INVERSE RELATIONS FOR LUCAS SEQUENCES

HANS J. H. TUENTER

Abstract. We give new identities for the Fibonacci and Lucas polynomials that can be seen as inverses of the well-known, closed-form expressions for these polynomials, and generalize an identity that was posed as a problem for solution by Paul S. Bruckman. We also give the corresponding inverse identities for Lucas and other sequences.

1. Introduction

The Fibonacci and Lucas polynomials are defined by the recurrences

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x) \quad \text{and} \quad L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \tag{1.1}$$

with the initial values $F_0(x) = 0$ and $F_1(x) = 1$, and $L_0(x) = 2$ and $L_1(x) = x$. For $x = 1$, these polynomials reduce to the Fibonacci and Lucas numbers. The first few Fibonacci and Lucas polynomials are easily determined as

$$F_0(x) = 0,\ F_1(x) = 1,\ F_2(x) = x,\ F_3(x) = x^2 + 1,\ F_4(x) = x^3 + 2x,$$

and

$$L_0(x) = 2,\ L_1(x) = x,\ L_2(x) = x^2 + 2,\ L_3(x) = x^3 + 3x,\ L_4(x) = x^4 + 4x^2 + 2.$$

As with the Fibonacci and Lucas numbers, one can reverse the recurrences to derive the Fibonacci and Lucas polynomials at negative indices, and easily verify the reflection formulas

$$F_{-n}(x) = -(-1)^n F_n(x) \quad \text{and} \quad L_{-n}(x) = (-1)^n L_n(x). \tag{1.2}$$

Closed-form expressions are given by

$$F_{n+1}(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} x^{n-2i} \quad \text{for } n = 0, 1, 2, \ldots, \tag{1.3}$$

and

$$L_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i} \binom{n-i}{i} x^{n-2i} \quad \text{for } n = 1, 2, 3, \ldots. \tag{1.4}$$

These definitions and identities can be found in various textbooks on the Fibonacci numbers and associated sequences, such as Koshy [3].

2. Motivation

In the elementary problem section of the August 2010 issue of *The Fibonacci Quarterly*, Paul Bruckman [1] posed Problem B-1075 and asked to show that

$$x^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} F_{n+1-2i}(x) \quad \text{for } n = 0, 1, 2, \ldots. \tag{2.1}$$

This identity can be seen as an inverse relation to (1.3) and allows one to express the monomial x^n as a weighted sum over the Fibonacci polynomials. A year passed with no solutions being received, other than the proposer's, and prompted the editors to extend the deadline by another three months. This "last call" was answered by Ángel Plaza and Sergio Falcón [8], who provided a succinct proof in the November 2011 issue using an induction argument.

3. Generalization

It turns out that with little effort one can generalize Bruckman's result and gave the impetus to write this short note. Let p and q be nonzero constants. For $n = 0, 1, 2, \ldots$, define the recurrent sequence μ_n by

$$\mu_{n+2} = p\mu_{n+1} - q\mu_n, \tag{3.1}$$

with initial values μ_0 and μ_1, not both being zero. Note that one can reverse the recurrence definition and determine μ_n as a function of μ_{n+1} and μ_{n+2}, so that μ_n is also defined for negative indices and thus, (3.1) holds true for all integers n. We shall refer to μ_n as the general Lucas sequence.

Theorem 3.1. *For nonnegative integers n and all integers r,*

$$p^n\mu_r = \sum_{i=0}^{n}\binom{n}{i}q^i\mu_{n+r-2i} \quad \text{and} \quad q^n\mu_r = \sum_{i=0}^{n}(-1)^i\binom{n}{i}p^{n-i}\mu_{n+r+i}. \tag{3.2}$$

