GRAPH-THEORETIC CONFIRMATIONS OF FOUR SUMS OF JACOBSTHAL POLYNOMIAL PRODUCTS OF ORDER 4

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Abstract. Using graph-theoretic tools, we establish four identities involving sums of Jacobsthal polynomial products of order 4.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary complex variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 0$ [1, 2, 5].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial* [1, 2]. Clearly, $J_n(1) = F_n$ and $j_n(1) = L_n$.

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. We also omit a lot of basic algebra.

Table 1 lists some well known fundamental Jacobsthal identities [3]. We will employ them in our discourse.

$J_{n+1} + xJ_{n-1} = j_n$	$J_{2n} = J_n j_n$
$J_{n+1}^2 + xJ_n^2 = J_{2n+1}$	$J_{n+2} + x^2 J_{n-2} = (2x+1)J_n$
$J_{m+n} = J_{m+1}J_n + xJ_m J_{n-1}$	

Table 1: Fundamental Jacobsthal Identities

1.1. **Sums of Jacobsthal Polynomial Products of Order 4.**

Several sums of gibonacci polynomial products of order 4 are investigated in [5]; the following are six of them. Identities (1.1), (1.2), (1.5), and (1.6) form the cornerstone for our discourse:

$$\begin{aligned} J_{4n} &= J_{n+2}^3 J_n - 2xJ_{n+2}^2 J_n^2 - x^2 J_{n+2}^2 J_n J_{n-2} + 2(x^2+x)J_{n+2}J_n^3 \\ &\quad + x^4 J_{n+2}J_n J_{n-2}^2 - 2(x^4+x^3)J_n^3 J_{n-2} + 2x^5 J_n^2 J_{n-2}^2 - x^6 J_n J_{n-2}^3; \end{aligned} \tag{1.1}$$

$$\begin{aligned} J_{4n+1} &= J_{n+2}^4 - 4xJ_{n+2}^3 J_n + 2(3x^2+2x)J_{n+2}^2 J_n^2 - (4x^3+6x^2+x)J_{n+2}J_n^3 \\ &\quad - 2x^3 J_{n+2}J_n^2 J_{n-2} + (x^2+x)^2 J_n^4 + (2x^4+x^3)J_n^3 J_{n-2}; \end{aligned} \tag{1.2}$$

$$J_{4n+2} = J_{n+2}^4 - 3x^2 J_{n+2}^2 J_n^2 + 2x^4 J_{n+2}J_n^2 J_{n-2} + x^4 J_n^4 - x^6 J_n^2 J_{n-2}^2; \tag{1.3}$$

$$\begin{aligned} J_{4n+3} &= (x+1)J_{n+2}^4 - 4x^2 J_{n+2}^3 J_n + (6x^3+x^2)J_{n+2}^2 J_n^2 - (4x^4+6x^3+x^2)J_{n+2}J_n^3 \\ &\quad + (x^5+3x^4+x^3)J_n^4 + (2x^5+x^4)J_n^3 J_{n-2} - x^6 J_n^2 J_{n-2}^2; \end{aligned} \tag{1.4}$$

$$\begin{aligned} j_{4n+2} &= (2x+1)J_{n+2}^4 - 8x^2 J_{n+2}^3 J_n + (12x^3+5x^2)J_{n+2}^2 J_n^2 - 2(4x^4+6x^3+x^2)J_{n+2}J_n^3 \\ &\quad - 2x^4 J_{n+2}J_n^2 J_{n-2} + (2x^5+5x^4+2x^3)J_n^4 \\ &\quad + 2(2x^5+x^4)J_n^3 J_{n-2} - x^6 J_n^2 J_{n-2}^2; \end{aligned} \tag{1.5}$$

$$\begin{aligned} j_{4n+3} &= (3x+1)J_{n+2}^4 - 4x^2J_{n+2}^3J_n + x^2J_{n+2}^2J_n^2 - (4x^4+6x^3+x^2)J_{n+2}J_n^3 \\ &\quad + 4x^5J_{n+2}J_n^2J_{n-2} + (3x^5+3x^4+x^3)J_n^4 + (2x^5+x^4)J_n^3J_{n-2} \\ &\quad - (2x^7+x^6)J_n^2J_{n-2}^2, \end{aligned} \tag{1.6}$$

where $J_n = J_n(x)$ and $j_n(x) = j_n(x)$.

