INFINITE SUMS INVOLVING JACOBSTHAL POLYNOMIAL PRODUCTS

THOMAS KOSHY

Abstract. We explore infinite sums involving Jacobsthal polynomial products and their Jacobsthal-Lucas counterparts, and then extract the corresponding gibonacci versions.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. They can also be defined by the *Binet-like* formulas

$$J_n(x) = \frac{u^n(x) - v^n(x)}{D} \text{ and } j_n(x) = u^n(x) + v^n(x),$$

where $2u(x) = 1 + D$, $2v(x) = 1 - D$, and $D = \sqrt{4x+1}$. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$.

Fibonacci and Jacobsthal polynomials, and Lucas and Jacobsthal-Lucas polynomials are closely related by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [2].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. We let $\Delta = \sqrt{x^2+4}$, $2\alpha(x) = x + \Delta$, and $2\beta(x) = x - \Delta$, and omit a lot of basic algebra.

2. Sums Involving Gibonacci Polynomial Products

The following sums of gibonacci polynomial products are studied in [3].

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n}^2 + 1} = \frac{1+\sqrt{5}}{2}, \qquad \sum_{n=0}^{\infty} \frac{1}{F_{2n+1}^2 + 1} = \frac{\sqrt{5}}{3},$$

$$\sum_{n=0}^{\infty} \frac{1}{F_n^2 + 1} = \frac{3+5\sqrt{5}}{6}, \qquad \sum_{n=3}^{\infty} \frac{1}{F_n^4 - 1} = \frac{35}{18} - \frac{5\sqrt{5}}{6},$$

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n}^2 + 1} = \frac{\alpha}{5}, \qquad \sum_{k=0}^{\infty} \frac{1}{L_{2n+1}^2 + 9} = \frac{\sqrt{5}}{15},$$

$$\sum_{n=3}^{\infty} \frac{1}{L^4 - 25} = \frac{5}{63} - \frac{\sqrt{5}}{30}.$$

We will revisit them in our investigations.

3. Sums Involving Jacobsthal Polynomial Products

Our discourse hinges on the identities $J_{n+1} + xJ_{n-1} = j_n$, $J_{n+2} + x^2 J_{n-2} = (2x+1)J_n$, $J_{2n} = J_n j_n$, $J_{n+k}J_{n-k} - J_n^2 = -(-x)^{n-k}J_k^2$, $-(-x)^b J_{a-b} = J_{a+1}J_b - J_a J_{b+1}$, $j_n^2 - D^2 J_n^2 = 4(-x)^n$, $J_{n+3} + x^2 J_{n-1} = (2x+1)J_{n+1}$ [2].

With this background, we begin our explorations with the first infinite sum.

Theorem 3.1. *Let $J_n = J_n(x)$. Then,*

$$\sum_{n=0}^{\infty} \frac{x^{2n-1}}{J_{2n}^2 + x^{2n-1}} = u(x). \tag{3.1}$$

Proof. First, we will establish the summation formula

$$\sum_{n=0}^{m} \frac{x^{2n-1}}{J_{2n}^2 + x^{2n-1}} = \frac{J_{2m+2}}{J_{2m+1}}, \tag{3.2}$$

using recursion [2, 3]. To this end, let A_m denote the left side of equation (3.2) and B_m its right side. Using the Jacobsthal addition formula and the Cassini-like identity, we have

$$\begin{aligned} B_m - B_{m-1} &= \frac{J_{2m+2}}{J_{2m+1}} - \frac{J_{2m}}{J_{2m-1}} \\ &= \frac{J_{2m+2}J_{2m-1} - J_{2m+1}J_{2m}}{J_{2m+1}J_{2m-1}} \\ &= \frac{x^{2m-1}J_{(2m+2)-2m}}{J_{2m}^2 + x^{2m-1}} \\ &= \frac{x^{2m-1}}{J_{2m}^2 + x^{2m-1}} \\ &= A_m - A_{m-1}. \end{aligned}$$

Thus, $A_m - A_{m-1} = B_m - B_{m-1}$; so $A_m - B_m = A_{m-1} - B_{m-1} = \cdots = A_0 - B_0 = 1 - 1 = 0$. This implies, $A_m = B_m$.

