AMICABLE HERON TRIANGLES

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Abstract. A Heron triangle is a triangle whose side lengths and area are integers. Two Heron triangles are *amicable* if the perimeter of one is the area of the other. We show, using elementary techniques, that there is only one pair of amicable Heron triangles.

1. Introduction

A Heron triangle is a triangle whose side lengths and area are integers. They are named after the Greek mathematician Heron (or Hero) of Alexandria, who is usually credited with inventing the formula for the area A of a triangle in terms of its side lengths a, b, c:

$$A = \sqrt{s(s-a)(s-b)(s-c)};$$

here s is the semiperimeter $\frac{1}{2}(a+b+c)$.

Heron triangles form a popular topic (e.g., [1, 2, 6, 7]), and new facts about them are still being discovered. For example, Hirakawa and Matsumura [5] showed that there is a unique pair (up to scaling) of right and isosceles Heron triangles with the same perimeter and the same area. The proof uses sophisticated tools from the theory of hyperelliptic curves. The result has been featured in a Numberphile video [4].

The video primarily focuses on *equable* Heron triangles (called Super-Hero triangles in the video), i.e., Heron triangles where the perimeter is equal to the area. This seems analogous to *perfect numbers*. Recall that a perfect number is a positive integer whose aliquot sum is equal to itself. (The aliquot sum of n is the sum of the divisors of n, excluding n.) In equable Heron triangles, the perimeter and area play the roles of n and its aliquot sum.

Venturing beyond a single number, recall that a pair of positive integers n and m form an *amicable pair* if the aliquot sum of n is equal to m and the aliquot sum of m is equal to n. Analogously, we define two Heron triangles H_1 and H_2 to be *amicable* if the area of H_1 is equal to the perimeter of H_2 and the perimeter of H_1 is equal to the area of H_2.

Amicable Heron triangles exist: the triangles with side lengths $(3, 25, 26)$ and $(9, 12, 15)$ form an example. They are an unusual looking pair (Figure 1).

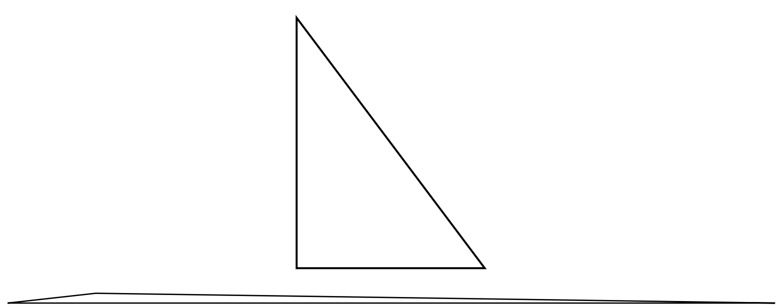

Figure 1. A unique pair of triangles

However, there is no other example.

Theorem. *There is only one pair of amicable Heron triangles: the* $(3, 25, 26)$ *and* $(9, 12, 15)$ *triangles.*

In contrast to [5], we use completely elementary methods to prove this result.

2. Proofs

We begin by establishing our notation. Suppose we have a Heron triangle with side lengths a, b, c. Its semiperimeter $s = (a + b + c)/2$ is an integer. We define $x = s - a$, $y = s - b$, $z = s - c$, and set $x \le y \le z$. Then Heron's formula for the area of the triangle becomes $\sqrt{sxyz}$.

Lemma 1. *Suppose H is one of a pair of amicable Heron triangles with perimeter p and area A. Then, A divides $2p^2$.*

Proof. For any Heron triangle,

$$\frac{2A^2}{p} = \frac{2sxyz}{2s} = xyz$$

is an integer. Apply this result to the partner triangle of H. Because the perimeter of H is equal to the area of its partner and the area of H is equal to the perimeter of its partner, we get the result of the lemma. □

We now take care of the case where both amicable triangles are equable. It is well-known that there are only five equable Heron triangles [3]: triangles with side lengths $(5, 12, 13)$, $(6, 8, 10)$, $(6, 25, 29)$, $(7, 15, 20)$, and $(9, 10, 17)$. Their perimeters (and thus their areas) are all different, so none of them form an amicable pair. Equable triangles are not amicable.