Our objective is to confirm the Jacobsthal identities (1.1), (1.2), (1.5), and (1.6) using graph-theoretic techniques.

2. Some Graph-theoretic Tools

To confirm these Jacobsthal results, consider the *weighted Jacobsthal digraph* D_1 in Figure 1 with vertices v_1 and v_2 [3, 4].

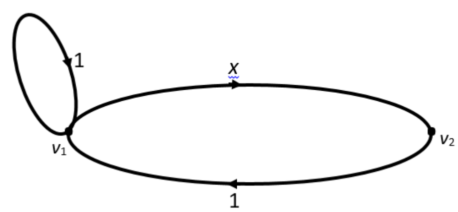

Figure 1. Weighted Fibonacci Digraph D_1

It follows from its *weighted adjacency matrix* $M = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix}$ that

$$M^n = \begin{bmatrix} J_{n+1} & xJ_n \\ J_n & xJ_{n-1} \end{bmatrix},$$

where $J_n = J_n(x)$ and $n \geq 1$.

It then follows that the sum of the weights of closed walks of length n originating at v_1 is J_{n+1}, and that of those originating at v_2 is xJ_{n-1}. So, the sum of the weights of all closed walks of length n in the digraph is $J_{n+1} + xJ_{n-1} = j_n$. These facts play a major role in the graph-theoretic proofs.

Let A, B, C, and D denote the sets of closed walks of varying lengths originating at vertex v, respectively. Then, the sum of the weights of the elements in the product set $A \times B \times C \times D$ is *defined* as the product the sums of the walks in each component [4].

With these tools at our disposal, we are now ready to explore the graph-theoretic proofs.

3. Graph-theoretic Confirmations

3.1. **Proof of Identity** (1.1).

Proof. Let S denote the sum of the weights of closed walks of length $4n-1$ originating at v_1. Clearly, $S = J_{4n}$.

We will now compute the sum S in a different way. To this end, let w be an arbitrary closed walk of length $4n-1$ originating at v_1. It can land at v_1 or v_2 at the nth, $2n$th, and $3n$th steps:

$$w = \underbrace{v_1 - \cdots - v}_{\text{subwalk of length } n} \quad \underbrace{v - \cdots - v}_{\text{subwalk of length } n} \quad \underbrace{v - \cdots - v}_{\text{subwalk of length } n} \quad \underbrace{v - \cdots - v_1}_{\text{subwalk of length } n-1},$$

where $v = v_1$ or v_2.

Table 2 shows the possible cases and the sums of weights of the corresponding walks w, where $J_n = J_n(x)$.

w lands at v_1 at the nth step?	w lands at v_1 at the $2n$th step?	w lands at v_1 at the $3n$th step?	w lands at v_1 at the $(4n-1)$st step?	sum of the weights of walks w
yes	yes	yes	yes	$J_{n+1}^3 J_n$
yes	yes	no	yes	$xJ_{n+1}^2 J_n J_{n-1}$
yes	no	yes	yes	$xJ_{n+1}J_n^3$
yes	no	no	yes	$x^2 J_{n+1} J_n J_{n-1}^2$
no	yes	yes	yes	$xJ_{n+1}J_n^3$
no	yes	no	yes	$x^2 J_n^3 J_{n-1}$
no	no	yes	yes	$x^2 J_n^3 J_{n-1}$
no	no	no	yes	$x^3 J_n J_{n-1}^3$