Proof. By induction. For $n = 0$, the first identity in (3.2) holds trivially for all values of r. Now, assume that the identity holds for n and all values of r, and consider the sum

$$\begin{aligned}\sum_{i=0}^{n+1}\binom{n+1}{i}q^k\mu_{n+1+r-2i} &= \sum_{i=0}^{n+1}\left(\binom{n}{i}+\binom{n}{i-1}\right)q^i\mu_{n+1+r-2i}\\ &= \sum_{i=0}^{n}\binom{n}{i}q^i\mu_{n+r+1-2i} + \sum_{i=1}^{n+1}\binom{n}{i-1}q^i\mu_{n+1+r-2i}\\ &= p^n\mu_{r+1} + qp^n\mu_{r-1} = p^{n+1}\mu_r,\end{aligned}$$

where we use the property that the binomial coefficient $\binom{n}{i}$ is zero, when $i > n$ or $i < 0$. This shows that the identity also holds for $n+1$ and all values of r, completes the induction step, and proves the first identity in (3.2). The proof of the second identity proceeds mutatis mutandis along the same lines and proves the Theorem. $\square$

Taking $r = -n$ in the second identity of (3.2) gives

$$q^n\mu_{-n} = \sum_{i=0}^{n}(-1)^i\binom{n}{i}p^{n-i}\mu_i, \tag{3.3}$$

and an explicit expression for μ_n with a negative index in terms of the original sequence μ_n, defined on the nonnegative indices. More succinct expressions are provided later.

4. Applications

Theorem 3.1 can be used to derive results for a variety of special cases. The Fibonacci and Lucas polynomials are obtained from the recurrent sequence μ_n by taking the constants as $p = x$, $q = -1$ and the initial conditions as $\mu_0 = 0$ and $\mu_1 = 1$, and $\mu_0 = 2$ and $\mu_1 = x$, respectively. Applying this to the first identity in (3.2) gives the following corollary.

Corollary 4.1. *For nonnegative integers n and all integers r,*

$$x^n F_r(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} F_{n+r-2i}(x) \quad \text{and} \quad x^n L_r(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} L_{n+r-2i}(x). \tag{4.1}$$

For $r = 1$, the former gives Bruckman's inversion formula (2.1). For $r = 0$, the latter gives the identity

$$2x^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} L_{n-2i}(x), \tag{4.2}$$

and can be seen as an inverse relation to (1.4). Similar identities and inverse relations for the Fibonacci and Lucas polynomials are obtained from (4.1) by taking $r = 2$ and $r = 1$, respectively. Note that the Lucas identity in (4.1) also follows from the Fibonacci identity in (4.1), as $L_n(x) = F_{n+1}(x) + F_{n-1}(x)$. Using the reflection formulas in (1.2) for the Fibonacci and Lucas polynomials, one can derive condensed versions of the identities (2.1) and (4.2), containing about half the number of summands, as

$$x^n = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \frac{n+1-2i}{n+1} \binom{n+1}{i} F_{n+1-2i}(x) \tag{4.3}$$

and

$$x^n = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n}{i} L_{n-2i}(x) - (-1)^{n/2} \binom{n}{n/2} \delta_{(n \text{ is even})}, \tag{4.4}$$

where δ_x is the Kronecker delta, which is 1 when x is true and 0 when x is false. Rather than proving these identities separately, we note that they are special cases of the more general identities (5.7) and (5.8). Although (4.3) and (4.4) are computationally more efficient than their counterparts (2.1) and (4.2), the latter are more elegant and can sometimes be easier to work with in proofs and the manipulation of formulas.

