\left(\sqrt{\frac{\varphi}{2}}\right) + \log\left(\frac{\varphi\sqrt{5}}{2}\right) - \frac{1}{4}\log^2\left(\frac{\varphi\sqrt{5}}{2}\right)\right]$$
$$- \frac{2\varphi}{\sqrt{5}} \left[\operatorname{arctanh}^2\left(\frac{1}{\sqrt{2\varphi}}\right) - \sqrt{\frac{2}{\varphi}}\operatorname{arctanh}\left(\frac{1}{\sqrt{2\varphi}}\right) + \log\left(\frac{2\varphi}{\sqrt{5}}\right) + \frac{1}{4}\log^2\left(\frac{2\varphi}{\sqrt{5}}\right)\right].$$

Neither of these series could be said to be interesting due to their extended forms.

4. Other Ordinary Generating Functions and Series

As our method for finding ordinary generating functions containing the product between harmonic numbers and Fibonacci numbers makes clear, many series containing such products are possible, provided the ordinary generating function for the sequence containing the harmonic number of interest can be found. For example, ordinary generating functions for the sequences $\{H_n/n\}_{n\geqslant 1}$, $\{H_n/n^2\}_{n\geqslant 1}$, and $\{H_n/n^3\}_{n\geqslant 1}$ are all known [2, 7, 11], meaning ordinary generating functions for the sequences $\{H_nF_n/n\}_{n\geqslant 1}$, $\{H_nF_n/n^2\}_{n\geqslant 1}$, and $\{H_nF_n/n^3\}_{n\geqslant 1}$ can be found. Turning these into interesting series is problematic. As all the ordinary generating functions in these cases contain polylogarithmic functions of order two (dilogarithms), three (trilogarithms), or four (tetralogarithms), the difficultly lies in singling out any series of interest as the dilogarithm and trilogarithm are only known to be reducible to more fundamental constants for a limited number of values. Recall, the polylogarithm function $\operatorname{Li}_s(x)$ of order s is defined by $\sum_{n=1}^{\infty} x^n/n^s$ for $|x| \leqslant 1$, provided $s > 1$.

As an example of the difficulties faced when it comes to finding interesting series containing the product between the harmonic numbers and the Fibonacci numbers, let us find the sum for the simple but intriguing looking series

$$\sum_{n=1}^{\infty} \left(\frac{H_nF_n}{2^n}\right)^2. \tag{4.1}$$

We start by first giving as a lemma, the ordinary generating function for the sequence $\{H_n^2\}_{n\geqslant 1}$. It seems this result was first given without proof in [3]. An alternative proof to the one we are about to give can be found in [10].

Lemma 4.1. *For $|x| < 1$, the ordinary generating function for the sequence $\{H_n^2\}_{n\geqslant 1}$ is*

$$\sum_{n=1}^{\infty} H_n^2 x^n = \frac{\operatorname{Li}_2(x) + \log^2(1-x)}{1-x}. \tag{4.2}$$

Here, $\operatorname{Li}_2(x)$ is the dilogarithm function $\sum_{n=1}^{\infty} x^n/n^2$.

Proof. Noting that

$$H_{n+1}^2 - H_n^2 = (H_{n+1} - H_n)(H_{n+1} + H_n) = \frac{2H_n}{n+1} + \frac{1}{(n+1)^2},$$

where the recurrence relation of $H_{n+1} = H_n + \frac{1}{n+1}$ for the harmonic numbers has been used, we have

$$\sum_{n=1}^{\infty} H_{n+1}^2 x^n - \sum_{n=1}^{\infty} H_n^2 x^n = 2\sum_{n=1}^{\infty} \frac{H_n x^n}{n} + \sum_{n=1}^{\infty} \frac{x^n}{(n+1)^2},$$

or

$$\sum_{n=1}^{\infty} H_n^2 x^{n-1} - \sum_{n=1}^{\infty} H_n^2 x^n = 2\sum_{n=1}^{\infty} \frac{H_n x^n}{n+1} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n^2}, \tag{4.3}$$

after the index in the leftmost sum on the left of the equality and the rightmost sum on the right of the equality have been shifted by $n \mapsto n-1$. The first sum to the right of the equality in (4.3) is (2.12), whereas the second sum is $\mathrm{Li}_2(x)/x$. Thus,

$$\left(\frac{1-x}{x}\right) \sum_{n=1}^{\infty} H_n^2 x^n = \frac{\log^2(1-x)}{x} + \frac{\mathrm{Li}_2(x)}{x},$$