Table 2: Sums of the Weights of Closed Walks Originating at v_1

It follows from the table that the sum S of the weights of such walks w is given by

$$\begin{aligned} S &= J_{n+1}^3 J_n + xJ_{n+1}^2 J_n J_{n-1} + 2xJ_{n+1}J_n^3 + x^2 J_{n+1} J_n J_{n-1}^2 + 2x^2 J_n^3 J_{n-1} + x^3 J_n J_{n-1}^3 \\ &= A + B + C + D + E + F, \end{aligned}$$

where

$$\begin{aligned} A &= J_{n+1}^3 J_n \\ &= (J_{n+2} - xJ_n)^3 J_n \\ &= J_{n+2}^3 J_n - 3xJ_{n+2}^2 J_n^2 + 3x^2 J_{n+2} J_n^3 - x^3 J_n^4; \\ B &= xJ_{n+1}^2 J_n J_{n-1} \\ &= x(J_{n+2} - xJ_n)^2 J_n (J_n - xJ_{n-2}) \\ &= xJ_{n+2}^2 J_n^2 - x^2 J_{n+2}^2 J_n J_{n-2} - 2x^2 J_{n+2} J_n^3 + 2x^3 J_{n+2} J_n^2 J_{n-2} + x^3 J_n^4 - x^4 J_n^3 J_{n-2}; \\ C &= 2xJ_{n+1} J_n^3 \\ &= 2xJ_n^3 (J_{n+2} - xJ_n) \\ &= 2xJ_{n+2} J_n^3 - 2x^2 J_n^4; \\ D &= x^2 J_{n+1} J_n J_{n-1}^2 \\ &= x^2 (J_{n+2} - xJ_n) J_n (J_n - xJ_{n-2})^2 \\ &= x^2 J_{n+2} J_n^3 - 2x^3 J_{n+2} J_n^2 J_{n-2} + x^4 J_{n+2} J_n J_{n-2}^2 - x^3 J_n^4 + 2x^4 J_n^3 J_{n-2} - x^5 J_n^2 J_{n-2}^2; \\ E &= 2x^2 J_n^3 J_{n-1} \\ &= 2x^2 J_n^3 (J_n - xJ_{n-2}) \\ &= 2x^2 J_n^4 - 2x^3 J_n^3 J_{n-2}; \\ F &= x^3 J_n J_{n-1}^3 \\ &= x^3 J_n (J_n - xJ_{n-2})^3 \\ &= x^3 J_n^4 - 3x^4 J_n^3 J_{n-2} + 3x^5 J_n^2 J_{n-2}^2 - x^6 J_n J_{n-2}^3, \end{aligned}$$

where $J_n = J_n(x)$.

Thus,

$$\begin{aligned} S = \; & J_{n+2}^3 J_n - 2xJ_{n+2}^2 J_n^2 - x^2 J_{n+2}^2 J_n J_{n-2} + 2(x^2 + x) J_{n+2} J_n^3 + x^4 J_{n+2} J_n J_{n-2}^2 \\ & - 2(x^4 + x^3) J_n^3 J_{n-2} + 2x^5 J_n^2 J_{n-2}^2 - x^6 J_n J_{n-2}^3. \end{aligned}$$

This value of S, coupled with its earlier value, yields identity (1.1), as desired. □

3.2. **Proof of Identity** (1.2).

Proof. Let S' denote the sum of the weights of closed walks of length $4n$ originating at v_1 in the digraph. Then $S' = J_{4n+1}$.

To compute S' in a different way, we first let w be an arbitrary closed walk of length $4n$ originating at v_1. It can land at v_1 or v_2 at the nth, $2n$th, and $3n$th steps:

$$w = \underbrace{v_1 - \cdots - v}_{\text{subwalk of length } n} \quad \underbrace{v - \cdots - v}_{\text{subwalk of length } n} \quad \underbrace{v - \cdots - v}_{\text{subwalk of length } n} \quad \underbrace{v - \cdots - v_1}_{\text{subwalk of length } n},$$

where $v = v_1$ or v_2.

Table 3 summarizes the possible cases and the sums of the weights of the respective walks w, where $J_n = J_n(x)$.

w lands at v_1 at the nth step?	w lands at v_1 at the $2n$th step?	w lands at v_1 at the $3n$th step?	w lands at v_1 at the $4n$st step?	sum of the weights of walks w
yes	yes	yes	yes	J_{n+1}^4
yes	yes	no	yes	$xJ_{n+1}^2J_n^2$
yes	no	yes	yes	$xJ_{n+1}^2J_n^2$
yes	no	no	yes	$x^2J_{n+1}J_n^2J_{n-1}$
no	yes	yes	yes	$xJ_{n+1}^2J_n^2$
no	yes	no	yes	$x^2J_n^4$
no	no	yes	yes	$x^2J_{n+1}J_n^2J_{n-1}$
no	no	no	yes	$x^3J_n^2J_{n-1}^2$