Because $\lim_{m\to\infty} \frac{J_{m+1}}{J_m} = u(x)$, it follows from equation (3.2) that

$$\sum_{n=0}^{\infty} \frac{x^{2n-1}}{J_{2n}^2 + x^{2n-1}} = u(x), \tag{3.3}$$

as desired. □

It follows from equations (3.2) and (3.1) that

$$\begin{aligned} \sum_{n=0}^{m} \frac{1}{F_{2n}^2 + 1} &= \frac{F_{2m+2}}{F_{2m+1}}; \\ \sum_{n=0}^{\infty} \frac{1}{F_{2n}^2 + 1} &= \frac{1+\sqrt{5}}{2}, \end{aligned}$$

respectively, as in [3, 4, 6]; we also have

$$\begin{aligned}\sum_{n=0}^{m} \frac{2^{2n-1}}{J_{2n}^2 + 2^{2n-1}} &= \frac{J_{2m+2}}{J_{2m+1}};\\ \sum_{n=0}^{\infty} \frac{2^{2n-1}}{J_{2n}^2 + 2^{2n-1}} &= 2.\end{aligned}$$

Next, we explore a corresponding result for odd-numbered Jacobsthal polynomials.

Theorem 3.2.

$$\sum_{n=0}^{\infty} \frac{(2x+1)x^{2n-1}}{J_{2n+1}^2 + x^{2n-1}} = \sqrt{4x+1}. \tag{3.4}$$

Proof. First, we will establish that

$$\sum_{n=0}^{m} \frac{(2x+1)x^{2n-1}}{J_{2n+1}^2 + x^{2n-1}} = \frac{J_{4m+4}}{J_{2m+3}J_{2m+1}}, \tag{3.5}$$

using recursion. Let A_m and B_m denote the left and right side of equation (3.5), respectively. Using the addition formula, Cassini-like identity, and the identities $J_{2n} = J_n j_n$, $j_n = J_{n+1} + xJ_{n-1}$, and $J_{n+3} + x^2 J_{n-1} = (2x+1)J_{n+1}$, we get

$$\begin{aligned}
B_m - B_{m-1} &= \frac{J_{4m+4}}{J_{2m+3}J_{2m+1}} - \frac{J_{4m}}{J_{2m+1}J_{2m-1}}\\
&= \frac{J_{2m+2}(J_{2m+3} + xJ_{2m+1})}{J_{2m+3}J_{2m+1}} - \frac{J_{2m}(J_{2m+1} + xJ_{2m-1})}{J_{2m+1}J_{2m-1}}\\
&= \frac{J_{2m+3}(J_{2m+2}J_{2m-1} - J_{2m+1}J_{2m}) - xJ_{2m-1}(J_{2m+3}J_{2m} - J_{2m+2}J_{2m+1})}{J_{2m+3}J_{2m+1}J_{2m-1}}\\
&= \frac{x^{2m-1}J_{2m+3}J_2 - x(-x^{2m})J_{2m-1}J_2}{J_{2m+3}J_{2m+1}J_{2m-1}}\\
&= \frac{x^{2m-1}(J_{2m+3} + x^2 J_{2m-1})}{J_{2m+3}J_{2m+1}J_{2m-1}}\\
&= \frac{(2x+1)x^{2m-1}}{J_{2m+3}J_{2m-1}}\\
&= \frac{(2x+1)x^{2m-1}}{J_{2m+1}^2 + x^{2m-1}}\\
&= A_m - A_{m-1}.
\end{aligned}$$

Consequently, $A_m - B_m = A_{m-1} - B_{m-1} = \cdots = A_0 - B_0 = \dfrac{2x+1}{x+1} - \dfrac{2x+1}{x+1} = 0$. So, $A_m = B_m$.