from which the required result follows. □

Theorem 4.2. *For $|x| < 1/\varphi^2$, the ordinary generating function for the sequence $\{H_n^2 F_n^2\}_{n \geqslant 1}$ is*

$$\begin{aligned} \sum_{n=1}^{\infty} H_n^2 F_n^2 x^n = {} & \frac{\mathrm{Li}_2(\varphi^2 x) + \log^2(1-\varphi^2 x)}{5(1-\varphi^2 x)} - \frac{2\left(\mathrm{Li}_2(-x) + \log^2(1+x)\right)}{5(1+x)} \\ & + \frac{\varphi^2}{5(\varphi^2 - x)} \left[\mathrm{Li}_2\left(\frac{x}{\varphi^2}\right) + \log^2\left(\frac{\varphi^2 - x}{\varphi^2}\right)\right]. \end{aligned} \tag{4.4}$$

Proof. In the ordinary generating function given by (4.2), enforcing substitutions of $x \mapsto \varphi^2 x$ and $x \mapsto \frac{x}{\varphi^2}$ gives

$$\sum_{n=1}^{\infty} H_n^2 \varphi^{2n} x^n = \frac{\mathrm{Li}_2(\varphi^2 x) + \log^2(1-\varphi^2 x)}{1-\varphi^2 x}, \tag{4.5}$$

and

$$\sum_{n=1}^{\infty} \frac{H_n^2 x^n}{\varphi^{2n}} = \frac{\varphi^2}{\varphi^2 - x} \left[\mathrm{Li}_2\left(\frac{x}{\varphi^2}\right) + \log^2\left(\frac{\varphi^2 - x}{\varphi^2}\right)\right], \tag{4.6}$$

respectively. Also, if x is replaced with $-x$ in (4.2), one has

$$\sum_{n=1}^{\infty} (-1)^n H_n^2 x^n = \frac{\mathrm{Li}_2(-x) + \log^2(1+x)}{1+x}. \tag{4.7}$$

From the square of Binet's formula

$$F_n^2 = \frac{1}{5}\left(\varphi^{2n} + \frac{1}{\varphi^{2n}}\right) - \frac{2(-1)^n}{5},$$

we have

$$\sum_{n=1}^{\infty} H_n^2 F_n^2 x^n = \frac{1}{5}\sum_{n=1}^{\infty} H_n^2 \varphi^{2n} x^n + \frac{1}{5}\sum_{n=1}^{\infty} \frac{H_n^2 x^n}{\varphi^{2n}} - \frac{2}{5}\sum_{n=1}^{\infty} (-1)^n H_n^2 x^n. \tag{4.8}$$

Combining (4.5), (4.6), and (4.7) with (4.8) yields the desired result. □

The sum for (4.1) can now be found by setting $x = \frac{1}{4}$ in (4.4). Doing so yields

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{H_n F_n}{2^n}\right)^2 = {} & \frac{4\varphi}{5\sqrt{5}} \left[\mathrm{Li}_2\left(\frac{\varphi^2}{4}\right) + \log^2\left(\frac{4\varphi}{\sqrt{5}}\right)\right] - \frac{8}{25}\left[\mathrm{Li}_2\left(-\frac{1}{4}\right) + \log^2\left(\frac{5}{4}\right)\right] \\ & + \frac{4}{5\sqrt{5}\varphi}\left[\mathrm{Li}_2\left(\frac{1}{4\varphi^2}\right) + \log^2\left(\frac{\varphi\sqrt{5}}{4}\right)\right]. \end{aligned}$$

The sum found for this simple looking series is not interesting. The real problem here in finding interesting series from ordinary generating functions that contain the dilogarithm, or the higher order polylogarithms, is the dilogarithm is only known to be reducible to simpler constants for the eight arguments corresponding to: $0, \frac{1}{2}, \pm 1, -\varphi, \pm\frac{1}{\varphi}$, and $\frac{1}{\varphi^2}$ [9, pp. 4, 6–7].

5. Conclusion

We have shown how ordinary generating functions involving the product between the harmonic numbers and the Fibonacci numbers can be found. These depended on the ordinary generating function for the sequence of interest containing the harmonic numbers being known. From these ordinary generating functions, some interesting series containing the product between the harmonic numbers and the Fibonacci numbers, that are almost nonexistent in the literature, were then found.

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