WINNING STRATEGY FOR MULTIPLAYER AND MULTIALLIANCE ZECKENDORF GAMES

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Abstract. Edouard Zeckendorf [5] proved that every positive integer n can be uniquely written as the sum of nonadjacent Fibonacci numbers, known as the Zeckendorf decomposition. Based on Zeckendorf's decomposition, we have the Zeckendorf game for multiple players. We show that when the Zeckendorf game has at least three players, none of the players have a winning strategy for $n \geq 5$. Then we extend the multiplayer game to the multialliance game, finding some interesting situations in which no alliance has a winning strategy. This includes the two-alliance game, and some cases in which one alliance always has a winning strategy.

1. Introduction

1.1. Rules of Zeckendorf Game.

The Fibonacci sequence is one of the most fabulous sequences with a number of beautiful properties. Among these properties is a theorem by Edouard Zeckendorf [5]. Zeckendorf proved that every positive integer n can be uniquely written as the sum of distinct nonconsecutive Fibonacci numbers. This sum is also known as the *Zeckendorf decomposition* of n. We define the Fibonacci sequence as $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, and $F_{n+1} = F_n + F_{n-1}$. If we stuck with $F_1 = F_2 = 1$, we lose uniqueness.

Baird-Smith, Epstein, Flint, and Miller [1, 2] created a game based on the Zeckendorf decomposition. We quote from [2] to describe the game.

We first introduce some notation. By $\{1^n\}$ or $\{{F_1}^n\}$, we mean n copies of 1, or F_1. If we have three copies of F_1, two copies of F_2, and seven copies of F_4, we write $\{{F_1}^3 + {F_2}^2 + {F_4}^7\}$ or $\{1^3 + 2^2 + 5^7\}$.

Definition 1.1 (The Zeckendorf Game). *At the beginning of the game, there is an unordered list of n 1's. Let $F_1 = 1$, $F_2 = 2$, and $F_{i+1} = F_i + F_{i-1}$; therefore the initial list is $\{{F_1}^n\}$. On each turn, a player can do one of the following moves.*

1. *If the list contains two consecutive Fibonacci numbers, F_{i-1}, F_i, these can be combined to create F_{i+1}. We denote this move by $F_{i-1} + F_i = F_{i+1}$.*
2. *If the list has two of the same Fibonacci number, F_i, F_i, then*
   - (a) *if $i = 1$, F_1, F_1 can be combined to create F_2, denoted by $1 + 1 = 2$,*
   - (b) *if $i = 2$, a player can change F_2, F_2 to F_1, F_3, denoted by $2 + 2 = 1 + 3$, and*
   - (c) *if $i \geq 3$, a player can change F_i, F_i to F_{i-2}, F_{i+1}, denoted by $F_i + F_i = F_{i-2} + F_{i+1}$.*

The players alternate moving. The game ends when one player moves to create the Zeckendorf decomposition.

In the following results, p represents the number of players in the Zeckendorf game.

1.2. Previous Results.

Baird-Smith, Epstein, Flint, and Miller [2] proved the following results in the Zeckendorf game.

Theorem 1.2. *Every game terminates in a finite number of moves at the Zeckendorf decomposition.*

Theorem 1.3. *In the two-player game ($p = 2$), for any $n > 2$, player 2 always has a winning strategy.*

It is worth noting that the proof of Theorem 1.3 is nonconstructive, and it is still an open problem to find a constructive winning strategy for player 2.

1.3. New Results.

Theorem 1.4. *When $n \geq 5$, for any $p \geq 3$, no player has a winning strategy.*

Now we extend the multiplayer games to the game of more than two alliances, or teams. We use t to represent the number of teams. We then have the following theorems.

Theorem 1.5. *For any $n \geq 2k^2 + 4k$ and $t \geq 3$, if each team has exactly $k = t - 1$ consecutive players, then no team has a winning strategy.*

Theorem 1.6. *Let $n \geq 30$ and $p = 6$. If one alliance has four players and the other alliance has two players, then the 4-player alliance always has a winning strategy.*

Theorem 1.7. *Let $n \geq 32$ and $p \geq 7$. If there are two alliances, one alliance with $p - 2$ players (called the big alliance), and the other with exactly two players (called the small alliance), then the big alliance always has a winning strategy.*

Finally, we extend this to larger alliances with the following theorems.