Table 3: Sums of the Weights of Closed Walks Originating at v_1

It follows from the table that

$$\begin{aligned} S' &= J_{n+1}^4 + 3xJ_{n+1}^2J_n^2 + 2x^2J_{n+1}J_n^2J_{n-1} + x^2J_n^4 + x^3J_n^2J_{n-1}^2 \\ &= G + H + I + J + K, \end{aligned}$$

where

$$\begin{aligned} G &= J_{n+1}^4 \\ &= (J_{n+2} - xJ_n)^4 \\ &= J_{n+2}^4 - 4xJ_{n+2}^3J_n^2 + 6x^2J_{n+2}^2J_n^2 - 4x^3J_{n+2}J_n^3 + x^4J_n^4; \\ H &= 3xJ_{n+1}^2J_n^2 \\ &= 3xJ_n^2(J_{n+2} - xJ_n)^2 \\ &= 3xJ_{n+2}^2J_n^2 - 6x^2J_{n+2}J_n^3 + 3x^3J_n^4; \\ I &= 2x^2J_{n+1}J_n^2J_{n-1} \\ &= 2x^2J_n^2(J_{n+2} - xJ_n)(J_n - xJ_{n-2}) \\ &= 2x^2J_{n+2}J_n^3 - 2x^3J_{n+2}J_n^2J_{n-2} - 2x^3J_n^4 + 2x^4J_n^3J_{n-2}; \\ J &= x^2J_n^4; \\ K &= x^3J_n^2J_{n-1}^2 \\ &= x^3J_n^2(J_n - xJ_{n-2})^2 \\ &= x^3J_n^4 - 2x^4J_n^3J_{n-2} + x^5J_n^2J_{n-2}^2. \end{aligned}$$

Consequently,

$$\begin{aligned} S' &= J_{n+2}^4 - 4xJ_{n+2}^3 J_n + 3(2x^2+x)J_{n+2}^2 J_n^2 - 2x^3 J_{n+2} J_n^2 J_{n-2} \\ &\quad - 4(x^3+x^2)J_{n+2}J_n^3 + (x^2+x)^2 J_n^4 + x^5 J_n^2 J_{n-2}^2. \end{aligned} \tag{3.1}$$

To get the desired form for S', consider

$$L = xJ_{n+2}^2 J_n^2 - (2x^2+x)J_{n+2}J_n^3 + (2x^4+x^3)J_n^3 J_{n-2} - x^5 J_n^2 J_{n-2}^2. \tag{3.2}$$

Using the identity $J_{n+2} = (2x+1)J_n - x^2 J_{n-2}$, we have

$$\begin{aligned} L &= xJ_{n+2}J_n^2\left[J_{n+2} - (2x+1)J_n\right] + x^3 J_n^2 J_{n-2}\left[(2x+1)J_n - x^2 J_{n-2}\right] \\ &= 0. \end{aligned}$$

Thus, adding L in equation (3.2) to S' in equation (3.1) yields

$$\begin{aligned} S' &= J_{n+2}^4 - 4xJ_{n+2}^3 J_n + 2(3x^2+2x)J_{n+2}^2 J_n^2 - (4x^3+6x^2+x)J_{n+2}J_n^3 \\ &\quad - 2x^3 J_{n+2}J_n^2 J_{n-2} + (x^2+x)^2 J_n^4 + (2x^4+x^3)J_n^2 J_{n-2}^2. \end{aligned}$$

By equating the two values of S', we get the desired result, as expected. □

3.3. **Proof of Identity** (1.5).

Proof. Let S^* denote the sum of the weights of all closed walks of length $4n+2$ in the digraph. Clearly, $S^* = j_{4n+2}$.

We will now compute S^* in a different way, and then equate the two values. To this end, let w be an arbitrary closed walk of length $4n+2$.