5. Lucas Sequences

The Lucas sequences are named after the French mathematician François Édouard Anatole Lucas (1842–1891), who first studied them in 1878 in a seminal article (in French) in the first volume of the *American Journal of Mathematics* [4], established earlier that year at the Johns Hopkins University by the English mathematician James Joseph Sylvester (1814–1897).[1]

Let p and q be nonzero and coprime integers. The particular Lucas sequences $U_n(p, q)$ and $V_n(p, q)$ are defined by the recurrences

$$U_n = pU_{n-1} - qU_{n-2} \quad \text{and} \quad V_n = pV_{n-1} - qV_{n-2}, \tag{5.1}$$

with the initial values $U_0 = 0$ and $U_1 = 1$, and $V_0 = 2$ and $V_1 = p$. For notational brevity, we omit the arguments p and q, and use U_n and V_n. The first few elements of these Lucas sequences are easily derived as

$$U_0 = 0,\ U_1 = 1,\ U_2 = p,\ U_3 = p^2 - q,\ U_4 = p^3 - 2pq,$$

and

$$V_0 = 2,\ V_1 = p,\ V_2 = p^2 - 2q,\ V_3 = p^3 - 3pq,\ V_4 = p^4 - 4p^2q + 2q^2.$$

[1] An English translation was commissioned by The Fibonacci Association and published in 1969. The electronic version can be found on its website [6].

As with the Fibonacci and Lucas polynomials, one can reverse the recurrence relations to derive the Lucas sequences at negative indices, and easily verify the reflection formulas

$$U_{-n} = -q^{-n}U_n \quad \text{and} \quad V_{-n} = q^{-n}V_n. \tag{5.2}$$

Lucas [4, Section XII] gave the closed-form expressions

$$U_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}(-q)^i p^{n-2i} \quad \text{for } n = 0, 1, 2, \ldots, \tag{5.3}$$

and

$$V_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i}\binom{n-i}{i}(-q)^i p^{n-2i} \quad \text{for } n = 1, 2, 3, \ldots. \tag{5.4}$$

Taking the initial conditions in Theorem 3.1 as $\mu_0 = 0$ and $\mu_1 = 1$, and $\mu_0 = 2$ and $\mu_1 = p$ yields the Lucas sequences. The first identity in (3.2) then gives the following corollary.

Corollary 5.1. *For nonnegative integers n and all integers r,*

$$p^n U_r = \sum_{i=0}^{n} \binom{n}{i} q^i U_{n+r-2i} \quad \text{and} \quad p^n V_r = \sum_{i=0}^{n} \binom{n}{i} q^i V_{n+r-2i}. \tag{5.5}$$

Taking $r = 1$ in the former and $r = 0$ in the latter gives the identities

$$p^n = \sum_{i=0}^{n} \binom{n}{i} q^i U_{n+1-2i} \quad \text{and} \quad 2p^n = \sum_{i=0}^{n} \binom{n}{i} q^i V_{n-2i}. \tag{5.6}$$

These can be seen as inverse relations to (5.3) and (5.4). Note that the second identity in (5.5) also follows from the first as $V_n = U_{n+1} - qU_{n-1}$. Using the reflection formulas for U_n and V_n, these identities can be condensed to

$$p^n = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n+1-2i}{n+1}\binom{n+1}{i} q^i U_{n+1-2i} \tag{5.7}$$

and

$$p^n = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{i} q^i V_{n-2i} - \binom{n}{n/2} q^{n/2}\delta_{(n \text{ is even})}. \tag{5.8}$$

A proof for these identities is provided in Section 8. Note that taking $p = x$ and $q = -1$ turns U_n into the Fibonacci polynomial $F_n(x)$ and V_n into the Lucas polynomial $L_n(x)$, so that identities (4.3) and (4.4) are seen to be special cases of (5.7) and (5.8). Using the same initial conditions in Theorem 3.1 as before, the second identity in (3.2) gives the following corollary.

Corollary 5.2. *For nonnegative integers n and all integers r,*

$$q^n U_r = \sum_{i=0}^{n} (-1)^i \binom{n}{i} p^{n-i} U_{n+r+i} \quad \text{and} \quad q^n V_r = \sum_{i=0}^{n} (-1)^i \binom{n}{i} p^{n-i} V_{n+r+i}. \tag{5.9}$$

Taking $r = 1$ in the former and $r = 0$ in the latter gives the identities

$$q^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} p^{n-i} U_{n+1+i} \quad \text{and} \quad 2q^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} p^{n-i} V_{n+i} \tag{5.10}$$

as companions to (5.6).