Theorem 1.8. *For any $n \geq 4pb + 2p - 2b$, if one alliance contains more than two-thirds of the players and there is some integer b such that if player i is not in the alliance, player $(i - b) \bmod p$ is in it, and the alliance has at least $2b$ players in a row, then that alliance has a winning strategy.*

Theorem 1.9. *Let $n \geq 2p + 4b$. If there is some integer b such that if player i is not in the alliance, then player $(i - b) \bmod p$ is in the alliance, and the alliance has at least $3b$ players in a row, then that alliance has a winning strategy.*

Theorem 1.10. *Assume we have one big alliance (size $2d$) and one small alliance (size d). In this two-alliance game consisting of $3d$ players (d can be any positive integer), when the small alliance consists of d consecutive players and the big alliance consists of $2d$ consecutive players, if $n \geq 12d^2 + 4d$, then the big alliance always has a winning strategy.*

2. Winning Strategy for the Zeckendorf Game

2.1. Proof of Theorem 1.4.

Note: In all the following proofs of this section, player 0 is player p modulo p.

To prove Theorem 1.4, we introduce the following property.

Property 1. *Suppose player m has a winning strategy ($1 \leq m \leq p$). For any $p \geq 3$, if player m is not the player who takes Step 2 listed below, then any winning path of player m does not contain the following three consecutive steps:*

Step 1: $1 + 1 = 2$.
Step 2: $1 + 1 = 2$.
Step 3: $2 + 2 = 1 + 3$.

Proof. Suppose player m has a winning strategy and there is a winning path that contains these three consecutive steps. Then, there exists a player a where $1 \leq a \leq p$, $a \neq m$, such that player $a-1 \pmod{p}$ can take Step 1, player a can take Step 2, and player $a+1 \pmod{p}$ can take Step 3.

Note that instead of doing $1+1=2$, player a can do $1+2=3$. Then, player $m-1 \pmod{p}$ has a winning strategy, which is a contradiction.

Therefore, by using the stealing strategy, Property 1 holds. □

We now prove Theorem 1.4 by splitting it into the following two lemmas, Lemma 2.1 and Lemma 2.2.

Lemma 2.1. *When $n \geq 13$, for any $p \geq 4$, no player has a winning strategy.*

Proof. Suppose player m has a winning strategy ($1 \leq m \leq p$).

Consider the following two cases.

Case 1. If $m \geq 4$, then players 1, 2, 3 can do the following:

Player 1: $1+1=2$.
Player 2: $1+1=2$.
Player 3: $2+2=1+3$.

This contradicts Property 1, so player m does not have a winning strategy for any $m \geq 4$.

Case 2. If $m \leq 3$, then after player m's first move, players $m+1$, $m+2$, $m+3$ can do the following:

Player $m+1$: $1+1=2$.
Player $m+2$: $1+1=2$.
Player $m+3$: $2+2=1+3$.

This contradicts Property 1, so player m does not have a winning strategy for any $m \leq 3$.

By Case 1 and Case 2, Lemma 2.1 is proved. □

Lemma 2.2. *When $n \geq 13$, for $p = 3$, no player has a winning strategy.*

Proof. Suppose player m has a winning strategy ($1 \leq m \leq 3$).

After player m's first move, players $m+1$ and $m+2$ can do the following (if $m = 3$, we can start the following process from the first step of the game):

Player $m+1$: $1+1=2$ (Step 1).
Player $m+2$: $1+1=2$ (Step 2).
Player m: Player m can do any valid move (Step 3).

Note that if player m does $2+2=1+3$, then Steps $1, 2$, and 3 violate Property 1, which is a contradiction.

If player m does any valid move other than $2+2=1+3$, then after player m's first move, the other two players can do the following (continuing after the first three steps listed above with two more steps; if $m = 3$, player $m+1$ is player 1):

Player $m+1$: $1+1=2$ (Step 1).
Player $m+2$: $1+1=2$ (Step 2).
Player m: Player m can do any valid move (Step 3).
Player $m+1$: $1+1=2$ (Step 4).