It then follows from equation (3.5) that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(2x+1)x^{2n-1}}{J_{2n+1}^2+x^{2n-1}} &= \lim_{m\to\infty}\frac{J_{2m+2}j_{2m+2}}{J_{2m+3}J_{2m+1}}\\ &= \lim_{m\to\infty}\frac{J_{2m+2}}{J_{2m+3}}\cdot\lim_{n\to\infty}\frac{j_{2m+2}}{J_{2m+1}}\\ &= \frac{1}{u(x)}\cdot u(x)D\\ &= D,\end{aligned}$$

as expected. □

It follows from equations (3.5) and (3.4) that

$$\begin{aligned}\sum_{n=0}^{m}\frac{3}{F_{2n+1}^2+1} &= \frac{F_{4m+4}}{F_{2m+3}F_{2m+1}};\\ \sum_{n=0}^{\infty}\frac{1}{F_{2n+1}^2+1} &= \frac{\sqrt{5}}{3};\\ \sum_{n=0}^{m}\frac{5\cdot 2^{2n-1}}{J_{2n+1}^2+2^{2n-1}} &= \frac{J_{4m+4}}{J_{2m+3}J_{2m+1}};\\ \sum_{n=0}^{\infty}\frac{2^{2n-1}}{J_{2n+1}^2+2^{2n-1}} &= \frac{3}{5};\end{aligned}$$

see [3, 6].

3.1. **Additional Implications.** Using the relationship $J_n(x) = x^{(n-1)/2}f_n(1/\sqrt{x})$, we can extract in two steps the Fibonacci versions of equations (3.2) and (3.5), and hence, equations (3.1) and (3.4).

Consider equation (3.2). Replacing $1/\sqrt{x}$ with x and then x with $1/x^2$, we get

$$\begin{aligned}\text{LHS} &= \sum_{n=0}^{m}\frac{x^{2n-1}}{\left[x^{(2n-1)/2}f_{2n}(1/\sqrt{x})\right]^2+x^{2n-1}}\\ &= \sum_{n=0}^{m}\frac{x^{2n-1}}{x^{2n-1}f_{2n}^2(1/\sqrt{x})+x^{2n-1}}\\ &= \sum_{n=0}^{m}\frac{1}{x^{4n-2}\left[\frac{1}{x^{4n-2}}f_{2n}^2(x)+\frac{1}{x^{4n-2}}\right]}\\ &= \sum_{n=0}^{m}\frac{1}{f_{2n}^2(x)+1};\\ \text{RHS} &= \frac{x^{(2m+1)/2}f_{2m+2}(1/\sqrt{x})}{x^{(2m)/2}f_{2m+1}(1/\sqrt{x})}\\ &= \frac{f_{2m+2}(x)}{xf_{2m+1}(x)}.\end{aligned}$$

Combining the two sides, we get

$$\begin{aligned} \sum_{n=0}^{m} \frac{x}{f_{2n}^2(x)+1} &= \frac{f_{2m+2}}{f_{2m+1}}; \\ \sum_{n=0}^{\infty} \frac{x}{f_{2n}^2(x)+1} &= \alpha(x), \end{aligned}$$

as in equations (2.1) and (2.2) of [3].

Next, consider equation (3.5). Using the Jacobsthal-Fibonacci relationship and the two steps, we get

$$\begin{aligned} \text{LHS} &= \sum_{n=0}^{m} \frac{(2x+1)x^{2n-1}}{\left[x^{(2n)/2} f_{2n+1}(1/\sqrt{x})\right]^2 + x^{2n-1}} \\ &= \sum_{n=0}^{m} \frac{(2x+1)x^{2n-1}}{x^{2n} f_{2n+1}^2(1/\sqrt{x}) + x^{2n-1}} \\ &= \sum_{n=0}^{m} \frac{x^2+2}{x^2\left[\dfrac{1}{x^2} f_{2n+1}^2(x)+1\right]} \\ &= \sum_{n=0}^{m} \frac{x^2+2}{f_{2n+1}^2(x)+x^2}; \\ \text{RHS} &= \frac{x^{(4m+3)/2} f_{4m+4}(1/\sqrt{x})}{x^{(2m+2)/2} f_{2m+3}(1/\sqrt{x}) \cdot x^{(2m)/2} f_{2m+1}(1/\sqrt{x})} \\ &= \frac{\sqrt{x} f_{4m+4}(1/\sqrt{x})}{f_{2m+3}(1/\sqrt{x}) f_{2m+1}(1/\sqrt{x})} \\ &= \frac{f_{4m+4}(x)}{x f_{2m+3}(x) f_{2m+1}(x)}. \end{aligned}$$

Equating the two sides then yields

$$\begin{aligned} \sum_{n=0}^{m} \frac{x^3+2x}{f_{2n+1}^2+x^2} &= \frac{f_{4m+4}}{f_{2m+3}f_{2m+1}}; \\ \sum_{n=0}^{\infty} \frac{x^3+2x}{f_{2n+1}^2+x^2} &= \alpha(x)-\beta(x), \end{aligned}$$

as in equations (2.4) and (2.3) of [3], respectively.