Case 1. Suppose w originates (and ends) at v_1. It can land at v_1 or v_2 at the $(n+1)$st, $(2n+2)$nd, and $(3n+2)$nd steps:

$$w = \underbrace{v_1 - \cdots - v}_{\text{subwalk of length } n+1} \quad \underbrace{v - \cdots - v}_{\text{subwalk of length } n+1} \quad \underbrace{v - \cdots - v}_{\text{subwalk of length } n} \quad \underbrace{v - \cdots - v_1}_{\text{subwalk of length } n},$$

where $v = v_1$ or v_2.

It follows from Table 4 that the sum S_1^* of the weights of all such walks w is given by

$$\begin{aligned} S_1^* &= J_{n+2}^2 J_{n+1}^2 + xJ_{n+2}^2 J_n^2 + xJ_{n+2}J_{n+1}^2 J_n + x^2 J_{n+2}J_{n+1}J_nJ_{n-1} + xJ_{n+1}^4 \\ &\quad + 2x^2 J_{n+1}^2 J_n^2 + x^3 J_{n+1}J_n^2 J_{n-1} \\ &= \left(J_{n+2}^2 + xJ_{n+1}^2\right)\left(J_{n+1}^2 + xJ_n^2\right) + xJ_{n+1}J_n(J_{n+2}+xJ_n)(J_{n+1}+xJ_{n-1}) \\ &= J_{2n+3}J_{2n+1} + xJ_{2n+2}J_{2n} \\ &= J_{4n+3}. \end{aligned}$$

w lands at v_1 at the $(n+1)$st step?	w lands at v_1 at the $(2n+2)$nd step?	w lands at v_1 at the $(3n+2)$nd step?	w lands at v_1 at the $(4n+2)$nd step?	sum of the weights of walks w
yes	yes	yes	yes	$J_{n+2}^2 J_{n+1}^2$
yes	yes	no	yes	$xJ_{n+2}^2 J_n^2$
yes	no	yes	yes	$xJ_{n+2}J_{n+1}^2 J_n$
yes	no	no	yes	$x^2 J_{n+2}J_{n+1}J_nJ_{n-1}$
no	yes	yes	yes	xJ_{n+1}^4
no	yes	no	yes	$x^2 J_{n+1}^2 J_n^2$
no	no	yes	yes	$x^2 J_{n+1}^2 J_n^2$
no	no	no	yes	$x^3 J_{n+1}J_n^2 J_{n-1}^2$

Table 4: Sums of the Weights of Closed Walks Originating at v_1

Case 2. Suppose w originates at v_2. It also can land at v_1 or v_2 at the $(n+1)$st, $(2n+2)$nd, and $(3n+2)$nd steps:

$$w = \underbrace{v_2 - \cdots - v}_{\text{subwalk of length } n+1} \quad \underbrace{v - \cdots - v}_{\text{subwalk of length } n+1} \quad \underbrace{v - \cdots - v}_{\text{subwalk of length } n} \quad \underbrace{v - \cdots - v_2}_{\text{subwalk of length } n},$$

where $v = v_1$ or v_2.

It follows from Table 5 that the sum S_2^* of the weights of all such walks w is given by

$$\begin{aligned} S_2^* &= xJ_{n+2}J_{n+1}^2J_n + x^2J_{n+2}J_{n+1}J_nJ_{n-1} + 2x^2J_{n+1}^2J_n^2 + x^3J_{n+1}^2J_{n-1}^2 \\ &\quad + x^3J_{n+1}J_n^2J_{n-1} + x^3J_n^4 + x^4J_n^2J_{n-1}^2 \\ &= xJ_{n+1}J_n(J_{n+2} + xJ_n)(J_{n+1} + xJ_{n-1}) + x^2\left(J_{n+1}^2 + xJ_n^2\right)\left(J_n^2 + xJ_{n-1}^2\right) \\ &= xJ_{2n+2}J_{2n} + x^2J_{2n+1}J_{2n-1} \\ &= xJ_{4n+1}. \end{aligned}$$

w lands at v_1 at the $(n+1)$st step?	w lands at v_1 at the $(2n+2)$nd step?	w lands at v_1 at the $(3n+2)$nd step?	w lands at v_2 at the $(4n+2)$nd step?	sum of the weights of walks w
yes	yes	yes	yes	$xJ_{n+2}J_{n+1}^2J_n$
yes	yes	no	yes	$x^2J_{n+2}J_{n+1}J_nJ_{n-1}$
yes	no	yes	yes	$x^2J_{n+1}^2J_n^2$
yes	no	no	yes	$x^3J_{n+1}^2J_{n-1}^2$
no	yes	yes	yes	$x^2J_{n+1}^2J_n^2$
no	yes	no	yes	$x^3J_{n+1}J_n^2J_{n-1}$
no	no	yes	yes	$x^3J_n^4$
no	no	no	yes	$x^4J_n^2J_{n-1}^2$