Player $m+2$: $2+2=1+3$ (Step 5).

Note that Step 3 removes at most one 2, but Step 1 and Step 2 generate two 2's in total, so there will be at least one 2 remaining after Step 3. Therefore, player $m+1$ can do $1+2=3$ instead in Step 4. By doing so, now player $m-1 \pmod p$ has a winning strategy, which is a contradiction.

Thus by using the stealing strategy, Lemma 2.2 is proved. □

By Lemmas 2.1 and 2.2, and brute force computations for $n < 13$, Theorem 1.4 is proved.

2.2. **Proof of Theorem 1.5.** For the following proofs, team 0 is team t modulo t.

Note that player tk's next player is player 1, and we regard player tk and player 1 as two consecutive players. Therefore, without loss of generality, in all the following proofs, we assume that team 1 has player 1, 2, 3, …, k; team 2 has player $k+1$, $k+2$, …, $2k$; team 3 has player $2k+1$, $2k+2$, …, $3k$ and so on.

For this proof, we utilize the following property.

Property 2. *Suppose team m has a winning strategy ($1 \le m \le t$). For any $t \ge 3$ and $k = t-1$, if none of the middle k players listed below belong to team m, then any winning path for team m does not contain the following $3k$ consecutive steps:*

First k players all do: $1+1=2$.

Middle k players all do: $1+1=2$.

Last k players all do: $2+2=1+3$.

Proof. Suppose team m has a winning strategy and there is a winning path for team m that contains such $3k$ consecutive steps. Then there exists q ($1 \le q \le p$) such that player q belongs to team m and takes the last step of the game.

For the middle k players, instead of doing $1+1=2$, they can all do $1+2=3$.

By doing so, player $q-k$ now becomes the player who takes the last step.

Note that team m has k players, so player $q-k$ belongs to team $m-1 \pmod t$.

So team $m-1 \pmod t$ has a winning strategy, which contradicts our assumption.

Therefore, by using the stealing strategy, it is proved that Property 2 holds. □

After proving Property 2, we prove Theorem 1.5 by splitting it into the following two lemmas: Lemmas 2.3 and 2.4.

Lemma 2.3. *When $n \ge 2k^2+4k$, for any $t \ge 4$ and $k = t-1$, no team has a winning strategy.*

Proof. Suppose team m has a winning strategy ($1 \le m \le t$).

Note that the last player in team m is player mk, so the first player after team m is player $mk+1 \pmod p$.

Also, there are $t-1=k$ other teams, and each team has k players, where $k \ge 4-1=3$. Therefore, there are $k^2 \ge 3k$ consecutive players from other teams. After all the members of team m's first move, the consecutive $t-1=k$ other teams can do the following:

(If $m=t$, we start the following steps from the first step of player 1.) In all the following, all players' numbers are mod p.

From player $mk+1$ to $(m+1)k$, all do $1+1=2$.

From player $(m+1)k+1$ to player $(m+2)k$, all do $1+1=2$.

From player $(m+2)k+1$ to player $(m+3)k$, all do $2+2=1+3$.

Because all these $3k$ players are not from team m, it contradicts Property 2, so team m does not have a winning strategy. Therefore, Lemma 2.3 is proved. □

Lemma 2.4. *When $n \ge 30$, for any $t=3$ and $k=2$, no team has a winning strategy.*

Proof. Suppose team m has a winning strategy ($1 \leq m \leq 3$). Note that the game has three teams and six players in total, so all the players' numbers listed below are modulo 6, and all the teams' numbers listed below are modulo 3. Team $m+1$ has players $2m+1$ and $2m+2$; team $m-1$ has players $2m+3$ and $2m+4$; team m has players $2m-1$ and $2m$.

After player $2m$'s (last player from team m) first move, let us do the following first:
(If $m=3$, the same following process can start from first step of player 1.)
Player $2m+1$: $1+1=2$ (Step 1).
Player $2m+2$: $1+1=2$ (Step 2).
Player $2m+3$: $1+1=2$ (Step 3).
Player $2m+4$: $1+1=2$ (Step 4).
Player $2m-1$: Any valid move (Step 5).
Player $2m$: Any valid move (Step 6).
Player $2m+1$: $1+1=2$ (Step 7).
Player $2m+2$: $1+1=2$ (Step 8).
Player $2m+3$: $1+1=2$ (Step 9).
Player $2m+4$: $1+1=2$ (Step 10).
Player $2m-1$: Any valid move (Step 11).
Player $2m$: Any valid move (Step 12).