We conclude that if we have an amicable pair, then for one of the triangles, the perimeter is larger than the area. These triangles are long and skinny, similar to the second triangle in Figure 1. There are not many such triangles. From now on, let H denote a triangle of this kind.

Lemma 2. *For H as above,*

$$4(x + y + z) > xyz. \tag{$*$}$$

Proof. This is a consequence of the perimeter of H being larger than the area of H. □

These two lemmas are our main tool for cutting down the possible values of x, y, and z. They suffice to show that there are only finitely many.

Lemma 3. *Let H be as above. Then, there are only a finite number of x, y, z values that satisfy Lemmas 1 and 2.*

Proof. We first show that $x \le 3$. If $x \ge 4$, then by Lemma 2,

$$4(z + z + z) \ge 4(x + y + z) > xyz \ge 4 \cdot 4 \cdot z = 16z,$$

a contradiction, so $x \le 3$.

Now, we show that $y \le 9$. If $y \ge 10$, then

$$4(3 + z + z) \ge 4(x + y + z) > xyz \ge 10z \implies 12 + 8z > 10z \implies 6 > z,$$

which is a contradiction because $z \ge y \ge 10$. We conclude that there are only finitely many values of x and y.

We now tackle z. Lemma 1 states that $2p^2/A$ is an integer, which means

$$\frac{8s^2}{\sqrt{sxyz}} \in \mathbb{N} \implies \frac{64s^4}{sxyz} \in \mathbb{N} \implies \frac{64(x+y+z)^3}{xyz} \in \mathbb{N} \implies \frac{64(x+y+z)^3}{z} \in \mathbb{N}.$$

Let $c = x+y$. Then, $64(z+c)^3/z = 64(z^2+3zc+3c^2+c^3/z)$ is an integer, which implies that $64c^3/z$ is an integer. So, z must be a divisor of $64(x+y)^3$, and because there are only finitely many values of $64(x+y)^3$, there are only finitely many values of z also. $\square$

We now need to investigate only a finite number of cases. It turns out that the possibilities can be cut down considerably by using the requirement that the area of H is an integer. We provide an example here; other cases are similar.

Suppose $x = 1$ and $y = 4$. The $2p^2/A$ calculation in Lemma 3 shows that $4z$ divides $64(z+5)^3$; we conclude that z is a divisor of $2^4 \cdot 5^3$. Because $z \geq 4$, there are 18 possibilities for z. The area of H is $\sqrt{4z(z+5)}$; among the 18 possible values of z, only $z = 4$ produces an integer area. Thus, this case produces only one possibility: $x = 1$, $y = 4$, and $z = 4$.

Indeed, when we check all possible values of x, y, z, we come up with just four cases that satisfy all the requirements mentioned above:

- $x = 1$, $y = 4$, $z = 4$, producing H with side lengths 5, 5, 8.
- $x = 1$, $y = 2$, $z = 3$, producing H with side lengths 3, 4, 5.
- $x = 1$, $y = 2$, $z = 24$, producing H with side lengths 3, 25, 26.
- $x = 1$, $y = 2$, $z = 864$, producing H with side lengths 3, 865, 866.

The first two cases are easy to eliminate. The first case produces a triangle of area 12 and perimeter 18. Its amicable partner, if it exists, must have semiperimeter 6. This yields three possibilities: $x = 1$, $y = 2$, $z = 3$ or $x = 1$, $y = 1$, $z = 4$ or $x = 2$, $y = 2$, $z = 2$, none of which yields an area of 18. The second case produces a triangle of area 6, but it is impossible to have a partner triangle of perimeter 6, the only possibility being an equilateral triangle with side-length 2 that has an irrational area.

If H is the fourth triangle listed above, then its perimeter is 1734 and its area is 1224. Thus, its partner triangle has perimeter 1224 and area 1734. Therefore, for the partner triangle, we have $xyz = 1734^2/612 = 4913$, an odd integer, which implies that x, y, and z are all odd. This contradicts that the semiperimeter of the partner triangle is 612. Therefore, H has no partner triangle.

The third case produces the amicable pair mentioned in the Theorem. This concludes the proof of the Theorem: there is a unique pair of amicable Heron triangles.

References

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