6. Reflection Formula

In the proof of Theorem 3.1, we tacitly assumed that we have a way of determining the values of μ_n for negative indices. Although we showed that these values are indeed defined and gave an explicit formula to determine them in (3.3), one still needs an efficient and more practical way of deriving them. The particular Lucas sequences U_n and V_n can serve as a basis to express the general Lucas sequence μ_n in and is captured by the identity

$$\mu_n = \left(\mu_1 - \tfrac{1}{2}\mu_0 p\right) U_n + \tfrac{1}{2}\mu_0 V_n. \tag{6.1}$$

This identity is well known and goes back to Lucas [5, p. 309, eqn. (5)]. It is easy to prove. Observe that the expressions on both sides of (6.1) define a Lucas sequence, and that both expressions have the same initial values, as shown by taking $n = 0$ and $n = 1$. Thus, they define the same sequence and proves the identity. In a similar fashion, one can show that

$$\mu_n = \mu_1 U_n - \mu_0 q U_{n-1} \quad \text{and} \quad \mu_n = \frac{p\mu_1 - 2q\mu_0}{p^2 - 4q} V_n + \frac{p\mu_0 - 2\mu_1}{p^2 - 4q} q V_{n-1}. \tag{6.2}$$

For the first identity, use the values $U_{-1} = -q^{-1}$, $U_0 = 0$, and $U_1 = 1$ to show that the right side and the left side agree for $n = 0$ and $n = 1$, and thus define the same Lucas sequence. For the second identity, use the values $V_{-1} = q^{-1}p$, $V_0 = 2$, and $V_1 = p$ to show that both sides define the same Lucas sequence. This argument can also be reversed. One can express the particular Lucas sequences in terms of the general Lucas sequence and show that

$$U_n = \frac{\mu_1\mu_n - \mu_0\mu_{n+1}}{\mu_1^2 - \mu_0\mu_2} \quad \text{and} \quad V_n = \frac{p\mu_1 - 2\mu_2}{\mu_1^2 - \mu_0\mu_2}\mu_n + \frac{2\mu_1 - p\mu_0}{\mu_1^2 - \mu_0\mu_2}\mu_{n+1}. \tag{6.3}$$

It is easily verified that these identities hold for $n = 0$ and $n = 1$, and thus for all n, as they define the same Lucas sequences.

Theorem 6.1 (Reflection Formula)**.** *Let μ_n be the recurrence defined by (3.1) with the initial conditions μ_0 and μ_1, not both being zero, and $p^2 \neq 4q$. Then,*

$$q^n \mu_{-n} = \frac{(2\mu_1 - p\mu_0)\mu_0}{\mu_1^2 - p\mu_0\mu_1 + q\mu_0^2}\mu_{n+1} - \frac{\mu_1^2 - q\mu_0^2}{\mu_1^2 - p\mu_0\mu_1 + q\mu_0^2}\mu_n. \tag{6.4}$$

Proof. Start with the first identity in (6.2), negate n, and apply the reflection formula for U_n from (5.2) to derive $q^n\mu_{-n} = \mu_0 U_{n+1} - \mu_1 U_n$. Now, use the first identity in (6.3) and substitute the corresponding expressions for U_n and U_{n+1}. Simplifying the resulting expression gives the reflection formula

$$\left(\mu_0\mu_2 - \mu_1^2\right) q^n \mu_{-n} = \mu_0^2\mu_{n+2} - 2\mu_0\mu_1\mu_{n+1} + \mu_1^2\mu_n. \tag{6.5}$$

Note that the condition $\mu_0\mu_2 \neq \mu_1^2$ is the same as $p^2 \neq 4q$. Making the substitutions $\mu_{n+2} = p\mu_{n+1} - q\mu_n$ and $\mu_2 = p\mu_1 - q\mu_0$, and rearranging terms give (6.4) and proves the Theorem. □

Note that (6.4) shows that there are only two fundamental cases, where μ_{-n} is proportional to μ_n for all integers n. The first corresponds to $\mu_0 = 0$ and the second to $\mu_1 = \frac{1}{2}p\mu_0$. These initial conditions give scaled versions of U_n and V_n, respectively.