(Note: Steps 5, 6, 11, and 12 can be any valid move because they are controlled by team m.)

Now, we prove this lemma in two cases.

<u>Case 1</u>. If Steps 5 and 6 are $2+2=1+3$, then look at Steps 1, 2, 3, 4, 5, and 6.

This contradicts Property 2, so team m has no winning strategy.

Similarly, if Step 11 and 12 are both $2+2=1+3$, then look at Steps 7, 8, 9, 10, 11, and 12.

This contradicts Property 2, so team m has no winning strategy.

<u>Case 2</u>. Otherwise, if one of Steps 5 or 6 is not $2+2=1+3$, and one of Steps 11 and 12 is not $2+2=1+3$, then let us do the following after player $2m$'s first move (continuing after the first 12 steps with four more steps; if $m=3$, the same process can start from the first step of player 1):

Player $2m+1$: $1+1=2$ (Step 1).
Player $2m+2$: $1+1=2$ (Step 2).
Player $2m+3$: $1+1=2$ (Step 3).
Player $2m+4$: $1+1=2$ (Step 4).
Player $2m-1$: Any valid move (Step 5).
Player $2m$: Any valid move (Step 6).
Player $2m+1$: $1+1=2$ (Step 7).
Player $2m+2$: $1+1=2$ (Step 8).
Player $2m+3$: $1+1=2$ (Step 9).
Player $2m+4$: $1+1=2$ (Step 10).
Player $2m-1$: Any valid move (Step 11).
Player $2m$: Any valid move (Step 12).
Player $2m+1$: $1+1=2$ (Step 13).
Player $2m+2$: $1+1=2$ (Step 14).
Player $2m+3$: $2+2=1+3$ (Step 15).
Player $2m+4$: $2+2=1+3$ (Step 16).

Note that one of Steps 5 and 6 is not $2 + 2 = 1 + 3$, and one of Steps 11 and 12 is not $2 + 2 = 1 + 3$, so Steps 5 and 6 will take away at most three 2's in total, and Steps 11 and 12 will take away at most three 2's in total. Also note that Steps 1, 2, ..., 10 generate eight 2's in total. So after Step 12, there will be at least two 2's remaining.

Therefore, for players $2m + 1$ and $2m + 2$, instead of doing $1 + 1 = 2$, they can do $1 + 2 = 3$.

Because team m has a winning strategy, player $2m - 1$ or player $2m$ takes the last step.

If player $2m - 1$ originally takes the last step, by using the stealing strategy mentioned above, player $2m - 1 - 2 = 2m - 3$ now takes the last step, which means that player $2m + 3$ takes the last step, so team $m - 1$ has a winning strategy, which contradicts our assumption.

If player $2m$ originally takes the last step, by using the stealing strategy mentioned above, player $2m - 2$ now takes the last step, which means that player $2m + 4$ takes the last step, so team $m - 1$ has a winning strategy, which contradicts our assumption.

In both cases, we can find a contradiction by using the stealing strategy, so Lemma 2.4 is proved. □

Therefore, by Lemmas 2.3 and 2.4, Theorem 1.5 is proved.

2.3. **Proof of Theorem 1.6.**
Note: In the following proof, because player 6's next player is player 1, player 6 and player 1 are considered to be consecutive players.

The 4-player alliance has three possible cases.

Case 1. If the 4-player alliance consists of four consecutive players, then the 2-player alliance will also have two consecutive players.

To show that the 2-player alliance does not actually have a winning strategy, the four consecutive players in the big alliance can be regarded as two teams, where each team has two consecutive players.

Therefore, according to Lemma 2.4, the 2-player alliance does not have a winning strategy.

Therefore, the 4-player alliance always has a winning strategy in this case.

Case 2. Assume the 4-player alliance is separated into two parts, and each part has two consecutive players.