Next, we explore the Jacobsthal counterpart of Theorem 2.3 in [3].

Theorem 3.3. *Let $u = u(x)$ and $J_n = J_n(x)$. Then,*

$$\sum_{n=3}^{\infty} \frac{(x+1)(2x+1)x^{2n}}{J_n^4 + (x-1)(-x)^{n-2} - x^{2n-3}} = (x^2+1)\left(\frac{2x^2+4x+1}{x+1} - \frac{4x+1}{u}\right) - \frac{x^6}{2x+1}. \tag{3.6}$$

Proof. From the proof of Theorem 2.3 in [3], we have

$$\sum_{n=1}^{m-1} \frac{x^3+x}{f_n f_{n+1} f_{n+2} f_{n+3}} = \frac{x^4+4x^2+2}{x^3+x} - \left(\frac{f_{m-1}}{f_m} + \frac{(x^2+2)f_m}{f_{m+1}} + \frac{f_{m+1}}{f_{m+2}}\right).$$

Using the relationship $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$, this yields

$$\begin{aligned}
\text{LHS} &= \frac{1}{x\sqrt{x}} \sum_{n=1}^{m-1} \frac{(x+1)x^{2n+1}}{\left[x^{(n-1)/2} f_n\right]\left[x^{n/2} f_{n+1}\right]\left[x^{(n+1)/2} f_{n+2}\right]\left[x^{(n+2)/2} f_{n+3}\right]} \\
&= \frac{1}{\sqrt{x}} \sum_{n=1}^{m-1} \frac{(x+1)x^{2n}}{J_n J_{n+1} J_{n+2} J_{n+3}}; \\
\text{RHS} &= \frac{(2x^2+4x+1)\sqrt{x}}{x+1} \\
&\quad - \left[\frac{x^{3/2}}{x} \cdot \frac{x^{(m-2)/2} f_{m-1}}{x^{(m-1)/2} f_m} + \frac{(2x+1)x}{x\sqrt{x}} \cdot \frac{x^{(m-1)/2} f_m}{x^{m/2} f_{m+1}} + \sqrt{x} \cdot \frac{x^{m/2} f_{m+1}}{x^{(m+1)/2} f_{m+2}}\right] \\
&= \frac{(2x^2+4x+1)\sqrt{x}}{x+1} - \left(\sqrt{x} \cdot \frac{J_{m-1}}{J_m} + \frac{2x+1}{\sqrt{x}} \cdot \frac{J_m}{J_{m+1}} + \sqrt{x} \cdot \frac{J_{m+1}}{J_{m+2}}\right) \\
&= \frac{(2x^2+4x+1)\sqrt{x}}{x+1} - \sqrt{x}\left(\frac{J_{m-1}}{J_m} + \frac{2x+1}{x} \frac{J_m}{J_{m+1}} + \frac{J_{m+1}}{J_{m+2}}\right),
\end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Combining the two sides, we then get

$$\sum_{n=1}^{m-1} \frac{(x+1)x^{2n}}{J_n J_{n+1} J_{n+2} J_{n+3}} = \frac{2x^2+4x+1}{x+1} - \left[x\frac{J_{m-1}}{J_m} + (2x+1)\frac{J_m}{J_{m+1}} + x\frac{J_{m+1}}{J_{m+2}}\right].$$

Consequently,

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{(x+1)x^{2n}}{J_{n-2} J_{n-1} J_n J_{n+1}} &= \frac{2x^2+4x+1}{x+1} - \left(\frac{x}{u} + \frac{2x+1}{u} + \frac{x}{u}\right) \\
&= \frac{2x^2+4x+1}{x+1} - \frac{4x+1}{u}; \\
\sum_{n=3}^{\infty} \frac{(x+1)x^{2n}}{J_{n-1} J_n J_{n+1} J_{n+2}} &= \frac{2x^2+4x+1}{x+1} - \frac{4x+1}{u} - \frac{x^4}{2x+1}.
\end{aligned}$$