7. Miscellaneous

Bruckman's problem B-1075 motivated the brothers Dence [2] to study generalizations of the Fibonacci and Lucas polynomials, defined in analogy to the closed-form expressions (1.3) and (1.4). Their definitions [2, eqn. (2.1), (4.2)] amount to a slightly different parametrization of the closed-form expressions (5.3) and (5.4) for the Lucas sequences, and can be obtained by replacing p by Px and q by Q. The authors posit inverse formulas and prove them by means of Binet's formula and an induction argument. These inverse formulas correspond to our condensed identities (5.7) and (5.8). It might have been easier and more general to start with Lucas' expressions (5.3) and (5.4), prove the inverse formulas, and then apply the parametrization.[2]

7.1. **Chebyshev Polynomials.** Although the initial setting of our note was that of the Fibonacci polynomials, the identities in Section 5 have further application. Consider the Chebyshev polynomials, named after the Russian mathematician Pafnuty Lvovich Chebyshev (1821–1894). These polynomials feature prominently in numerical analysis and can be defined by the recurrence

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x),$$

with the initial values $T_0(x) = 1$ and $T_1(x) = x$. It is easily verified that $T_n(x) = \frac{1}{2}V_n(2x, 1)$ and thus, corresponds to a scaled Lucas sequence.[3] We can use (5.4) to derive the closed-form expression

$$T_n(x) = \frac{1}{2}\sum_{i=0}^{\lfloor n/2\rfloor} \frac{n}{n-i}\binom{n-i}{i}(-1)^i(2x)^{n-2i} \quad \text{for } n = 1, 2, 3, \ldots.$$

An application of (5.8) gives the inverse relation

$$(2x)^n = 2\sum_{i=0}^{\lfloor n/2\rfloor}\binom{n}{i}T_{n-2i}(x) - \binom{n}{n/2}\delta_{(n \text{ is even})}.$$

These two formulas are, of course, well known and agree with the standard references, such as Mason and Handscomb [7], and Rivlin [9]. Note that, by (5.2), we have the reflection formula $T_{-n}(x) = T_n(x)$. Applying the second identity in Corollaries 5.1 and 5.2, with $p = 2x$ and $q = 1$, gives

$$(2x)^nT_r(x) = \sum_{i=0}^{n}\binom{n}{i}T_{n+r-2i}(x) \quad \text{and} \quad T_r(x) = \sum_{i=0}^{n}(-1)^i\binom{n}{i}(2x)^{n-i}T_{n+r+i}(x),$$

valid for nonnegative integers n and all integers r. These identities appear to be new.

7.2. **Jacobsthal Polynomials.** The reader will have noticed that we focused on the first identity in Theorem 3.1, driven by the context of the Fibonacci [and Lucas polynomials] in Bruckman's question. The second identity in Theorem 3.1 is no less useful and takes us to inversion formulas for the Jacobsthal and Jacobsthal-Lucas polynomials. These polynomials are named after the German mathematician Ernst Jacobsthal (1882–1965) and defined by the recurrences

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x) \quad \text{and} \quad j_n(x) = j_{n-1}(x) + xj_{n-2}(x),$$

[2]There is a typographical error in formula (1.1) of the Dence article [2]. The term $F_n(x)$ should be $F_{n+1}(x)$. In their analysis, the authors impose the restriction $(Px)^2 - 4Q > 0$. This is an artifact of basing their proof on Binet's formula and the requirement for the roots of the characteristic equation $t^2 - (Px)t + Q = 0$ to be different and real. Our derivation shows that this restriction is not necessary.