Note that this situation is equivalent to the 3-player game situation, where two of them are on the same team.

According to Lemma 2.2, the single player in the 3-player game does not have a winning strategy.

Equivalently, the 2-player alliance does not have a winning strategy.

Therefore, the 4-player alliance always has a winning strategy.

Case 3. Assume the 4-player alliance is separated into two parts, where one part has three consecutive players and the other part only has one player.

Then, the players in the 2-player alliance are separated from each other (if they are not separated, then the players of the 4-player alliance will be four consecutive players).

If the 2-player alliance has a winning strategy, then there always exists a player q from the 2-player alliance who takes the last step.

Assume the three consecutive players in the 4-player alliance are player a, $a+1 \pmod 6$, $a+2 \pmod 6$. Then, let us do the following from player a's first move:

Player a: $1+1=2$.

Player $a+1$: $1+1=2$.

Player $a+2$: $2+2=1+3$.

Note that if player $a+1$ does $1+2=3$ instead, then player $q-1$ will now be the player who takes the last step. Because two players of the 2-player alliance are separated, player $q-1$ belongs to the 4-player alliance. Therefore, the 4-player alliance now has a winning strategy, which contradicts our assumption.

Therefore, by using the stealing strategy, we prove that the 4-player alliance has a winning strategy.

Thus, by Case 1, Case 2, and Case 3, Theorem 1.6 follows.

2.4. **Proof of Theorem 1.7.** We look at the case of 7-player game first.

Lemma 2.5. *When $n \geq 32$, if one alliance has five players and the other alliance has two players, then the 5-player alliance always has a winning strategy.*

Proof. We prove this lemma by considering two cases.

Case 1. If the players in the 2-player alliance are not consecutive players, then the 5-player alliance will be separated into two parts (considering player 7 and player 1 as consecutive players).

According to the pigeonhole principle, one of these two parts will contain at least three consecutive players (we call this part the "large part").

If the 2-player alliance has a winning strategy, then there exists a player q from the 2-player alliance who takes the last step.

Suppose the first player in the large part is player a. Then, starting from player a's first move, let us do the following:

Player a: $1+1=2$.

Player $a+1$: $1+1=2$.

Player $a+2$: $2+2=1+3$.

Note that instead of doing $1+1=2$, player $a+1$ can do $1+2=3$. Then, player $q-1$ is the player who takes the last step of the winning path.

Because the two players in the 2-player alliance are not consecutive, player $q-1$ belongs to the 5-player alliance.

As a result, the 5-player alliance now has a winning strategy, which contradicts our assumption.

Therefore, by using the stealing strategy, we proved that the 5-player alliance has the winning strategy.

Case 2. If the players in the 2-player alliance are consecutive, then the players in the 5-player are also consecutive.

Suppose that the five consecutive players of the 5-player alliance starts with player a.

Then, starting with player a's first move, let us do the following:

Player a: $1+1=2$ (Step 1).

Player $a+1$: $1+1=2$ (Step 2).
Player $a+2$: $1+1=2$ (Step 3).
Player $a+3$: $1+1=2$ (Step 4).
Player $a+4$: $1+1=2$ (Step 5).
Player $a+5$: Any valid move (Step 6).
Player $a+6$: Any valid move (Step 7).
(Note that player $a+5$ and player $a+6$ are in the 2-player alliance.)
Player a: $1+1=2$ (Step 8).
Player $a+1$: $1+1=2$ (Step 9).
Player $a+2$: $1+1=2$ (Step 10).
Player $a+3$: $2+2=1+3$ (Step 11).
Player $a+4$: $2+2=1+3$ (Step 12).

If the 2-player alliance has a winning strategy, then there always exists a player q from the 2-player alliance who takes the last step.

Note that Step 6 and Step 7 can take away at most four 2's in total, and Steps 1, 2, 3, 4, 5, and 8 have generated six 2's in total.

As a result, after Step 8, there will be at least two 2's remaining.

Therefore, player $a+1$ in Step 9 and player $a+2$ in Step 10 can do $1+2=3$ instead. Then, player $q-2$ becomes the player who takes the last step.