Because $J_{n+2} + x^2 J_{n-2} = (2x+1)J_n$, $J_{n+k}J_{n-k} - J_n^2 = -(-x)^{n-k} J_k^2$ [2] and

$$\begin{aligned}
J_{n-2}J_{n-1}J_{n+1}J_{n+2} &= (J_{n+1}J_{n-1})(J_{n+2}J_{n-2}) \\
&= \left[J_n^2 - (-x)^{n-1}\right]\left[J_n^2 - (-x)^{n-2}\right] \\
&= J_n^4 + (x-1)(-x)^{n-2} J_n^2 - x^{2n-3},
\end{aligned}$$

we then have

$$\begin{aligned}
\frac{2x+1}{J_n^4 + (x-1)(-x)^{n-2} J_n^2 - x^{2n-3}} &= \frac{(2x+1)J_n}{J_{n-2}J_{n-1}J_nJ_{n+1}J_{n+2}} \\
&= \frac{J_{n+2} + x^2 J_{n-2}}{J_{n-2}J_{n-1}J_nJ_{n+1}J_{n+2}} \\
&= \frac{1}{J_{n-2}J_{n-1}J_nJ_{n+1}} + \frac{x^2}{J_{n-1}J_nJ_{n+1}J_{n+2}}, \\
\sum_{n=3}^{\infty} \frac{(x+1)(2x+1)x^{2n}}{J_n^4 + (x-1)(-x)^{n-2} J_n^2 - x^{2n-3}} &= (x^2+1)\left(\frac{2x^2+4x+1}{x+1} - \frac{4x+1}{u}\right) - \frac{x^6}{2x+1},
\end{aligned}$$

as desired. □

It follows from equation (3.6) that

$$\sum_{n=3}^{\infty} \frac{1}{F_n^4 - 1} = \frac{35}{18} - \frac{5\sqrt{5}}{6},$$

as in [3, 4, 6]. In addition,

$$\sum_{n=3}^{\infty} \frac{2^{2n}}{J_n^4 + (-2)^{n-2} J_n^2 - 2^{2n-3}} = -\frac{209}{450}.$$

Next, we investigate the Jacobsthal-Lucas consequences of the above Jacobsthal polynomial sums. Our investigation hinges on the identity $j_n^2 - D^2 J_n^2 = 4(-x)^n$, where $D = \sqrt{4x+1}$ [2].

4. Jacobsthal-Lucas Implications

4.1. Counterparts of Equations (3.2) and (3.5). It follows from equation (3.2) that

$$\begin{aligned}
\sum_{n=0}^{m} \frac{x^{2n-1}}{D^2 J_{2n}^2 + D^2 x^{2n-1}} &= \frac{J_{2m+2}}{D^2 J_{2m+1}};\\
\sum_{n=0}^{m} \frac{x^{2n-1}}{j_{2n}^2 - 4(-x)^{2n} + D^2 x^{2n-1}} &= \frac{J_{2m+2}}{D^2 J_{2m+1}};\\
\sum_{n=0}^{m} \frac{x^{2n-1}}{j_{2n}^2 + x^{2n-1}} &= \frac{J_{2m+2}}{(4x+1) J_{2m+1}}. \qquad (4.1)
\end{aligned}$$

Similarly, equation (3.5) yields

$$\sum_{n=0}^{m} \frac{(2x+1)x^{2n-1}}{j_{2n+1}^2 + (2x+1)^2 x^{2n-1}} = \frac{J_{4m+4}}{(4x+1) J_{2m+3} J_{2m+1}}. \qquad (4.2)$$