Table 5: Sums of the Weights of Closed Walks Originating at v_2

Combining the two cases and using identities (1.2) and (1.4), we get

$$\begin{aligned} S^* &= S_1^* + S_2^* \\ &= [(x+1)J_{n+2}^4 - 4x^2J_{n+2}^3J_n + (6x^3+x^2)J_{n+2}^2J_n^2 - (4x^4+6x^3+x^2)J_{n+2}J_n^3 \\ &\quad + (x^5+3x^4+x^3)J_n^4 + (2x^5+x^4)J_n^3J_{n-2} - x^6J_n^2J_{n-2}^2] \\ &\quad + x[J_{n+2}^4 - 4xJ_{n+2}^3J_n + 2(3x^2+2x)J_{n+2}^2J_n^2 - (4x^3+6x^2+x)J_{n+2}J_n^3 \\ &\quad - 2x^3J_{n+2}J_n^2J_{n-2} + (x^2+x)^2J_n^4 + (2x^4+x^3)J_n^3J_{n-2}] \\ &= (2x+1)J_{n+2}^4 - 8x^2J_{n+2}^3J_n + (12x^3+5x^2)J_{n+2}^2J_n^2 - 2(4x^4+6x^3+x^2)J_{n+2}J_n^3 \\ &\quad - 2x^4J_{n+2}J_n^2J_{n-2} + (2x^5+5x^4+2x^3)J_n^4 + 2(2x^5+x^4)J_n^3J_{n-2} - x^6J_n^2J_{n-2}^2. \end{aligned}$$

Equating this value of S^* with its earlier value yields identity (1.3), as desired. □

Finally, we explore the graph-theoretic confirmation of identity (1.6).

3.4. **Proof of Identity** (1.6)**.**

Proof. Let S denote the sum of the weights of all closed walks of length $4n+3$ in the digraph. Then $S = j_{4n+3}$.

We will now compute S in a different way. To this end, let w be an arbitrary walk of length $4n+3$.

Case 1. Suppose w originates (and ends) at v_1. It can land at v_1 or v_2 at the $(n+1)$st, $(2n+2)$nd, and $(3n+3)$rd steps:

$$w = \underbrace{v_1 - \cdots - v}_{\text{subwalk of length } n+1} \quad \underbrace{v - \cdots - v}_{\text{subwalk of length } n+1} \quad \underbrace{v - \cdots - v}_{\text{subwalk of length } n+1} \quad \underbrace{v - \cdots - v_1}_{\text{subwalk of length } n},$$

where $v = v_1$ or v_2.

It follows from Table 6 that the sum S_1 of the weights of all such walks w is given by

$$\begin{aligned} S_1 &= J_{n+2}^3 J_{n+1} + xJ_{n+2}^2 J_{n+1} J_n + 2xJ_{n+2} J_{n+1}^3 + x^2 J_{n+2} J_{n+1} J_n^2 + 2x^2 J_{n+1}^3 J_n + x^3 J_{n+1} J_n^3 \\ &= J_{n+1}\left(J_{n+2}^2 + 2xJ_{n+1}^2 + x^2 J_n^2\right)(J_{n+2} + xJ_n) \\ &= J_{2n+2}\left(J_{n+2}^2 + 2xJ_{n+1}^2 + x^2 J_n^2\right) \\ &= J_{2n+2}(J_{2n+3} + xJ_{2n+1}) \\ &= J_{4n+4}. \end{aligned}$$

w lands at v_1 at the $(n+1)$st step?	w lands at v_1 at the $(2n+2)$nd step?	w lands at v_1 at the $(3n+3)$rd step?	w lands at v_1 at the $(4n+3)$rd step?	sum of the weights of walks w
yes	yes	yes	yes	$J_{n+2}^3 J_{n+1}$
yes	yes	no	yes	$xJ_{n+2}^2 J_{n+1} J_n$
yes	no	yes	yes	$xJ_{n+2} J_{n+1}^3$
yes	no	no	yes	$x^2 J_{n+2} J_{n+1} J_n^2$
no	yes	yes	yes	$xJ_{n+2} J_{n+1}^3$
no	yes	no	yes	$x^2 J_{n+1}^3 J_n$
no	no	yes	yes	$x^2 J_{n+1}^3 J_n$
no	no	no	yes	$x^3 J_{n+1} J_n^3$