Because two players of the 2-player alliance are consecutive, player $q-2$ belongs to the 5-player alliance. Therefore, the 5-player alliance now has a winning strategy, which contradicts our assumption.

Therefore, by using the stealing strategy, Case 2 is proved.

Thus, by our analysis in Case 1 and Case 2, Lemma 2.5 is proved. □

Now, let us look at the game of eight or more players.

Lemma 2.6. *In a p-player game with two alliances, when n is significantly large ($n \geq 22$) and $p \geq 8$, if one alliance has $p-2$ players (let us call this the big alliance, which has at least six players), and the other alliance has two players, then the big alliance always has a winning strategy.*

Proof. We prove this lemma by considering two cases.

Case 1. If the players of the 2-player alliance are not consecutive, then the big alliance will be separated into two parts.

Note that the big alliance has at least six players. By the pigeonhole principle, there will be at least one part having at least three players (let us call this the big part).

Suppose 2-player alliance has a winning strategy. Then for any winning path, there exists a player q in the 2-player alliance who takes the last step.

Suppose the first player in the big part is player a, and let us do the following starting from player a's first move:
Player a: $1+1=2$ (Step 1).
Player $a+1$: $1+1=2$ (Step 2).
Player $a+2$: $2+2=1+3$ (Step 3).

Note that instead of doing $1+1=2$, player $a+1$ can do $1+2=3$ instead in Step 2. Now, player $q-1$ becomes the player who takes the last step. Because the players in the 2-player alliance are not consecutive, player $q-1$ belongs to the big alliance, so the big alliance now has a winning strategy, which contradicts our assumption.

Therefore, we proved Case 1 by using the stealing strategy.

Case 2. If the players of the 2-player alliance are consecutive, then the $p-2$ players of the big alliance are consecutive.

If the 2-player alliance has a winning strategy, then for any winning path, there exists a player q from the 2-player alliance who takes the last step.

Suppose the big alliance's $p-2$ consecutive players start with player a.

(Note the big alliance has at least six players, so players a, $a+1$, $a+2$, $a+3$, $a+4$, and $a+5$ are all in the big alliance).

Let us do the following, starting from player a's first move:

Player a: $1+1=2$ (Step 1).

Player $a+1$: $1+1=2$ (Step 2).

Player $a+2$: $1+1=2$ (Step 3).

Player $a+3$: $1+1=2$ (Step 4).

Player $a+4$: $2+2=1+3$ (Step 5).

Player $a+5$: $2+2=1+3$ (Step 6).

Note that player $a+2$ in Step 3 and player $a+3$ in Step 4 can do $1+2=3$ instead.

If they do so, then player $q-2$ becomes the player who takes the last step. Because the players in the 2-player alliance are consecutive, player $q-2$ belongs to the big alliance. Therefore, the big alliance now has a winning strategy, which contradicts our assumption.

By using the stealing strategy, we proved Case 2.

Thus, by our analysis in Case 1 and Case 2, Lemma 2.6 is proved. □

By Lemmas 2.5 and 2.6, Theorem 1.7 is proved.

2.5. **Proof of Theorem 1.8.** Assume we have an alliance a with more than two-thirds of the players. For a sufficiently large n, alliance a can then force the creation of an arbitrary number of 2's eventually, as players in the alliance can each produce at least one 2 per round: if each player plays $1+1=2$, opposing players can each remove only two 2's per round by playing $2+2=1+3$, meaning each round, alliance a can have a net increase in the number of 2's by at least one.

As such, at some point, alliance a can force the creation of at least b 2's. By assumption, alliance a has at least $2b$ subsequent players. Assume we have at least b 2's and we are about to begin the turns of those players.

The first b players could all play $1+1=2$, and the next b players could all play $2+2=1+3$, as the first b players would create b 2's and the next b players would use up those and the preexisting b 2's. Let us suppose the turn after this is turn $2b$, changing what we call turn 0 to make this the case.

Now, assume the opposing players have a winning strategy. In that case, after this, there would be a winning strategy from the resultant state for an alliance that begins when the opposing players do. In particular, this alliance plays on turns $2b+p_1$, $2b+p_2$, $2b+p_3$, …, $2b+p_n$, where p_1, p_2, … are some numbers selected to make this the case.