It follows from equations (4.1) and (4.2) that

$$\begin{aligned}
\sum_{n=0}^{m} \frac{2^{2n-1}}{j_{2n}^2 + 2^{2n-1}} &= \frac{J_{2m+2}}{9 J_{2m+1}};\\
\sum_{n=0}^{\infty} \frac{x^{2n-1}}{j_{2n}^2 + x^{2n-1}} &= \frac{u(x)}{4x+1}; \qquad (4.3)\\
\sum_{n=0}^{\infty} \frac{2^{2n-1}}{j_{2n}^2 + 2^{2n-1}} &= \frac{2}{9}; \qquad (4.4)\\
\sum_{n=0}^{m} \frac{2^{2n-1}}{j_{2n+1}^2 + 25 \cdot 2^{2n-1}} &= \frac{J_{4m+4}}{45 J_{2m+3} J_{2m+1}};\\
\sum_{n=0}^{\infty} \frac{(2x+1)x^{2n-1}}{j_{2n+1}^2 + (2x+1)^2 x^{2n-1}} &= \frac{1}{D}; \qquad (4.5)\\
\sum_{n=0}^{\infty} \frac{2^{2n-1}}{j_{2n+1}^2 + 25 \cdot 2^{2n-1}} &= \frac{1}{15}. \qquad (4.6)
\end{aligned}$$

Equations (4.3) and (4.5) imply that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{L_{2n}^2+1} &= \frac{1+\sqrt{5}}{10};\\ \sum_{n=0}^{\infty}\frac{1}{L_{2n+1}^2+9} &= \frac{\sqrt{5}}{15},\end{aligned}$$

respectively, as found earlier.

4.2. **Lucas Versions of Equations** (4.1) **and** (4.2)**.** Using the Jacobsthal-Lucas relationship $j_n(x) = x^{n/2}l_n(1/\sqrt{x})$ and the two steps used earlier, we can extract the Lucas counterparts of equations (4.1) and (4.2).

First, consider equation (4.1). Following the two-step method, we get

$$\begin{aligned}\text{LHS} &= D^2\sum_{n=0}^{m}\frac{x^{2n-1}}{j_{2n}^2(x)+x^{2n-1}}\\ &= D^2\sum_{n=0}^{m}\frac{x^{2n-1}}{x^{2n}l_{2n}^2(1/\sqrt{x})+x^{2n-1}}\\ &= \frac{\Delta^2}{x^2}\sum_{n=0}^{m}\frac{1}{x^{4n-2}\left[\dfrac{1}{x^{4n}}l_{2n}^2(x)+\dfrac{1}{x^{4n-2}}\right]}\\ &= \sum_{k=0}^{m}\frac{\Delta^2}{l_{2n}^2(x)+x^2};\\ \text{RHS} &= \frac{J_{2m+2}(x)}{J_{2m+1}(x)}\\ &= \frac{x^{(2m+1)/2}f_{2m+2}(1/\sqrt{x})}{x^{(2m)/2}f_{2m+1}(1/\sqrt{x})}\\ &= \frac{\sqrt{x}f_{2m+2}(1/\sqrt{x})}{f_{2m+1}(1/\sqrt{x})}\\ &= \frac{f_{2m+2}(x)}{xf_{2m+1}(x)}.\end{aligned}$$

Equating the two sides yields

$$\begin{aligned}\sum_{n=0}^{m}\frac{x}{l_{2n}^2+x^2} &= \frac{f_{2m+2}}{\Delta^2 f_{2m+1}};\\ \sum_{n=0}^{\infty}\frac{x}{l_{2n}^2+x^2} &= \frac{\alpha(x)}{x^2+4},\end{aligned}$$

as in [3].

Next, we begin with equation (4.2). Using the two-step procedure, we get

$$
\begin{aligned}
\text{LHS} &= D^2(2x+1)\sum_{n=0}^{m}\frac{x^{2n-1}}{\left[x^{(2n+1)/2}l_{2n+1}(1/\sqrt{x})\right]^2+(2x+1)^2x^{2n-1}}\\
&= \frac{\Delta^2(x^2+2)}{x^4}\sum_{n=0}^{m}\frac{x^4}{l_{2n+1}^2(x)+(x^2+2)^2}\\
&= \sum_{n=0}^{m}\frac{\Delta^2(x^2+2)}{l_{2n+1}^2(x)+(x^2+2)^2};\\
\text{RHS} &= \frac{J_{4m+4}(x)}{J_{2m+3}(x)J_{2m+1}(x)}\\
&= \frac{x^{(4m+3)/2}f_{4m+4}(1/\sqrt{x})}{x^{(2m+2)/2}f_{2m+3}(1/\sqrt{x})x^{(2m)/2}f_{2m+1}(1/\sqrt{x})}\\
&= \frac{\sqrt{x}f_{4m+4}(1/\sqrt{x})}{f_{2m+3}(1/\sqrt{x})f_{2m+1}(1/\sqrt{x})}\\
&= \frac{f_{4m+4}(x)}{xf_{2m+3}f_{2m+1}}.
\end{aligned}
$$