[3]The scaling factor is to standardize the Chebyshev polynomials, so that $T_n(1) = 1$ for all integers n, and originates from the standard trigonometric definition as $T_n(x) = \cos(n\theta)$, when $x = \cos\theta$. Chebyshev is the English transliteration of the Russian name. Tchebycheff is the French transliteration, hence $T_n(x)$.

with the initial values $J_0(x) = 0$ and $J_1(x) = 1$, and $j_0(x) = 2$ and $j_1(x) = 1$. They have the closed-form expressions

$$J_{n+1}(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} x^i \quad \text{and} \quad j_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i} \binom{n-i}{i} x^i,$$

see Koshy [3]. We can now apply Theorem 3.1 with $p = 1$ and $q = -x$ to give the inversion formulas

$$x^n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} J_{n+1+i}(x) \quad \text{and} \quad 2x^n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} j_{n+i}(x).$$

These identities appear to be new.

8. Proofs

To prove (5.7), we start with the first identity in (5.6). The basic idea is to use the reflection property for the Lucas sequence U_n, that is $U_{-n} = -q^{-n} U_n$, and reduce the number of summands by converting negatively indexed items to positively indexed ones. The derivation and proof, followed by some explanatory comments, is as follows.

$$\begin{aligned}
p^n &= \sum_{i=0}^{n} \binom{n}{i} q^i U_{n+1-2i} = \sum_{i=0}^{n+1} \binom{n}{i} q^i U_{n+1-2i} \\
&= \sum_{i=0}^{\lfloor \frac{1}{2}(n+1) \rfloor} \binom{n}{i} q^i U_{n+1-2i} + \sum_{i=\lceil \frac{1}{2}(n+1) \rceil}^{n+1} \binom{n}{i} q^i U_{n+1-2i} - \binom{n}{\frac{1}{2}(n+1)} q^{\frac{1}{2}(n+1)} U_0 \delta_{(n \text{ is odd})} \\
&= \sum_{i=0}^{\lfloor \frac{1}{2}(n+1) \rfloor} \left[\binom{n}{i} - \binom{n}{n+1-i} \right] q^i U_{n+1-2i} = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n+1-2i}{n+1} \binom{n+1}{i} q^i U_{n+1-2i}.
\end{aligned}$$

We start the derivation by extending the summation to include the index $n + 1$. This does not change the value of the sum because the corresponding summand has value zero, by virtue of the property that the binomial coefficient $\binom{n}{i}$ is zero, when $i > n$ or $i < 0$. The purpose of this extension is to ensure that the number of negatively indexed items equals the number of positively indexed ones. We now split the summation into two parts and adjust for double counting the zero-indexed item when n is odd. We then perform a change of variables, apply the reflection formula $U_{-n} = -q^{-n} U_n$, collect terms with like indexed items, simplify the difference of the two binomial coefficients, and drop the correction term because $U_0 = 0$. Because the summand corresponding to the index value $\lfloor \frac{1}{2}(n+1) \rfloor$ is zero, we can replace this value by $\lfloor n/2 \rfloor$. This completes the derivation and proof of identity (5.7). The proof of identity (5.8) starts with the second identity in (5.6) and proceeds along the same lines. The only difference being that we do not extend the summation, use the reflection formula $V_{-n} = q^{-n} V_n$, and that the correction term does not disappear because $V_0 = 2$. We note that one can derive condensed versions of the identities in (5.5) for other values of r. For small values of r, this could potentially yield other formulae of interest. For large values of r, this will not yield much or any reduction in the number of summands, and the resulting formulae may be of little practical use.

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Mathematical Finance Program, University of Toronto, 720 Spadina Avenue, Suite 219, Toronto, Ontario, M5S 2T9, Canada
Email address: hans.tuenter@utoronto.ca