Now, note that the $2b$ players can instead play as follows: the first b can all play $1+2=3$, resulting in the same state as the one we reached after all $2b$ players played last time. As such, the same strategy as the one used previously results in a win for the alliance that plays on turns $b+p_1, b+p_2, b+p_3, \ldots, b+p_n$.

By assumption, alliance a has b players play before a player from the opposition plays. We know the players with turns $2b+p_1$, $2b+p_2$, $2b+p_3$, …, $2b+p_n$ are opposed to alliance a, so the players with turns $b+p_1$, $b+p_2$, $b+p_3$, …, $b+p_n$ must be in alliance a.

As such, this would be a winning strategy for alliance a, contradicting the assumption that the opposing players have a winning strategy. As such, alliance a must have a winning strategy.

2.6. **Proof of Theorem 1.9.** Assume we have an alliance a with $3b$ consecutive players at some point, and $n \geq 2p+4b$.

First, let us examine the first turn of the $3b$ players. As n is at least $2p+4b$, regardless of when they start, there will be enough 1's in the game for the first $2b$ of them to play $1+1=2$. If they do so, the next b could all play $2+2=1+3$. Let us assume the turn after this is turn $3b$, changing what we call turn 0 to make this the case.

Now, assume the opposing players have a winning strategy. In that case, after this, there would be a winning strategy from the resultant state for an alliance that begins when the opposed players do. In particular, this alliance plays on turns $3b+p_1$, $3b+p_2$, $3b+p_3$, …, $3b+p_n$, where p_1, p_2, … are some numbers selected to make this the case.

Now, note that the $3b$ players can instead play as follows: the first b players can all play $1+1=2$ and the next b players can all play $1+2=3$, resulting in the same state as the one we produced after all $3b$ players played last time.

As such, the same strategy as the one used previously gives the win to the alliance that plays on turns $2b+p_1, 2b+p_2, 2b+p_3, \ldots, 2b+p_n$. By assumption, alliance a has b players play before a player from the opposition. We know the players with turns $3b+p_1$, $3b+p_2$, $3b+p_3$, …, $3b+p_n$ are opposed to alliance a, so the players with turns $2b+p_1, 2b+p_2, 2b+p_3, \ldots, 2b+p_n$ must be in alliance a.

As such, this would be a winning strategy for alliance a, contradicting the assumption that the opposing players have a winning strategy. As such, alliance a must have a winning strategy.

2.7. **Proof of Theorem 1.10.** Let the $2d$ players in the big alliance all do $1+1=2$ for the first d rounds (one round means every player takes a move, and we define the first round starting from the big alliance's first move).

If in any of the first d round, the d consecutive players in the small alliance all do $2+2=1+3$, then we can directly let the second half of the $2d$ players in the big alliance (which is d consecutive players) all do $1+2=3$ instead in the next round.

In this case, suppose that the small alliance has a winning strategy. Then for any winning path, there exists a player q from the small alliance who takes the last step.

Note that player q belongs to the small alliance, so player $q-d$ belongs to the big alliance. Because player q's last winning step becomes player $q-d$'s last step, the big alliance now has a winning strategy, which contradicts our assumption.

Otherwise, if in each of the first d rounds, there is at least one player from the small alliance who does not play $2+2=1+3$, that means that this step will take away at most one 2. Then, there will be at least one 2 generated in each round.

As a result, after d rounds, there are at least d 2's generated from the stealing. After that, in the $(d+1)$th round, we can let the first half of the players in the big alliance (which are the first d consecutive players) all do $1+1=2$, and the second half of the big alliance (which are last d consecutive players) all do $2+2=1+3$.

In this case, suppose that the small alliance has a winning strategy. Then for any winning path, there exists a player q from the small alliance who takes the last step.

Note that player q belongs to the small alliance, so player $q-d$ belongs to the big alliance. In this round, the first half of the big alliance (which are the first d consecutive players) can all do $1 + 2 = 3$ instead, so player q's last winning step becomes player $q - d$'s last step. As a result, the big alliance now has a winning strategy, which contradicts our assumption.

Therefore, the big alliance always has a winning strategy, so we have proved this theorem.

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