Equating the two sides, we get

$$
\begin{aligned}
\sum_{n=0}^{m}\frac{x^3+2x}{l_{2n+1}^2+(x^2+2)^2} &= \frac{f_{4m+4}}{\Delta^2 f_{2m+3}f_{2m+1}};\\
\sum_{n=0}^{\infty}\frac{x^3+2x}{l_{2n+1}^2+(x^2+2)^2} &= \frac{1}{\sqrt{x^2+4}},
\end{aligned}
$$

as in [3].

Using the Lucas-Jacobsthal relationship $j_n(x)=x^{n/2}l_n(1/\sqrt{x})$, next we explore the Jacobsthal version of equation (4.5) in [3]:

$$
\sum_{n=3}^{\infty}\frac{(x^2+1)(x^3+2x)\Delta^4}{d(x)} = \frac{2(x^4+4x^2+2)}{x^3+x}+2\Delta^2\beta(x)-\frac{1}{x^3+2x}, \tag{4.7}
$$

where $d(x)=l_n^4-(-1)^n[(x^2-1)\Delta^2+8]l_n^2-(x^3+2x)^2$.

4.3. **Jacobsthal Version of Equation** (4.7)**.** Replacing x with $\sqrt{x}$ in equation (4.7) and then multiplying the resulting equation with x^{2n-4}, we get

$$
\begin{aligned}
\text{LHS} &= \frac{1}{\sqrt{x}}\sum_{n=3}^{\infty}\frac{(x+1)(2x+1)D^4}{\{x^4l_n^4+(-1)^n[(x-1)D^2-8x^2]x^2l_n^2-x(2x+1)^2\}}\\
&= \frac{1}{\sqrt{x}}\sum_{n=3}^{\infty}\frac{(x+1)(2x+1)D^4x^{2n-4}}{\{j_n^4+(-1)^n[(x-1)D^2-8x^2]x^{n-4}j_n^2-(2x+1)^2x^{2n-3}\}};\\
\text{RHS} &= \frac{2(2x^2+4x+1)\sqrt{x}}{x^2+x}+\frac{(1-D)D^2}{x\sqrt{x}}-\frac{x\sqrt{x}}{2x+1},
\end{aligned}
$$

where $l_n=l_n(1/\sqrt{x})$ and $j_n=j_n(x)$.

Combining the two sides yields

$$\sum_{n=3}^{\infty} \frac{(x+1)(2x+1)D^4x^{2n-4}}{k(x)} = \frac{2(2x^2+4x+1)}{x+1} + \frac{(1-D)D^2}{x} - \frac{x^2}{2x+1}, \tag{4.8}$$

where $k(x) = j_n^4 + (-1)^n[(x-1)D^2 - 8x^2]x^{n-4}j_n^2 - (2x+1)^2x^{2n-3}$.

In particular, we then get

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{1}{L_n^4 - 8(-1)^n L_n^2 - 9} &= \frac{7-3\sqrt{5}}{90}; \\ \sum_{n=3}^{\infty} \frac{2^{2n}}{j_n^4 - 23(-2)^{n-4}j_n^2 - 25 \cdot 2^{2n-3}} &= \frac{368}{18225}. \end{aligned}$$

4.4. **An Additional Jacobsthal Implication.** Finally, we develop the Jacobsthal consequence of equation (4.6) in [3]:

$$\sum_{n=3}^{\infty} \frac{(x^2+2)(x^4+x^2)}{l_n^4 + (-1)^n(x^2-1)\Delta^2 l_n^2 - \Delta^4 x^2} = \frac{(x^2+1)(x^6+6x^4+10x^2+3)}{(x^2+2)(x^2+3)(x^4+4x^2+2)} - \frac{x}{\Delta}. \tag{4.9}$$