Table 6: Sums of the Weights of Closed Walks Originating at v_1

Case 2. Suppose w originates at v_2. It also can land at v_1 or v_2 at the $(n+1)$st, $(2n+2)$nd, and $(3n+3)$rd steps:

$$w = \underbrace{v_2 - \cdots - v}_{\text{subwalk of length } n+1} \quad \underbrace{v - \cdots - v}_{\text{subwalk of length } n+1} \quad \underbrace{v - \cdots - v}_{\text{subwalk of length } n+1} \quad \underbrace{v - \cdots - v_2}_{\text{subwalk of length } n},$$

where $v = v_1$ or v_2.

It follows from Table 7 that the sum S_2 of the weights of all closed walks w originating at v_2 is given by

$$\begin{aligned} S_2 &= xJ_{n+2}^2 J_{n+1} J_n + x^2 J_{n+2} J_{n+1}^2 J_{n-1} + x^2 J_{n+2} J_{n+1} J_n^2 + x^2 J_{n+1}^3 J_n + 2x^3 J_{n+1}^2 J_n J_{n-1} \\ &\quad + x^3 J_{n+1} J_n^3 + x^4 J_n^3 J_{n-1} \\ &= xJ_{n+1}(J_{n+2}J_n + xJ_{n+1}J_{n-1})(J_{n+2} + xJ_n) + x^2\left(J_{n+1}^2 + xJ_n^2\right) J_n(J_{n+1} + xJ_{n-1}) \\ &= xJ_{2n+1}(J_{2n+2} + xJ_{2n}) \\ &= xJ_{4n+2}. \end{aligned}$$

w lands at v_1 at the $(n+1)$st step?	w lands at v_1 at the $(2n+2)$nd step?	w lands at v_1 at the $(3n+3)$rd step?	w lands at v_1 at the $(4n+3)$rd step?	sum of the weights of walks w
yes	yes	yes	yes	$xJ_{n+2}^2 J_{n+1} J_n$
yes	yes	no	yes	$x^2 J_{n+2} J_{n+1}^2 J_{n-1}$
yes	no	yes	yes	$x^2 J_{n+1}^3 J_n$
yes	no	no	yes	$x^3 J_{n+1}^2 J_n J_{n-1}$
no	yes	yes	yes	$x^2 J_{n+2} J_{n+1} J_n^2$
no	yes	no	yes	$x^3 J_{n+1}^2 J_n J_{n-1}$
no	no	yes	yes	$x^3 J_{n+1} J_n^3$
no	no	no	yes	$x^4 J_n^3 J_{n-1}$

Table 7: Sums of the Weights of Closed Walks Originating at v_2

Using equations (1.3) and (1.4), we then get

$$\begin{aligned} S &= S_1 + S_2 \\ &= J_{4n+4} + xJ_{4n+2} \\ &= J_{4n+3} + 2xJ_{4n+2} \\ &= (3x+1)J_{n+2}^4 - 4x^2J_{n+2}^3J_n + x^2J_{n+2}^2J_n^2 - (4x^4+6x^3+x^2)J_{n+2}J_n^3 + 4x^5J_{n+2}J_n^2J_{n-2} \\ &\quad + (3x^5+3x^4+x^3)J_n^4 + (2x^5+x^4)J_n^3J_{n-2} - (2x^7+x^6)J_n^2J_{n-2}^2. \end{aligned}$$

This value of S, coupled with its earlier version, yields the desired result, as expected. □

4. Conclusion

The graph-theoretic confirmations of the Jacobsthal identities (1.3) and (1.4) follow using similar arguments.

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