Theorem 4.1. *Let $l_n = l_n(x)$, $j_n = j_n(x)$, and $D = \sqrt{4x+1}$. Then,*

$$\sum_{n=3}^{\infty} \frac{(x+1)(2x+1)x^{2n-3}}{j(x)} = \frac{(x+1)(3x^3+10x^2+6x+1)}{(2x+1)(3x+1)(2x^2+4x+1)} - \frac{1}{D}, \tag{4.10}$$

where $j(x) = j_n^4 - (x-1)(-x)^{n-2}D^2j_n^2 - D^4x^{2n-3}$.

Proof. Replacing x with $1/\sqrt{x}$ in equation (4.9) and then using the relationship $j_n(x) = x^{n/2}l_n(1/\sqrt{x})$, we get

$$\begin{aligned} \text{LHS} &= \sum_{n=3}^{\infty} \frac{(x+1)(2x+1)}{x^3 l_n^4 - (-1)^n(x-1)xD^2l_n^2 - D^4} \\ &= \sum_{n=3}^{\infty} \frac{(x+1)(2x+1)x^{2n-3}}{(x^{n/2}l_n)^4 - (x-1)(-x)^{n-2}D^2(x^{n/2}l_n)^2 - D^4x^{2n-3}} \\ &= \sum_{n=3}^{\infty} \frac{(x+1)(2x+1)x^{2n-3}}{j_n^4 - (x-1)(-x)^{n-2}D^2j_n^2 - D^4x^{2n-3}}; \\ \text{RHS} &= \frac{(x+1)(3x^3+10x^2+6x+1)x^4}{x^4(2x+1)(3x+1)(2x^2+4x+1)} - \frac{1}{D} \\ &= \frac{(x+1)(3x^3+10x^2+6x+1)}{(2x+1)(3x+1)(2x^2+4x+1)} - \frac{1}{D}, \end{aligned}$$

where $l_n = l_n(1/\sqrt{x})$ and $j_n = j_n(x)$.

Combining the two sides, we get

$$\sum_{n=3}^{\infty} \frac{(x+1)(2x+1)x^{2n-3}}{j_n^4 - (x-1)(-x)^{n-2}D^2j_n^2 - D^4x^{2n-3}} = \frac{(x+1)(3x^3+10x^2+6x+1)}{(2x+1)(3x+1)(2x^2+4x+1)} - \frac{1}{D},$$

as desired. □

It follows from equation (4.10) that

$$\sum_{n=3}^{\infty} \frac{1}{L^4 - 25} = \frac{5}{63} - \frac{\sqrt{5}}{30},$$

as in [3, 5, 7]. In addition, we have

$$\sum_{n=3}^{\infty} \frac{2^{2n}}{j_n^4 - 9(-2)^{n-2} j_n^2 - 81 \cdot 2^{2n-3}} = \frac{112}{3825}.$$

5. Acknowledgment

The author thanks the reviewer for a careful reading of the article, constructive suggestions, and encouraging words.

References

[1] M. Bicknell, *A primer for the Fibonacci numbers: Part VII*, The Fibonacci Quarterly, **8.4** (1970), 407–420.
[2] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Volume II, Wiley, Hoboken, New Jersey, 2019.
[3] T. Koshy, *Infinite sums involving gibonacci polynomial products*, The Fibonacci Quarterly, **59.3** (2021), 237–245.
[4] H. Ohtsuka, *Problem H-783*, The Fibonacci Quarterly, **54.1** (2016), 87.
[5] À. Plaza, *Problem H-810*, The Fibonacci Quarterly, **55.3** (2017), 282.
[6] À. Plaza, *Solution to Problem H-783*, The Fibonacci Quarterly, **56.1** (2018), 90–91.
[7] À. Plaza, *Solution to Problem H-810*, The Fibonacci Quarterly, **57.3** (2019), 281.

MSC2020: Primary 11B37, 11B39, 11B83, 11C08

Department of Mathematics, Framingham State University, Framingham, MA 01701
Email address: `tkoshy@emeriti.framingham.edu`